Bergman Kernel and Pluripotential Theory

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Bergman Completeness

Ω bounded domain in $\mathbb{C}^n$

$H^2(\Omega) = \mathcal{O}(\Omega) \cap L^2(\Omega)$

$K_\Omega(\cdot, \cdot)$ Bergman kernel

$$f(w) = \int_\Omega f\, \overline{K_\Omega(\cdot, w)}\, d\lambda, \quad w \in \Omega,\ f \in H^2(\Omega)$$

$$\begin{aligned} K_\Omega(w) &= K_\Omega(w, w) \\ &= \sup\{|f(w)|^2 : f \in H^2(\Omega),\ \|f\| \le 1\} \end{aligned}$$

Ω is called Bergman complete if it is complete w.r.t. the Bergman metric $B_\Omega = i\partial\bar{\partial} \log K_\Omega$

Kobayashi Criterion (1959) If

$$\lim_{w \to \partial\Omega} \frac{|f(w)|^2}{K_\Omega(w)} = 0, \quad f \in H^2(\Omega),$$

then Ω is Bergman complete.

The opposite is not true even for $n = 1$ (Zwonek, 2001).

Kobayashi Criterion easily follows using the embedding

$$\iota : \Omega \ni w \longmapsto [K_\Omega(\cdot, w)] \in \mathbb{P}(H^2(\Omega))$$

and the fact that $\iota^* \omega_{FS} = B_\Omega$.

Since ι is distance decreasing,

$$dist^B_\Omega(z, w) \geq \arccos \frac{|K_\Omega(z, w)|}{\sqrt{K_\Omega(z) K_\Omega(w)}}.$$

Some Pluripotential Theory

Ω is called hyperconvex if it admits a negative plurisubharmonic (psh) exhaustion function ($u \in PSH^{-}(\Omega)$ s.th. $u = 0$ on $\partial\Omega$).

Demailly (1985) If Ω is pseudoconvex with Lipschitz boundary then it is hyperconvex.

Pluricomplex Green function For a pole $w \in \Omega$ we set

$$G_\Omega(\cdot, w) = G_w = \sup\{v \in PSH^{-}(\Omega) : \; v \leq \log|\cdot - w| + C\}$$

Lempert (1981) Ω convex $\Rightarrow$ G_Ω symmetric

Demailly (1985) Ω hyperconvex $\Rightarrow$ $e^{G_\Omega} \in C(\bar{\Omega} \times \Omega)$

Open Problem $e^{G_\Omega} \in C(\bar{\Omega} \times \bar{\Omega} \setminus \Delta_{\partial\Omega})$

Equivalently: $G(\cdot, w_k) \to 0$ loc. uniformly as $w_k \to \partial\Omega$?

True if $\partial\Omega \in C^2$ (Herbort, 2000)

Demailly (1985) If Ω is hyperconvex then $G_w = G_\Omega(\cdot, w)$ is the unique solution to

$$\begin{cases} u \in PSH(\Omega) \cap C(\bar{\Omega} \setminus \{w\}) \\ (dd^c u)^n = (2\pi)^n \delta_w \\ u = 0 \text{ on } \partial\Omega \\ u \leq \log|\cdot - w| + C \end{cases}$$

B. (2000) If Ω is smooth and strongly pseudoconvex then $G_w \in C^{1,1}(\bar{\Omega} \setminus \{w\})$.

B. (1995) If Ω is hyperconvex then $\exists!\ u = u_\Omega$ s.th.

$$\begin{cases} u \in PSH(\Omega) \cap C(\bar{\Omega}) \\ (dd^c u)^n = 1\, d\lambda \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

B.-Y. Chen, Pflug - B. / Herbort (1998)

Hyperconvex domains are Bergman complete

Herbort If Ω is pseudoconvex then

$$\frac{|f(w)|^2}{K_\Omega(w)} \leq c_n \int\limits_{\{G_w<-1\}} |f|^2 d\lambda, \quad w \in \Omega,\ f \in H^2(\Omega).$$

Corollary $\lim\limits_{w\to\partial\Omega} \lambda(\{G_w < -1\}) = 0 \ \Rightarrow\ \Omega$ is Bergman complete

Proposition If Ω is hyperconvex then

$$\lim_{w\to\partial\Omega} ||G_w||_{L^n(\Omega)} = 0.$$

Sketch of proof $||G_w||_n^n = \int_\Omega |G_w|^n (dd^c u_\Omega)^n$

$$\leq n!||u_\Omega||_\infty^{n-1} \int_\Omega |u_\Omega|(dd^c G_w)^n \leq C(n, \lambda(\Omega))\,|u_\Omega(w)|$$

Lower Bound for the Bergman Distance

Diederich-Ohsawa (1994), B. (2005) If Ω is pseudoconvex with C^2 boundary then

$$dist_\Omega^B(\cdot, w) \geq \frac{\log \delta_\Omega^{-1}}{C \log\log \delta_\Omega^{-1}},$$

where $\delta_\Omega(z) = dist_\Omega(z, \partial\Omega)$.

