Estimates for the Bergman Kernel and Logarithmic Capacity

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Basic Notation ($n = 1$)

$$\begin{cases} \Delta G_D(\cdot, w) = 2\pi\delta_w \\ G_D(\cdot, w) = 0 \text{ on } \partial D \quad \text{(if } D \text{ is regular)} \end{cases}$$

(Green function with pole at $w \in D \subset \mathbb{C}$)

$$c_D(w) := \exp \lim_{z \to w} \left(G_D(z, w) - \log|z - w| \right)$$

(logarithmic capacity of $\mathbb{C} \setminus D$ w.r.t. $w \in D$)

$A^2(D) := \mathcal{O} \cap L^2(D)$ (Bergman space)

$f(w) = \int_D f \, \overline{K_D(\cdot, w)} \, d\lambda, \quad w \in D, \ f \in A^2(D)$ (Bergman kernel)

Then

$$K_D(w) = K_D(w, w) = sup\{|f(w)|^2 : f \in \mathcal{O}(D), \ \int_D |f|^2 d\lambda \leq 1\}$$

and $K_D(z, w) = \dfrac{2}{\pi} \dfrac{\partial^2}{\partial z \partial \bar{w}} G_D(z, w), \ z \neq w$ (Schiffer)

Suita Conjecture (SC)

Suita (1972) conjectered the following estimate

$$(c_D(w))^2 \leq \pi K_D(w,w), \quad w \in D. \tag{1}$$

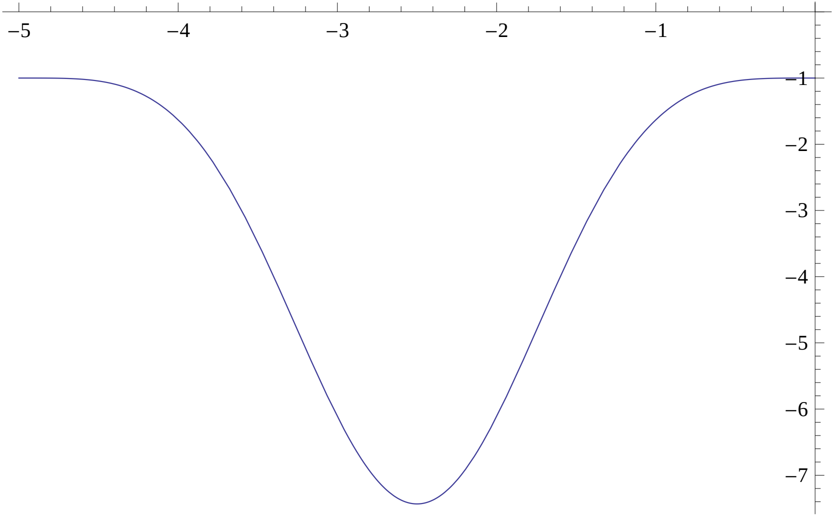
$c_D|dz|$ is an invariant metric (Suita metric)

$$Curv_{c_D|dz|} = -\frac{(\log c_D)_{z\bar{z}}}{c_D^2}$$

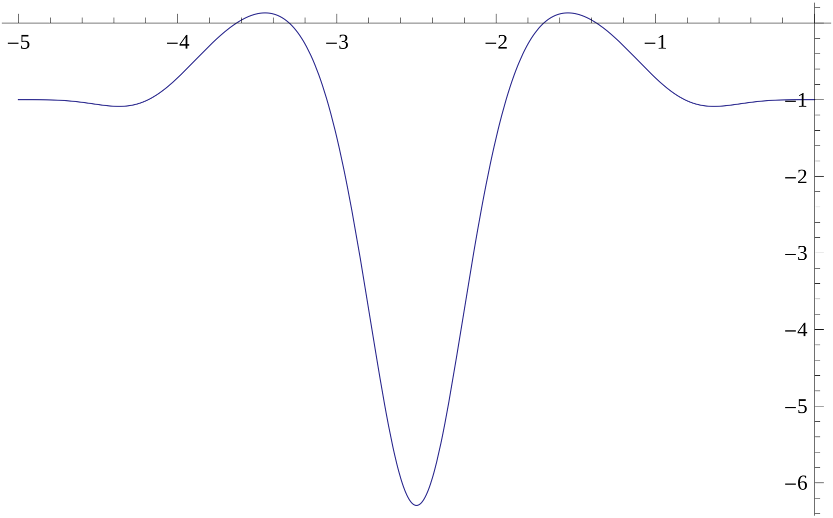
Since $(\log c_D)_{z\bar{z}} = \pi K_D$ (Suita), (1) is equivalent to

$$Curv_{c_D|dz|} \leq -1$$

- "$=$" if D is simply connected
- "$<$" if D is an annulus (Suita)
- Enough to prove for D with smooth boundary
- "$=$" on ∂D if D has smooth boundary



$Curv_{c_D|dz|}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log|z|$



$Curv_{(\log K_D)_{z\bar{z}}|dz|^2}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log |z|$

Ohsawa (1995) observed that SC can be treated as an extension problem: for $w \in D$ find $f \in \mathcal{O}(D)$ s.th. $f(w) = 1$ and

$$\int_D |f|^2 d\lambda \leq \frac{\pi}{(c_D(w))^2}.$$

Using the SCV methods of the Ohsawa-Takegoshi extension theorem he showed the estimate

$$c_D^2 \leq C\pi K_D$$

with $C = 750$.

