Bergman Kernel and Kobayashi Pseudodistance in Convex Domains

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$\Omega \subset \mathbb{C}^n$, $w \in \Omega$

$$K_\Omega(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 d\lambda \leq 1\}$$

(Bergman kernel on the diagonal)

$$\begin{aligned} G_w(z) &= G_\Omega(z, w) \\ &= \sup\{u(z) : u \in PSH^-(\Omega) : \overline{\lim_{z \to w}} \big(u(z) - \log|z - w|\big) < \infty\} \end{aligned}$$

(pluricomplex Green function)

Theorem 0 Assume Ω is pseudoconvex in $\mathbb{C}^n$. Then for $w \in \Omega$ and $t \leq 0$

$$K_\Omega(w) \geq \frac{1}{e^{-2nt}\lambda(\{G_w < t\})}.$$

Optimal constant: "=" if $\Omega = B(w, r)$.

Proof 1 Using Donnelly-Fefferman's estimate for $\bar\partial$ one can prove

$$K_\Omega(w) \geq \frac{1}{c(n,t)\lambda(\{G_w < t\})}, \tag{1}$$

where

$$c(n,t) = \left(1 + \frac{C}{Ei(-nt)}\right)^2, \quad Ei(a) = \int_a^\infty \frac{ds}{se^s}$$

(B. 2005). Now use the tensor power trick: $\widetilde\Omega = \Omega \times \cdots \times \Omega \subset \mathbb{C}^{nm}$, $\widetilde w = (w, \dots, w)$ for $m \gg 0$. Then

$$K_{\widetilde\Omega}(\widetilde w) = (K_\Omega(w))^m, \quad \lambda(\{G_{\widetilde w} < t\}) = (\lambda(\{G_w < t\}))^m,$$

and by (1) for $\widetilde\Omega$

$$K_\Omega(w) \geq \frac{1}{c(nm,t)^{1/m}\lambda(\{G_w < t\})}.$$

But $\lim_{m\to\infty} c(nm,t)^{1/m} = e^{-2nt}$. □

Proof 2 (Lempert) By Berndtsson's result on log-(pluri)subharmonicity of the Bergman kernel for sections of a pseudoconvex domain it follows that $\log K_{\{G_w<t\}}(w)$ is convex for $t \in (-\infty, 0]$. Therefore

$$t \longmapsto 2nt + \log K_{\{G_w<t\}}(w)$$

is convex and bounded, hence non-decreasing. It follows that

$$K_\Omega(w) \geq e^{2nt} K_{\{G_w<t\}}(w) \geq \frac{e^{2nt}}{\lambda(\{G_w < t\})}. \quad \square$$

Berndtsson-Lempert: This method can be improved to show the Ohsawa-Takegoshi extension theorem with optimal constant.

Theorem 0 Assume Ω is pseudoconvex in $\mathbb{C}^n$. Then for $w \in \Omega$ and $t \leq 0$

$$K_\Omega(w) \geq \frac{1}{e^{-2nt}\lambda(\{G_w < t\})}.$$

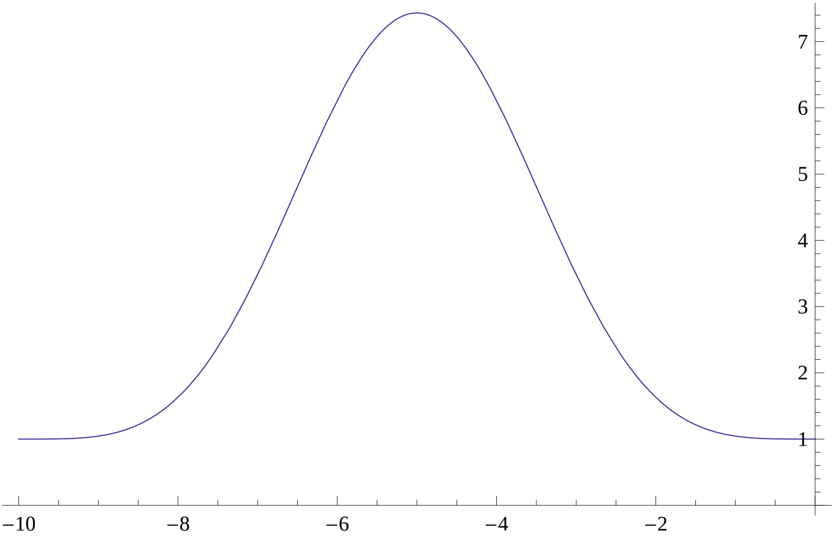
What happens when $t \to -\infty$? For $n = 1$ Theorem 0 immediately gives:

Theorem (Suita conjecture) For a domain $\Omega \subset \mathbb{C}$ one has

$$K_\Omega(w) \geq c_\Omega(w)^2/\pi, \quad w \in \Omega, \tag{2}$$

where $c_\Omega(w) = \exp\left(\lim_{z\to w}(G_\Omega(z,w) - log|z-w|)\right)$
(logarithmic capacity of $\mathbb{C} \setminus \Omega$ w.r.t. w).

Theorem (Guan-Zhou) Equality holds in (2) iff $\Omega \simeq \Delta \setminus F$, where Δ is the unit disk and F a closed polar subset.



$\frac{\pi K_\Omega}{c_\Omega^2}$ for $\Omega = \{e^{-5} < |z| < 1\}$ as a function of $2 \log |w|$

What happens with $e^{-2nt}\lambda(\{G_w < t\})$ as $t \to -\infty$ for arbitrary n? For convex Ω using Lempert's theory one can get

Proposition If Ω is bounded, smooth and strongly convex in $\mathbb{C}^n$ then for $w \in \Omega$

$$\lim_{t\to-\infty} e^{-2nt}\lambda(\{G_w < t\}) = \lambda(I^K_\Omega(w)),$$

where $I^K_\Omega(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$ (Kobayashi indicatrix).

Corollary If $\Omega \subset \mathbb{C}^n$ is convex then

$$K_\Omega(w) \geq \frac{1}{\lambda(I^K_\Omega(w))}, \quad w \in \Omega.$$

For general Ω one can prove

Theorem If Ω is bounded and hyperconvex in $\mathbb{C}^n$ and $w \in \Omega$ then

$$\lim_{t\to-\infty} e^{-2nt}\lambda(\{G_w < t\}) = \lambda(I^A_\Omega(w)),$$

where $I^A_\Omega(w) = \{X \in \mathbb{C}^n : \overline{\lim}_{\zeta\to 0}\left(G_w(w + \zeta X) - \log|\zeta|\right) \leq 0\}$
(Azukawa indicatrix)

Corollary (SCV version of the Suita conjecture) If $\Omega \subset \mathbb{C}^n$ is pseudoconvex and $w \in \Omega$ then

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega^A(w))}.$$

Remark 1. For $n = 1$ one has $\lambda(I_\Omega^A(w)) = \pi / c_\Omega(w)^2$.
2. If Ω is convex then $I_\Omega^A(w) = I_\Omega^K(w)$.

Conjecture For Ω pseudoconvex and $w \in \Omega$ the function

$$t \longmapsto e^{-2nt} \lambda(\{G_w < t\})$$

is non-decreasing in t.

It would easily follow if we knew that the function

$$t \longmapsto \log \lambda(\{G_w < t\})$$

is convex on $(-\infty, 0]$. Fornæss however constructed a counterexample to this (already for $n = 1$).

Theorem The conjecture is true for $n = 1$.

Proof It is be enough to prove that $f'(t) \geq 0$ where

$$f(t) := \log \lambda(\{G_w < t\}) - 2t$$

and t is a regular value of G_w. By the co-area formula

$$\lambda(\{G_w < t\}) = \int_{-\infty}^{t} \int_{\{G_w = s\}} \frac{d\sigma}{|\nabla G_w|} ds$$

and therefore

$$f'(t) = \frac{\displaystyle\int_{\{G_w = t\}} \frac{d\sigma}{|\nabla G_w|}}{\lambda(\{G_w < t\})} - 2.$$

By the Schwarz inequality

$$\int_{\{G_w = t\}} \frac{d\sigma}{|\nabla G_w|} \geq \frac{(\sigma(\{G_w = t\}))^2}{\displaystyle\int_{\{G_w = t\}} |\nabla G_w| d\sigma} = \frac{(\sigma(\{G_w = t\}))^2}{2\pi}.$$

The isoperimetric inequality gives

$$(\sigma(\{G_w = t\}))^2 \geq 4\pi\lambda(\{G_w < t\})$$

and we obtain $f'(t) \geq 0$. □

The conjecture for arbitrary n is equivalent to the following *pluricomplex isoperimetric inequality* for smooth strongly pseudoconvex Ω

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G_w|} \geq 2n\lambda(\Omega).$$

The conjecture also turns out to be closely related to the problem of symmetrization of the complex Monge-Ampère equation.