Pluripotential theory is the main tool in proving this estimate, in particular we have the following:

B. (2005) If Ω is pseudoconvex and $z, w \in \Omega$ are such that

$$\{G_z < -1\} \cap \{G_w < -1\} = \emptyset$$

then

$$dist_\Omega^B(z, w) \geq c_n > 0.$$

Open Problem $dist_\Omega^B(\cdot, w) \geq \frac{1}{C} \log \delta_\Omega^{-1}$

From Herbort's estimate

$$\frac{|f(w)|^2}{K_\Omega(w)} \le c_n \int\limits_{\{G_w<-1\}} |f|^2 d\lambda, \quad w \in \Omega,\ f \in H^2(\Omega),$$

for $f \equiv 1$ we get

$$K_\Omega(w) \ge \frac{1}{c_n \lambda(\{G_w < -1\})}.$$

To find the optimal constant c_n here turns out to have very interesting consequences!

Herbort (1999) $c_n = 1 + 4e^{4n+3+R^2}$, where $\Omega \subset B(z_0, R)$

(Main tool: Hörmander's estimate for $\bar\partial$)

B. (2005) $c_n = (1 + 4/Ei(n))^2$, where $Ei(t) = \int_t^\infty \frac{ds}{se^s}$

(Main tool: Donnelly-Fefferman's estimate for $\bar\partial$)

Suita Conjecture

D bounded domain in $\mathbb{C}$

$c_D(z) := \exp \lim_{\zeta \to z} \left(G_D(\zeta, z) - \log |\zeta - z| \right)$

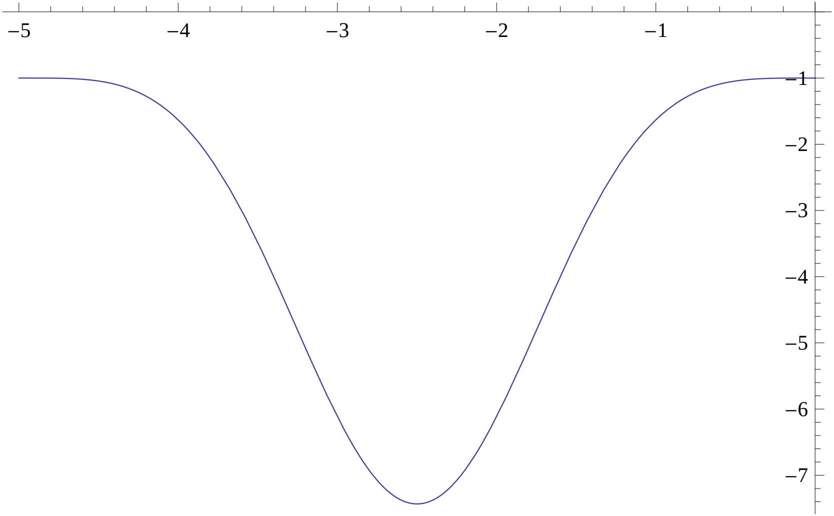
(logarithmic capacity of $\mathbb{C} \setminus D$ w.r.t. z)

$c_D|dz|$ is an invariant metric (Suita metric)

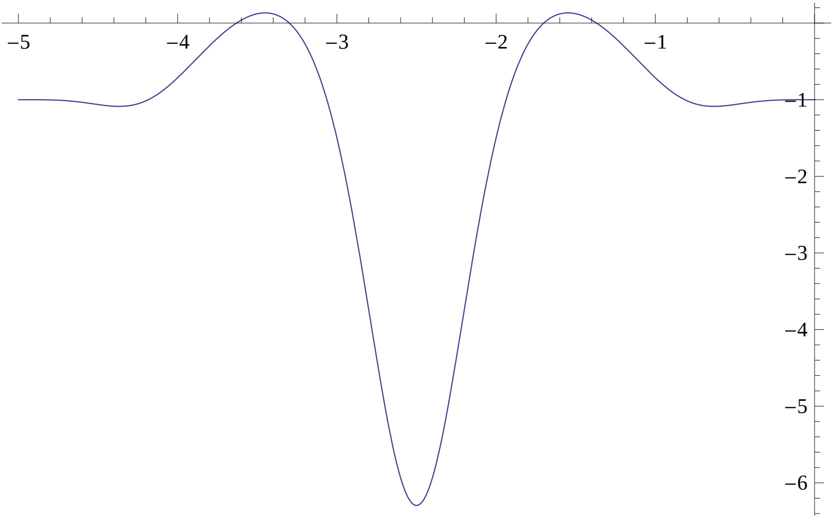
$$Curv_{c_D|dz|} = -\frac{(\log c_D)_{z\bar{z}}}{c_D^2}$$