$C = 2$ (B. 2007)
$C = 1.95388\ldots$ (Guan-Zhou-Zhu 2011)

Optimal estimate (SC) $C = 1$ (B. 2013)

Main tool: Hörmander's L^2-estimate for the $\bar\partial$-equation

Guan-Zhou 2015: "=" in SC $\Leftrightarrow$ $D \simeq \mathbb{D} \setminus F$, where $F \subset \mathbb{D}$ is polar (also for Riemann surfaces).

Carleson 1967: $A^2(D) = \{0\} \quad \Leftrightarrow \quad \mathbb{C} \setminus D$ is polar

The estimate $c_D^2 \le \pi K_D$ gives a quantitative version of $\Rightarrow$.
What about a quantitative version of $\Leftarrow$?

B.-Zwonek 2018 $w \in D$, $0 < r \le \delta_D(w) := \operatorname{dist}(w, \partial D)$. Then

$$K_D(w) \le \frac{1}{-2\pi r^2 \max_{z \in \overline{\Delta}(w,r)} G_D(z,w)}.$$

Corollary There exists $C > 0$ s.th.

$$K_D(w) \le \frac{C}{\delta_D(w)^2 \log\left(1/(\delta_D(w) c_D(w))\right)}.$$

Ohsawa-Takegoshi Extension Theorem

Ohsawa-Takegoshi 1987

Ω bounded pscvx domain in $\mathbb{C}^n$, φ psh in Ω

H complex affine subspace of $\mathbb{C}^n$

f holomorphic in $\Omega' := \Omega \cap H$

Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq C\pi \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda',$$

where C depends only on n and the diameter of Ω.

Siu / Berndtsson 1996

If $\Omega \subset \mathbb{C}^{n-1} \times \{|z_n| < 1\}$ and $H = \{z_n = 0\}$ then $C = 4$.

Problem Can we improve to $C = 1$?

Ohsawa-Takegoshi with Optimal Constant (B. 2013)

Ω pscvx in $\mathbb{C}^{n-1} \times D$, where $0 \in D \subset \mathbb{C}$,
φ psh in Ω, f holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

(For $n = 1$ and $\varphi \equiv 0$ we obtain the Suita Conjecture.)

Crucial ODE Problem Find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ s.th.
$h' < 0$, $h'' > 0$,

$$\lim_{t\to\infty} (g(t) + \log t) = \lim_{t\to\infty} (h(t) + \log t) = 0$$

and

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1.$$

Solution
$$h(t) := -\log(t + e^{-t} - 1)$$
$$g(t) := -\log(t + e^{-t} - 1) + \log(1 - e^{-t}).$$

Another Approach

$$K_\Omega(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 d\lambda \le 1\}$$

(Bergman kernel)

$$G_\Omega(\cdot, w) = \sup\{v \in PSH^-(\Omega),\ \overline{\lim_{z\to w}}(v(z) - \log|z-w|) < \infty\}$$

(pluricomplex Green function)

B. 2014 $\Omega \subset \mathbb{C}^n$ pscvx, $w \in \Omega$, $t \le 0$. Then

$$K_\Omega(w) \ge \frac{1}{e^{-2nt}\lambda(\{G_\Omega(\cdot, w) < t\})}.$$

For $n = 1$ letting $t \to -\infty$ this gives the Suita Conjecture:

$$K_\Omega(w) \ge \frac{c_\Omega(w)^2}{\pi}.$$

Proof 1 Using Donnelly-Fefferman's estimate for $\bar\partial$ one can prove

$$K_\Omega(w) \geq \frac{1}{c(n,t)\lambda(\{G_w < t\})}, \tag{2}$$

where $G_w = G_\Omega(\cdot, w)$ and

$$c(n,t) = \left(1 + \frac{C}{Ei(-nt)}\right)^2, \quad Ei(a) = \int_a^\infty \frac{ds}{se^s}$$