What about the corresponding upper bound in the Suita conjecture? Not true in general:

Proposition Let $\Omega = \{r < |z| < 1\}$. Then

$$\frac{K_\Omega(\sqrt{r})}{(c_\Omega(\sqrt{r}))^2} \geq \frac{-2\log r}{\pi^3}.$$

It would be interesting to find un upper bound of the Bergman kernel for domains in $\mathbb{C}$ in terms of logarithmic capacity which would in particular imply the $\Rightarrow$ part in the well known equivalence (due to Carleson)

$$K_\Omega > 0 \;\Leftrightarrow\; c_\Omega > 0$$

($c_\Omega^2 \leq \pi K_\Omega$ being a quantitative version of $\Leftarrow$).

The upper bound for the Bergman kernel holds for convex domains:

Theorem For a convex Ω and $w \in \Omega$ set

$$F_\Omega(w) := \left(K_\Omega(w)\lambda(I_\Omega^K(w))\right)^{1/n}.$$

Then $F_\Omega(w) \leq 4$.

Sketch of proof Denote $I := int\, I_\Omega^K(w)$ and assume that $w = 0$. One can show that $I \subset 2\,\Omega$. Then

$$K_\Omega(0)\lambda(I) \leq K_{I/2}(0)\lambda(I) = \frac{\lambda(I)}{\lambda(I/2)} = 4^n. \quad \square$$

If Ω is in addition symmetric w.r.t. w then $F_\Omega(w) \leq 16/\pi^2 = 1.621\ldots$.

Remark The proof of the optimal lower bound $F_\Omega \geq 1$ used $\bar\partial$.
The proof of the (probably) non-optimal upper bound $F_\Omega \leq 4$ is much more elementary!

For convex domains

$$F_{\Omega}(w) = \big(\lambda(I_{\Omega}(w))K_{\Omega}(w)\big)^{1/n}$$

is a biholomorphically invariant function satisfying $1 \leq F_{\Omega} \leq 4$.

- Find an example with $F_{\Omega} \not\equiv 1$.
- What are the properties of the function $w \longmapsto \lambda(I_{\Omega}(w))$?
- What is the optimal upper bound for F_{Ω}?

Formulas for some convex complex ellipsoids in $\mathbb{C}^2$

$$\mathcal{E}(p,q) = \{z \in \mathbb{C}^2 \colon |z_1|^{2p} + |z_2|^{2q} < 1\}, \quad p, q \geq 1/2.$$

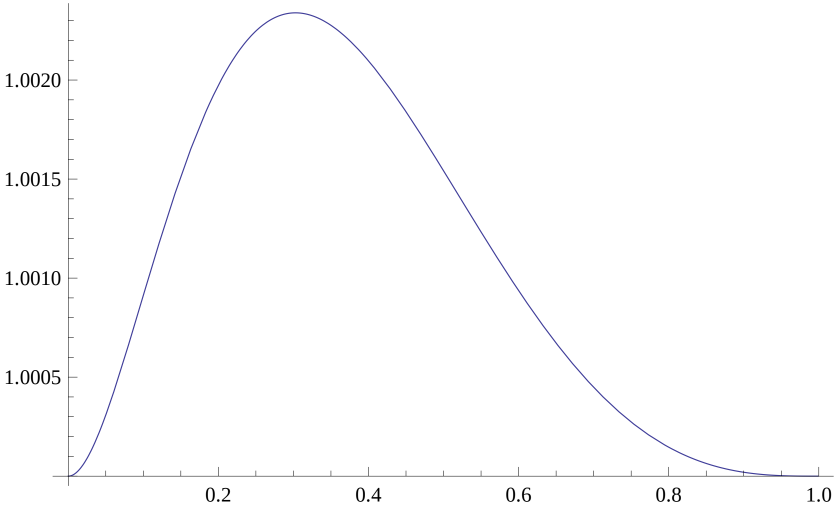
Blank-Fan-Klein-Krantz-Ma-Pang (1992) found implicit formulas for the Kobayashi function of $\mathcal{E}(m,1)$. They can be made explicit for $m = 1/2$. Using this one can prove