Suita Conjecture (1972): $\quad Curv_{c_D|dz|} \leq -1$

- "$=$" if D is simply connected
- "$<$" if D is an annulus (Suita)
- Enough to prove for D with smooth boundary
- "$=$" on ∂D if D has smooth boundary



$Curv_{c_D|dz|}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log |z|$



$Curv_{(\log K_D)_{z\bar{z}}|dz|^2}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log |z|$

$$\frac{\partial^2}{\partial z \partial \bar{z}}(\log c_D) = \pi K_D \qquad \text{(Suita)}$$

Therefore the Suita conjecture is equivalent to

$$c_D^2 \le \pi K_D.$$

Ohsawa (1995) observed that it is really an extension problem: for $z \in D$ find holomorphic f in D such that $f(z) = 1$ and

$$\int_D |f|^2 d\lambda \le \frac{\pi}{(c_D(z))^2}.$$

Using the methods of the Ohsawa-Takegoshi extension theorem he showed the estimate

$$c_D^2 \le C \pi K_D$$

with $C = 750$.

$C = 2$ (B., 2007)
$C = 1.95388\ldots$ (Guan-Zhou-Zhu, 2011)

Ohsawa-Takegoshi extension theorem (1987)
with optimal constant (B., 2013)

$0 \in D \subset \mathbb{C}, \quad \Omega \subset \mathbb{C}^{n-1} \times D, \quad \Omega$ pseudoconvex,
$\varphi \in PSH(\Omega)$
f holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{c_D(0)^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

For $n = 1$ and $\varphi \equiv 0$ we get the Suita conjecture.

Main tool: Hörmander's estimate for $\bar\partial$
B.-Y. Chen (2011) proved that the Ohsawa-Takegoshi theorem (without optimal constant) follows form Hörmander's estimate.

Tensor Power Trick

We have

$$K_\Omega(w) \geq \frac{1}{c_n \lambda(\{G_w < -1\})}$$

where $c_n = (1 + 4/Ei(n))^2$.

Take $m \gg 0$ and set $\widetilde{\Omega} := \Omega^m \subset \mathbb{C}^{nm}$, $\widetilde{w} := (w, \ldots, w)$. Then

$$K_{\widetilde{\Omega}}(\widetilde{w}) = (K_\Omega(w))^m, \quad \lambda_{2nm}(\{G_{\widetilde{w}} < -1\}) = (\lambda_{2n}(\{G_w < -1\})^m.$$

Therefore

$$K_\Omega(w) \geq \frac{1}{c_{nm}^{1/m} \lambda(\{G_w < -1\})}$$

but

$$\lim_{m \to \infty} c_{nm}^{1/m} = e^{2n}.$$

Repeating this argument for any sublevel set we get

Theorem 1 Assume Ω is pseudoconvex in $\mathbb{C}^n$. Then for $t \leq 0$ and $w \in \Omega$

$$K_\Omega(w) \geq \frac{1}{e^{-2nt}\lambda(\{G_w < t\})}.$$

Lempert recently noticed that this estimate can also be proved using Berndtsson's result on positivity of direct image bundles.

What happens when $t \to -\infty$?

For $n = 1$ we get $K_\Omega \geq c_\Omega^2/\pi$ (another proof of Suita Conjecture).

Theorem 2 If Ω is a convex domain in $\mathbb{C}^n$ then for $w \in \Omega$

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega(w))},$$

$I_\Omega(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$ (Kobayashi indicatrix).

Mahler Conjecture

K - convex symmetric body in $\mathbb{R}^n$

$$K' := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } x \in K\}$$

Mahler volume $:= \lambda(K)\lambda(K')$

Mahler volume is an invariant of the Banach space defined by K: it is independent of linear transformations and of the choice of inner product.

Santaló Inequality (1949) Mahler volume is maximized by balls

Mahler Conjecture (1938) Mahler volume is minimized by cubes

Hansen-Lima bodies: starting from an interval they are produced by taking products of lower dimensional HL bodies and their duals.
$n = 2$: square
$n = 3$: cube & octahedron
$n = 4$: ...