(Herbort 1999, B. 2005). Now use the tensor power trick:
$\widetilde\Omega = \Omega \times \cdots \times \Omega \subset \mathbb{C}^{nm}$, $\widetilde w = (w, \ldots, w)$ for $m \gg 0$. Then

$$K_{\widetilde\Omega}(\widetilde w) = (K_\Omega(w))^m, \quad \lambda(\{G_{\widetilde w} < t\}) = (\lambda(\{G_w < t\}))^m,$$

and by (2) for $\widetilde\Omega$

$$K_\Omega(w) \geq \frac{1}{c(nm,t)^{1/m}\lambda(\{G_w < t\})}.$$

But $\lim_{m\to\infty} c(nm,t)^{1/m} = e^{-2nt}$. □

Proof 2 (Lempert) By Berndtsson's result on log-(pluri)subharmonicity of the Bergman kernel for sections of a pseudoconvex domain (Maitani-Yamaguchi in dimension 2) it follows that $\log K_{\{G_w<t\}}(w)$ is convex for $t \in (-\infty, 0]$. Therefore

$$t \longmapsto 2nt + \log K_{\{G_w<t\}}(w)$$

is convex and bounded, hence non-decreasing. It follows that

$$K_\Omega(w) \geq e^{2nt} K_{\{G_w<t\}}(w) \geq \frac{e^{2nt}}{\lambda(\{G_w<t\})}. \quad \square$$

Berndtsson-Lempert 2016: This method can be improved to show the Ohsawa-Takegoshi extension theorem with optimal constant.

$$K_\Omega(w) \geq \frac{1}{e^{-2nt}\lambda(\{G_w < t\})}$$

B. 2014 If Ω is a convex domain in $\mathbb{C}^n$ then for $w \in \Omega$

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega(w))},$$

$I_\Omega(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\mathbb{D}, \Omega),\ \varphi(0) = w\}$ (Kobayashi indicatrix).

B.-Zwonek 2015 (SCV version of the Suita Conjecture) If $\Omega \subset \mathbb{C}^n$ is pscvx and $w \in \Omega$ then

$$K_\Omega(w) \geq \frac{1}{\lambda(I^A_\Omega(w))},$$

$I^A_\Omega(w) = \{X \in \mathbb{C}^n : \overline{\lim}_{\zeta\to 0}\left(G_w(w+\zeta X) - \log|\zeta|\right) \leq 0\}$
(Azukawa indicatrix)

Conjecture For pscvx Ω and $w \in \Omega$ the function

$$t \longmapsto e^{-2nt}\lambda(\{G_w < t\})$$

is non-decreasing in t.

B.-Zwonek 2015 True for $n = 1$.

B.-Zwonek 2015 For convex Ω and $w \in \Omega$ one has

$$\frac{1}{\lambda(I_\Omega(w))} \leq K_\Omega(w) \leq \frac{4^n}{\lambda(I_\Omega(w))}.$$

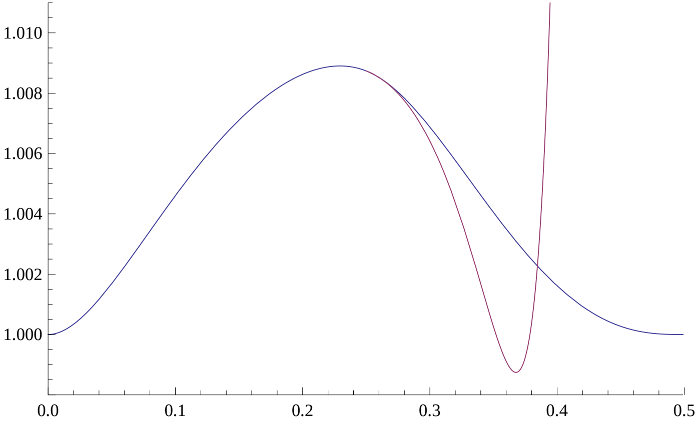
B.-Zwonek 2016 For $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1/4]$ one has

$$\lambda(I_\Omega((b,b))) = \frac{\pi^2}{6}\left(30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1\right).$$

For $b \in [1/4, 1/2)$

$$\begin{aligned}
\lambda(I_\Omega((b,b))) = {} & \frac{2\pi^2 b(1-2b)^3\left(-2b^3 + 3b^2 - 6b + 4\right)}{3(1-b)^2} \\
& + \frac{\pi\left(30b^{10} - 124b^9 + 238b^8 - 176b^7 - 260b^6 + 424b^5 - 76b^4 - 144b^3 + 89b^2 - 18b + 1\right)}{6(1-b)^2} \\
& \qquad \times \arccos\left(-1 + \frac{4b-1}{2b^2}\right) \\
& + \frac{\pi(1-2b)\left(-180b^7 + 444b^6 - 554b^5 + 754b^4 - 1214b^3 + 922b^2 - 305b + 37\right)}{72(1-b)}\sqrt{4b-1} \\
& + \frac{4\pi b(1-2b)^4\left(7b^2 + 2b - 2\right)}{3(1-b)^2}\arctan\sqrt{4b-1} \\
& + \frac{4\pi b^2(1-2b)^4(2-b)}{(1-b)^2}\arctan\frac{1-3b}{(1-b)\sqrt{4b-1}}.
\end{aligned}$$