Theorem For $\Omega = \{|z_1| + |z_2|^2 < 1\}$ and $b \in [0,1)$ one has

$$\lambda(I_\Omega((b,0))) = \frac{\pi^2}{3}(1-b)^3(1+3b+3b^2-b^3)$$

and

$$\lambda(I_\Omega((b,0)))K_\Omega((b,0)) = 1 + \frac{(1-b)^3 b^2}{3(1+b)^3}.$$



$F_\Omega((b,0))$ for $\Omega = \{|z_1| + |z_2|^2 < 1\}$

Although the Kobayashi function of $\mathcal{E}(m,1)$ is given by implicit formulas, it turns out that the volume of the Kobayashi indicatrix can be computed explicitly:

Theorem For $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$, $m \geq 1/2$, and $b \in [0,1)$ one has

$$\begin{aligned}&\lambda(I_\Omega((b,0)))\\&= \pi^2 \left[-\frac{m-1}{2m(3m-2)(3m-1)} b^{6m+2} - \frac{3(m-1)}{2m(m-2)(m+1)} b^{2m+2} \right.\\&\left. + \frac{m}{2(m-2)(3m-2)} b^6 + \frac{3m}{3m-1} b^4 - \frac{4m-1}{2m} b^2 + \frac{m}{m+1} \right].\end{aligned}$$

For $m = 2/3$

$$\lambda(I_\Omega((b,0))) = \frac{\pi^2}{80} \left(-65b^6 + 40b^6 \log b + 160b^4 - 27b^{10/3} - 100b^2 + 32 \right),$$

and $m = 2$

$$\lambda(I_\Omega((b,0))) = \frac{\pi^2}{240} \left(-3b^{14} - 25b^6 - 120b^6 \log b + 288b^4 - 420b^2 + 160 \right).$$

About the proof Main tool: Jarnicki-Pflug-Zeinstra (1993) formula for geodesics in convex complex ellipsoids. If

$$\mathbb{C} \supset U \ni z \longmapsto (f(z), g(z)) \in \partial I$$

is a parametrization of an S^1-invariant portion of ∂I then the volume of the corresponding part of I is given by

$$\frac{\pi}{2} \int_U |H(z)| d\lambda(z), \tag{3}$$

where

$$H = |f|^2(|g_{\bar{z}}|^2 - |g_z|^2) + |g|^2(|f_{\bar{z}}|^2 - |f_z|^2) + 2\mathrm{Re}\left(f\bar{g}(\overline{f_z}g_z - \overline{f_{\bar{z}}}g_{\bar{z}})\right).$$

Both H and the integral (3) are computed with the help of *Mathematica*. The same method is used for computations in other ellipsoids.

For $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ the formula for the Bergman kernel is well known:

$$K_\Omega(w) = \frac{1}{\pi^2}(1-|w_2|^2)^{1/m-2}\frac{(1/m+1)(1-|w_2|^2)^{1/m} + (1/m-1)|w_1|^2}{\left((1-|w_2|^2)^{1/m} - |w_1|^2\right)^3},$$

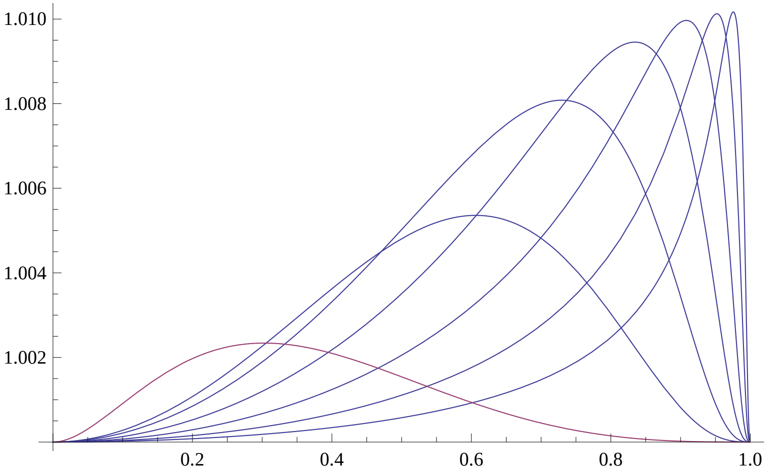
so that

$$K_\Omega((b,0)) = \frac{m+1+(1-m)b^2}{\pi^2 m(1-b^2)^3}.$$

Since for $t \in \mathbb{R}$ and $a \in \Delta$ the mapping

$$\Omega \ni z \longmapsto \left(e^{it}\frac{(1-|a|^2)^{1/2m}}{(1-\bar{a}z_2)^{1/m}}z_1, \frac{z_2-a}{1-\bar{a}z_2}\right)$$