Bourgain-Milman (1987) There exists $c > 0$ such that

$$\lambda(K)\lambda(K') \geq c^n \frac{4^n}{n!}.$$

Mahler Conjecture: $c = 1$

G. Kuperberg (2006) $c = \pi/4$

Nazarov (2012)

- equivalent SCV formulation of the Mahler Conjecture via the Fourier transform and the Paley-Wiener theorem
- proof of the Bourgain-Milman Inequality ($c = (\pi/4)^3$) using Hörmander's estimate for $\bar{\partial}$

K - convex symmetric body in $\mathbb{R}^n$

Nazarov: consider $T_K := \mathrm{int} K + i\mathbb{R}^n \subset \mathbb{C}^n$. Then

$$\left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda_n(K))^2} \le K_{T_K}(0) \le \frac{n!}{\pi^n} \frac{\lambda_n(K')}{\lambda_n(K)}.$$

Therefore

$$\lambda_n(K)\lambda_n(K') \ge \left(\frac{\pi}{4}\right)^{3n} \frac{4^n}{n!}.$$

To show the lower bound we can use Theorem 2:

$$K_{T_K}(0) \ge \frac{1}{\lambda_{2n}(I_{T_K}(0))}.$$

Proposition $I_{T_K}(0) \subset \dfrac{4}{\pi}(K + iK)$

In particular, $\lambda_{2n}(I_{T_K}(0)) \le \left(\dfrac{4}{\pi}\right)^{2n} (\lambda_n(K))^2$

Conjecture $\lambda_{2n}(I_{T_K}(0)) \le \left(\dfrac{4}{\pi}\right)^{n} (\lambda_n(K))^2$

Lempert Theory (1981)

Ω - bounded strongly convex domain in $\mathbb{C}^n$ with smooth boundary

$\varphi \in \mathcal{O}(\Delta, \Omega) \cap C(\bar{\Delta}, \bar{\Omega})$ is a geodesic if and only if $\varphi(\partial\Delta) \subset \partial\Omega$ and there exists $h \in \mathcal{O}(\Delta, \mathbb{C}^n) \cap C(\bar{\Delta}, \mathbb{C}^n)$ s.th. the vector $e^{it}\overline{h(e^{it})}$ is outer normal to $\partial\Omega$ at $\varphi(e^{it})$ for every $t \in \mathbb{R}$.

$\exists$ $F \in \mathcal{O}(\Omega, \Delta)$ a left-inverse to φ (i.e. $F \circ \varphi = id_\Delta$) s.th.

$$(z - \varphi(F(z))) \cdot h(F(z)) = 0, \quad z \in \Omega.$$

Lempert's Theory for Tube Domains (S. Zając, 2013)

$\Omega = T_K$, where K is smooth and strongly convex in $\mathbb{R}^n$

Since $\mathrm{Im}\,(e^{it}\overline{h(e^{it})}) = 0$, h must be of the form

$$h(\zeta) = \bar{w} + \zeta b + \zeta^2 w$$

for some $w \in \mathbb{C}^n$ and $b \in \mathbb{R}^n$. Therefore

$$\mathrm{Re}\,\varphi(e^{it}) = \nu^{-1}\left(\frac{b + 2\mathrm{Re}\,(e^{it}w)}{|b + 2\mathrm{Re}\,(e^{it}w)|}\right),$$

where $\nu : \partial K \to S^{n-1}$ is the Gauss map.

By the Schwarz formula

$$\varphi(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta}\,\nu^{-1}\left(\frac{b+2\mathrm{Re}\,(e^{it}w)}{|b+2\mathrm{Re}\,(e^{it}w)|}\right)dt + i\mathrm{Im}\,\varphi(0).$$

If K is in addition symmetric then all geodesics in T_K with $\varphi(0)=0$ are of the form

$$\varphi(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta}\,\nu^{-1}\left(\frac{\mathrm{Re}\,(e^{it}w)}{|\mathrm{Re}\,(e^{it}w)|}\right)dt$$

for some $w \in (\mathbb{C}^n)_*$. Then

$$\varphi'(0) = \frac{1}{\pi}\int_0^{2\pi} e^{it}\,\nu^{-1}\left(\frac{\mathrm{Re}\,(e^{it}\bar{w})}{|\mathrm{Re}\,(e^{it}\bar{w})|}\right)dt$$

parametrizes $\partial I_{T_K}(0)$ for $w \in S^{2n-1}$.

Conjecture $\lambda_{2n}(I_{T_K}(0)) \le \left(\frac{4}{\pi}\right)^n (\lambda_n(K))^2$