Suita Conjecture and the Ohsawa-Takegoshi Extension Theorem

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Green function for bounded domain D in $\mathbb{C}$:

$$\begin{cases} \Delta G_D(\cdot, z) = 2\pi\delta_z \\ G_D(\cdot, z) = 0 \text{ on } \partial D \text{ (if } D \text{ is regular)} \end{cases}$$

$$c_D(z) := \exp \lim_{\zeta \to z} (G_D(\zeta, z) - \log|\zeta - z|)$$

(logarithmic capacity of $\mathbb{C} \setminus D$ w.r.t. z)

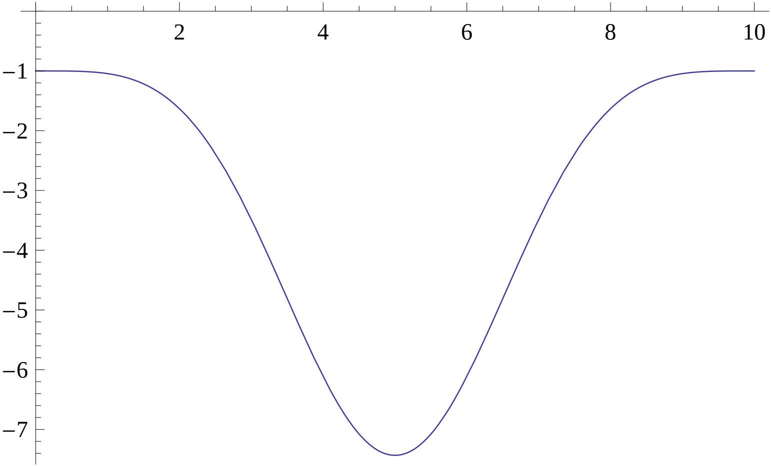
$c_D|dz|$ is an invariant metric (Suita metric)

$$Curv_{c_D|dz|} = -\frac{(\log c_D)_{z\bar{z}}}{c_D^2}$$

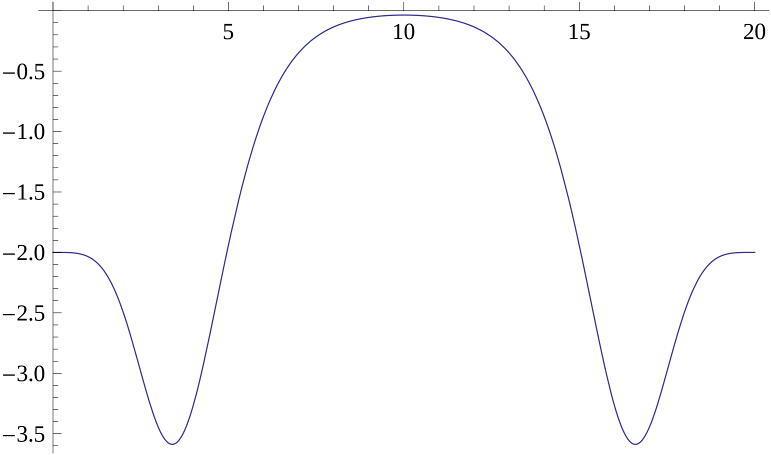
Suita conjecture (1972):

$$Curv_{c_D|dz|} \leq -1$$

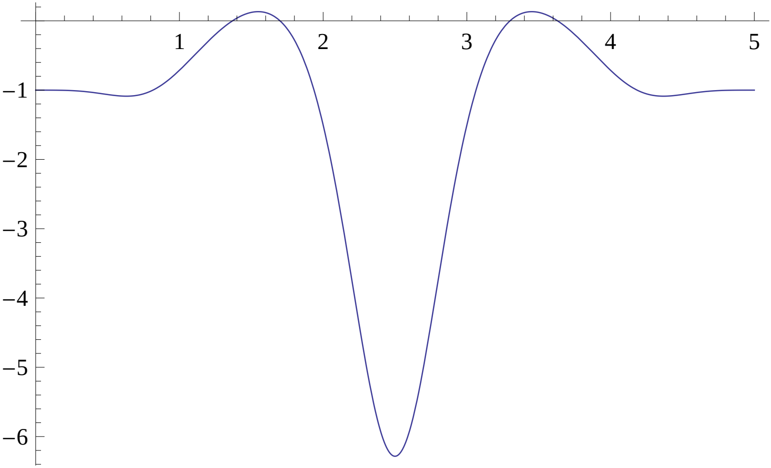
- "$=$" if D is simply connected
- "$<$" if D is an annulus (Suita)
- Enough to prove for D with smooth boundary
- "$=$" on ∂D if D has smooth boundary