is a holomorphic automorphism of Ω, $F_\Omega((b,0))$ for $b \in [0,1)$ attains all values of F_Ω in Ω.



$F_\Omega((b,0))$ in $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ for $m = 1/2, 4, 8, 16, 32, 64, 128$

$$\sup_{0<b<1} F_\Omega((b,0)) \to 1.010182\ldots \text{ as } m \to \infty$$

(highest value of F_Ω obtained so far in arbitrary dimension)

Theorem For $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in [0,1)$ one has

$$\lambda(I_\Omega((b,0))) = \frac{\pi^2}{6}(1-b)^4\big((1-b)^4 + 8b\big)$$

and

$$\lambda(I_\Omega((b,0)))K_\Omega((b,0)) = 1 + b^2\frac{(1-b)^4}{(1+b)^4}.$$

The Bergman kernel for this ellipsoid was found by Hahn-Pflug (1988):

$$K_\Omega(w) = \frac{2}{\pi^2}\cdot\frac{3(1-|w|^2)^2(1+|w|^2) + 4|w_1|^2|w_2|^2(5-3|w|^2)}{\big((1-|w|^2)^2 - 4|w_1|^2|w_2|^2\big)^3},$$

so that

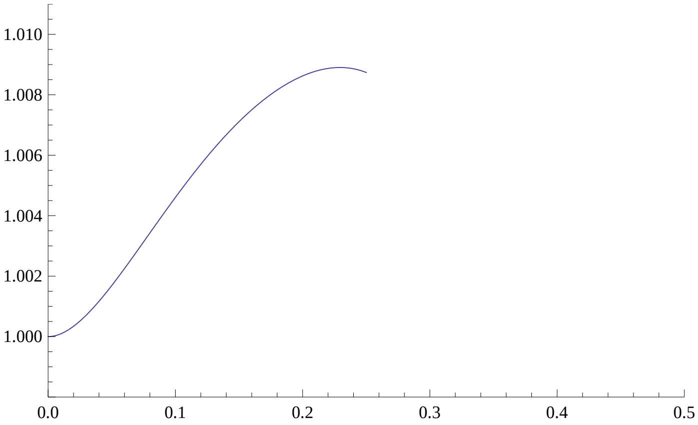
$$K_\Omega((b,0)) = \frac{6(1+b^2)}{\pi^2(1-b^2)^4}.$$

In all examples so far the function $w \mapsto \lambda(I_\Omega(w))$ is analytic. Is it true in general?

Theorem For $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1/4]$ one has

$$\lambda(I_\Omega((b, b))) = \frac{\pi^2}{6}\left(30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1\right).$$

Since $K_\Omega((b,b)) = \dfrac{2(3-6b^2+8b^4)}{\pi^2(1-4b^2)^3}$, we get the following picture:



$F_\Omega((b,b))$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $b \in [0, 1/4]$

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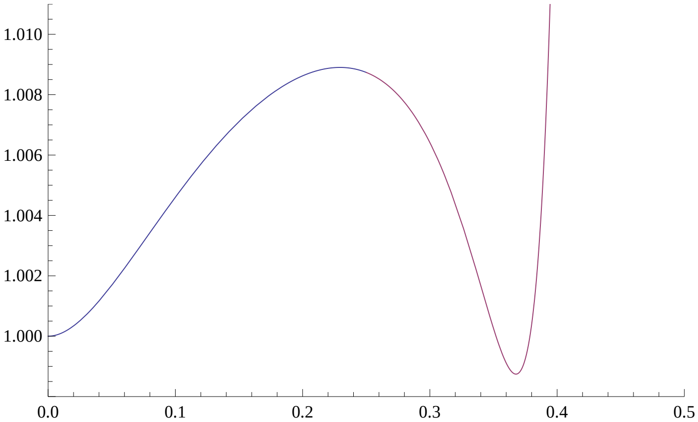


$F_\Omega((b,b))$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $b \in [0, 1/4]$

By either of the estimates $1 \le F_\Omega \le 4$, the function $b \mapsto F_\Omega((b,b))$ cannot be analytic on $(0, 1/2)$!