Since $K_\Omega((b,b)) = \dfrac{2(3-6b^2+8b^4)}{\pi^2(1-4b^2)^3}$ (Hahn-Pflug 1988), we get



$\sqrt{\lambda(I_\Omega(w))K_\Omega(w)}$ for $w=(b,b)\in\Omega=\{|z_1|+|z_2|<1\}$, $b\in[0,1/2)$

Mahler Conjecture

K - convex symmetric body in $\mathbb{R}^n$

$$K' := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } x \in K\}$$

Mahler volume $:= \lambda(K)\lambda(K')$

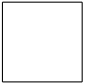
Mahler volume is an invariant of the Banach space defined by K: it is independent of linear transformations and of the choice of inner product.

Santaló Inequality (1949) Mahler volume is maximized by balls

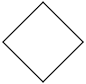
Mahler Conjecture (1938) Mahler volume is minimized by cubes

Hansen-Lima bodies: starting from an interval they are produced by taking products of lower dimensional HL bodies and their duals.

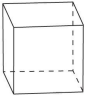
$n = 2$



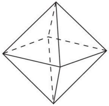
$\simeq$



$n = 3$



$\not\simeq$



Equivalent SCV Formulation (Nazarov, 2012)

For $u \in L^2(K')$ we have

$$|\widehat{u}(0)|^2 = \left| \int_{K'} u \, d\lambda \right|^2 \le \lambda(K') ||u||^2_{L^2(K')} = (2\pi)^{-n} \lambda(K') ||\widehat{u}||^2_{L^2(\mathbb{R}^n)}$$

with equality for $u = \chi_{K'}$. Therefore

$$\lambda(K') = (2\pi)^n \sup_{f \in \mathcal{P}} \frac{|f(0)|^2}{||f||^2_{L^2(\mathbb{R}^n)}},$$

where $\mathcal{P} = \{\widehat{u} \colon u \in L^2(K')\} \subset \mathcal{O}(\mathbb{C}^n)$. By the Paley-Wiener thm

$$\mathcal{P} = \{f \in \mathcal{O}(\mathbb{C}^n) \colon |f(z)| \le Ce^{q_K(\mathrm{Im}\, z)}, \ ||f||_{L^2(\mathbb{R}^n)} < \infty\},$$

where q_K is the Minkowski function for K. Therefore the Mahler Conjecture is equivalent to finding $f \in \mathcal{P}$ with $f(0) = 1$ and

$$\int_{\mathbb{R}^n} |f(x)|^2 d\lambda(x) \le n! \left(\frac{\pi}{2}\right)^n \lambda(K).$$

Bourgain-Milman Inequality

Bourgain-Milman (1987) There exists $c > 0$ such that

$$\lambda(K)\lambda(K') \geq c^n \frac{4^n}{n!}.$$

Mahler Conjecture: $c = 1$

G. Kuperberg (2006) $c = \pi/4$

Nazarov (2012) SCV proof using Hörmander's estimate $c = (\pi/4)^3$

Consider the tube domain $T_K := \mathrm{int} K + i\mathbb{R}^n \subset \mathbb{C}^n$. Then

$$\left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda(K))^2} \leq K_{T_K}(0) \leq \frac{n!}{\pi^n} \frac{\lambda(K')}{\lambda(K)}.$$

Therefore

$$\lambda(K)\lambda(K') \geq \left(\frac{\pi}{4}\right)^{3n} \frac{4^n}{n!}.$$

The upper bound $K_{T_K}(0) \leq \frac{n!}{\pi^n}\frac{\lambda(K')}{\lambda(K)}$ follows from Rothaus' formula (1968):

$$K_{T_K}(0) = (2\pi)^{-n}\int_{\mathbb{R}^n}\frac{d\lambda}{J_K},$$

where

$$J_K(y) = \int_K e^{-2x\cdot y}\,d\lambda(x).$$

To show the lower bound $K_{T_K}(0) \geq \left(\frac{\pi}{4}\right)^{2n}\frac{1}{(\lambda(K))^2}$ we can use our estimate:

$$K_{T_K}(0) \geq \frac{1}{\lambda_{2n}(I_{T_K}(0))}$$

and

Proposition $I_{T_K}(0) \subset \frac{4}{\pi}(K + iK)$

However, one can check that for $K = \{|x_1| + |x_2| + |x_3| \leq 1\}$ we have

$$K_{T_K}(0) > \left(\frac{\pi}{4}\right)^3 \frac{1}{(\lambda_3(K))^2}.$$

This shows that Nazarov's proof of the Bourgain-Milman inequality cannot give the Mahler conjecture directly.

Thank you!