$Curv_{c_D|dz|}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $t = -2\log|z|$



$Curv_{K_D|dz|^2}$ for $D = \{e^{-10} < |z| < 1\}$ as a function of $t = -2\log|z|$



$Curv_{(\log K_D)_{z\bar{z}}|dz|^2}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $t = -2\log|z|$

Suita showed that

$$\frac{\partial^2}{\partial z \partial \bar{z}}(\log c_D) = \pi K_D,$$

where

$$K_D(z) := \sup\{|f(z)|^2 : f \text{ holomorphic in D}, \int_D |f|^2 d\lambda \leq 1\}$$

is the Bergman kernel on the diagonal. Therefore the Suita conjecture is equivalent to the inequality

$$c_D^2 \leq \pi K_D.$$

It is thus an extension problem: for $z \in D$ find holomorphic f in D such that $f(z) = 1$ and

$$\int_D |f|^2 d\lambda \leq \frac{\pi}{(c_D(z))^2}.$$

Ohsawa (1995), using the methods of the Ohsawa-Takegoshi extension theorem, showed the estimate

$$c_D^2 \leq C\pi K_D$$

with $C = 750$. This was later improved to $C = 2$ (B., 2007) and to $C = 1.954$ (Guan-Zhou-Zhu, 2011).

Ohsawa-Takegoshi Extension Theorem, 1987

Ω - bounded pseudoconvex domain in $\mathbb{C}^n$, φ - psh in Ω
H - complex affine subspace of $\mathbb{C}^n$
f - holomorphic in $\Omega' := \Omega \cap H$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq C(n, \operatorname{diam}\Omega) \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Theorem (Berndtsson, 1996)

Ω - pseudoconvex in $\mathbb{C}^{n-1} \times \{|z_n < 1\}$, φ - psh in Ω
f - holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq 4\pi \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Theorem (Ohsawa, 2001, Ż. Dinew, 2007)

Ω - pseudoconvex in $\mathbb{C}^{n-1} \times D$, where $0 \in D \subset \mathbb{C}$, φ - psh in Ω,
f - holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{4\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

In 2011 B.-Y. Chen showed that the Ohsawa-Takegoshi extension theorem can be shown using directly Hörmander's estimate for $\bar\partial$-equation!

Hörmander's Estimate (1965)

Ω - pseudoconvex in $\mathbb{C}^n$, φ - smooth, strongly psh in Ω

$\alpha = \sum_j \alpha_j d\bar z_j \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$

Then one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda.$$

Here $|\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k} \varphi^{j\bar k}\bar\alpha_j \alpha_k$, where $(\varphi^{j\bar k}) = (\partial^2\varphi/\partial z_j \partial\bar z_k)^{-1}$ is the length of α w.r.t. the Kähler metric $i\partial\bar\partial\varphi$.

The estimate also makes sense for non-smooth φ: instead of $|\alpha|^2_{i\partial\bar\partial\varphi}$ one has to take any nonnegative $H \in L^\infty_{loc}(\Omega)$ with

$$i\bar\alpha \wedge \alpha \le H\, i\partial\bar\partial\varphi$$

(B., 2005).

Donnelly-Feffermann's Estimate (1982)

Ω - pseudoconvex, φ, ψ -psh in Ω

$|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le 1$ (that is $i\partial\psi \wedge \bar\partial\psi \le i\partial\bar\partial\psi$)

$\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$

Then one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le 4 \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{-\varphi} d\lambda.$$

Berndtsson's Estimate (1996)

Ω, φ, ψ, α as above

Then, if $0 \le \delta < 1$, one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{\delta\psi-\varphi} d\lambda \le \frac{4}{(1-\delta)^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\delta\psi-\varphi} d\lambda.$$

Theorem (B. 2004 & 2012)

The constants in the above estimates are optimal.