Theorem For $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1/4]$ one has

$$\lambda(I_\Omega((b,b))) = \frac{\pi^2}{6}\left(30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1\right).$$

For $b \in [1/4, 1/2)$

$$\begin{aligned}
\lambda(I_\Omega((b,b))) = & \frac{2\pi^2 b(1-2b)^3\left(-2b^3 + 3b^2 - 6b + 4\right)}{3(1-b)^2} \\
& + \frac{\pi\left(30b^{10} - 124b^9 + 238b^8 - 176b^7 - 260b^6 + 424b^5 - 76b^4 - 144b^3 + 89b^2 - 18b + 1\right)}{6(1-b)^2} \\
& \qquad \times \arccos\left(-1 + \frac{4b-1}{2b^2}\right) \\
& + \frac{\pi(1-2b)\left(-180b^7 + 444b^6 - 554b^5 + 754b^4 - 1214b^3 + 922b^2 - 305b + 37\right)}{72(1-b)}\sqrt{4b-1} \\
& + \frac{4\pi b(1-2b)^4\left(7b^2 + 2b - 2\right)}{3(1-b)^2}\arctan\sqrt{4b-1} \\
& + \frac{4\pi b^2(1-2b)^4(2-b)}{(1-b)^2}\arctan\frac{1-3b}{(1-b)\sqrt{4b-1}}.
\end{aligned}$$

By $\chi_-(b)$, resp. $\chi_+(b)$, denote $\lambda(I_\Omega((b,b)))$ for $b \le 1/4$, resp. $b \ge 1/4$. Then at $b = 1/4$

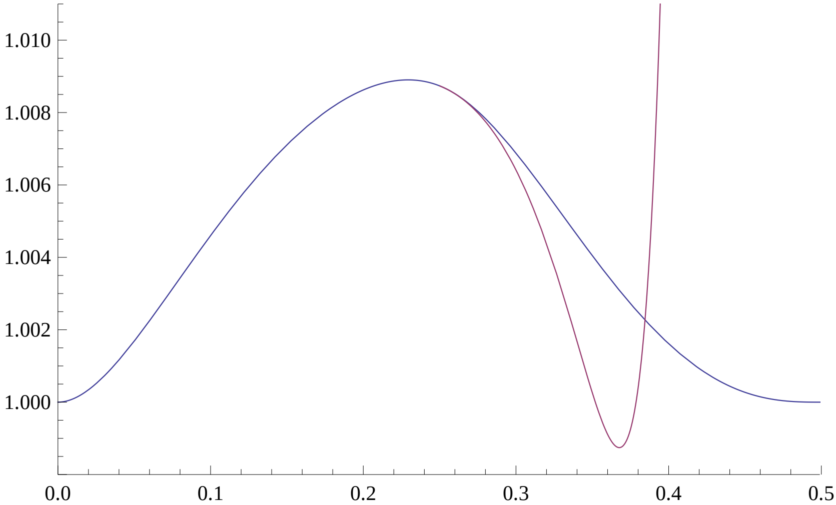
$$\chi_- = \chi_+ = \frac{15887}{196608}\pi^2, \qquad \chi'_- = \chi'_+ = -\frac{3521}{6144}\pi^2,$$

$$\chi''_- = \chi''_+ = -\frac{215}{1536}\pi^2, \quad \chi^{(3)}_- = \chi^{(3)}_+ = \frac{1785}{64}\pi^2,$$

but

$$\chi^{(4)}_- = \frac{1549}{16}\pi^2, \qquad \chi^{(4)}_+ = \infty.$$

Corollary For $\Omega = \{|z_1| + |z_2| < 1\}$ the function $w \mapsto \lambda(I_\Omega(w))$ is not $C^{3,1}$ at $w = (1/4, 1/4)$.



$F_\Omega((b,b))$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $b \in [0, 1/2)$

Mahler Conjecture

K - convex symmetric body in $\mathbb{R}^n$

$$K' := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } x \in K\}$$

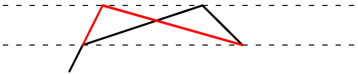
Mahler volume $:= \lambda(K)\lambda(K')$

Mahler volume is an invariant of the Banach space defined by K: it is independent of linear transformations and of the choice of inner product.

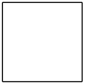
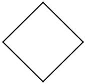
Blaschke-Santaló Inequality (1949) Mahler volume is maximized by balls

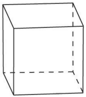
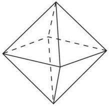
Mahler Conjecture (1938) Mahler volume is minimized by cubes

True for $n = 2$:



Hansen-Lima bodies: starting from an interval they are produced by taking products of lower dimensional HL bodies and their duals.

$n = 2$  $\simeq$ 

$n = 3$  $\not\simeq$ 

Equivalent SCV formulation (Nazarov, 2012)

For $u \in L^2(K')$ we have

$$|\widehat{u}(0)|^2 = \left| \int_{K'} u \, d\lambda \right|^2 \le \lambda(K') ||u||^2_{L^2(K')} = (2\pi)^{-n} \lambda(K') ||\widehat{u}||^2_{L^2(\mathbb{R}^n)}$$

with equality for $u = \chi_{K'}$. Therefore

$$\lambda(K') = (2\pi)^n \sup_{f \in \mathcal{P}} \frac{|f(0)|^2}{||f||^2_{L^2(\mathbb{R}^n)}},$$

where $\mathcal{P} = \{\widehat{u} \colon u \in L^2(K')\} \subset \mathcal{O}(\mathbb{C}^n)$. By the Paley-Wiener thm

$$\mathcal{P} = \{f \in \mathcal{O}(\mathbb{C}^n) \colon |f(z)| \le Ce^{C|z|}, \ \ |f(iy)| \le Ce^{q_K(y)}\},$$

where q_K is the Minkowski function for K. Therefore the Mahler conjecture is equivalent to finding $f \in \mathcal{P}$ with $f(0) = 1$ and

$$\int_{\mathbb{R}^n} |f(x)|^2 d\lambda(x) \le n! \left(\frac{\pi}{2}\right)^n \lambda(K).$$

Bourgain-Milman Inequality

Bourgain-Milman (1987) There exists $c > 0$ such that

$$\lambda(K)\lambda(K') \geq c^n \frac{4^n}{n!}.$$

Mahler Conjecture: $c = 1$

G. Kuperberg (2006) $c = \pi/4$

Nazarov (2012) SCV proof using Hörmander's estimate ($c = (\pi/4)^3$)

Consider the tube domain $T_K := \mathrm{int} K + i\mathbb{R}^n \subset \mathbb{C}^n$. Then

$$\left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda_n(K))^2} \leq K_{T_K}(0) \leq \frac{n!}{\pi^n} \frac{\lambda_n(K')}{\lambda_n(K)}.$$

Therefore

$$\lambda_n(K)\lambda_n(K') \geq \left(\frac{\pi}{4}\right)^{3n} \frac{4^n}{n!}.$$

The upper bound $K_{T_K}(0) \le \frac{n!}{\pi^n} \frac{\lambda_n(K')}{\lambda_n(K)}$ easily follows from Rothaus' formula (1968):

$$K_{T_K}(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{d\lambda}{J_K},$$

where

$$J_K(y) = \int_K e^{-2x\cdot y} d\lambda(x).$$

To show the lower bound $K_{T_K}(0) \ge \left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda_n(K))^2}$ we can use the estimate:

$$K_{T_K}(0) \ge \frac{1}{\lambda_{2n}(I_{T_K}(0))}$$

and

Proposition $I_{T_K}(0) \subset \frac{4}{\pi}(K + iK)$

Conjecture $K_{T_K}(0) \ge \left(\frac{\pi}{4}\right)^{n} \frac{1}{(\lambda_n(K))^2}$

This would be optimal, since we have equality for cubes.

However, one can check that for $K = \{|x_1| + |x_2| + |x_3| \leq 1\}$ we have

$$K_{T_K}(0) > \left(\frac{\pi}{4}\right)^3 \frac{1}{(\lambda_3(K))^2}.$$

This shows that Nazarov's proof of the Bourgain-Milman inequality cannot give the Mahler conjecture directly.

Thank you!