Theorem. Ω - pseudoconvex in $\mathbb{C}^n$, φ - psh in Ω
$\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
$\psi \in W^{1,2}_{loc}(\Omega)$ locally bounded from above, s.th.

$$|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \begin{cases} \leq 1 & \text{in } \Omega \\ \leq \delta < 1 & \text{on supp}\,\alpha. \end{cases}$$

Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2(1 - |\bar\partial\psi|^2_{i\partial\bar\partial\varphi})e^{2\psi-\varphi}d\lambda \leq \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}}\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi}e^{2\psi-\varphi}d\lambda.$$

Proof. (Ideas going back to Berndtsson and B.-Y. Chen.) By approximation we may assume that φ is smooth up to the boundary and strongly psh, and ψ is bounded.
u - minimal solution to $\bar\partial u = \alpha$ in $L^2(\Omega, e^{\psi-\varphi})$
$\Rightarrow u \perp \ker\bar\partial$ in $L^2(\Omega, e^{\psi-\varphi})$
$\Rightarrow v := ue^\psi \perp \ker\bar\partial$ in $L^2(\Omega, e^{-\varphi})$
$\Rightarrow v$ - minimal solution to $\bar\partial v = \beta := e^\psi(\alpha + u\bar\partial\psi)$ in $L^2(\Omega, e^{-\varphi})$
By Hörmander's estimate

$$\int_\Omega |v|^2 e^{-\varphi}d\lambda \leq \int_\Omega |\beta|^2_{i\partial\bar\partial\varphi}e^{-\varphi}d\lambda.$$

Therefore

$$\int_\Omega |u|^2 e^{2\psi-\varphi} d\lambda \leq \int_\Omega |\alpha + u\,\bar\partial\psi|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda$$
$$\leq \int_\Omega \left(|\alpha|^2_{i\partial\bar\partial\varphi} + 2|u|\sqrt{H}|\alpha|_{i\partial\bar\partial\varphi} + |u|^2 H \right) e^{2\psi-\varphi} d\lambda,$$

where $H = |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}$. For $t > 0$ we will get

$$\int_\Omega |u|^2 (1-H) e^{2\psi-\varphi} d\lambda$$
$$\leq \int_\Omega \left[|\alpha|^2_{i\partial\bar\partial\varphi} \left(1 + t^{-1} \frac{H}{1-H} \right) + t|u|^2(1-H) \right] e^{2\psi-\varphi} d\lambda$$
$$\leq \left(1 + t^{-1} \frac{\delta}{1-\delta} \right) \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda$$
$$+ t \int_\Omega |u|^2 (1-H) e^{2\psi-\varphi} d\lambda.$$

We will obtain the required estimate if we take $t := 1/(\delta^{-1/2} + 1)$.

Remark. This estimate implies Donnelly-Feffermann and Berndtsson's estimates: for psh φ, ψ with $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le 1$ and $\delta < 1$ set

$$\widetilde{\varphi} := (2-\delta)\psi + \varphi.$$

Then $2\psi - \widetilde{\varphi} = \delta\psi - \varphi$

$$|\bar\partial\psi|^2_{i\partial\bar\partial\widetilde{\varphi}} \le \frac{1}{2-\delta} =: \widetilde{\delta}$$

and

$$|\alpha|^2_{i\partial\bar\partial\widetilde{\varphi}} \le \widetilde{\delta}|\alpha|^2_{i\partial\bar\partial\psi}.$$

We will get Berndtsson's estimate with the constant

$$\frac{\widetilde{\delta}(1+\sqrt{\widetilde{\delta}})}{(1-\sqrt{\widetilde{\delta}})(1-\widetilde{\delta})} = \frac{1}{(\sqrt{2-\delta}-1)^2}.$$

Theorem (Ohsawa-Takegoshi with optimal constant)
Ω - pseudoconvex in $\mathbb{C}^{n-1} \times D$, where $0 \in D \subset \mathbb{C}$,
φ - psh in Ω, f - holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Sketch of proof. By approximation may assume that Ω is bounded, smooth, strongly pseudoconvex, φ is smooth up to the boundary, and f is holomorphic in a neighborhood of $\overline{\Omega'}$.

$\varepsilon > 0$

$$\alpha := \bar\partial\big(f(z')\chi(-2\log|z_n|)\big),$$

where $\chi(t) = 0$ for $t \leq -2\log\varepsilon$ and $\chi(\infty) = 1$.

$G := G_D(\cdot, 0)$
$\widetilde{\varphi} := \varphi + 2G + \eta(-2G)$
$\psi := \gamma(-2G)$

$F := f(z')\chi(-2\log|z_n|) - u$, where u is a solution of $\bar\partial u = \alpha$ given by the previous thm.

Crucial ODE Problem

Find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ such that $h' < 0$, $h'' > 0$,

$$\lim_{t\to\infty}(g(t) + \log t) = \lim_{t\to\infty}(h(t) + \log t) = 0$$

and

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1.$$

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$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1.$$

Solution:

$$h(t) := -\log(t + e^{-t} - 1)$$
$$g(t) := -\log(t + e^{-t} - 1) + \log(1 - e^{-t}).$$