Hörmander's $\bar{\partial}$-estimate, Some Generalizations, and New Applications

Zbigniew Błocki
(Uniwersytet Jagielloński, Kraków, Poland)

http://gamma.im.uj.edu.pl/~blocki

Analysis, Complex Geometry, and Mathematical Physics:
A Conference in Honor of Duong H. Phong
Columbia University, May 7-11, 2013

We will discuss applications of Hörmander's L^2-estimate for $\bar\partial$ in the following problems:

1. Suita Conjecture (1972) from potential theory
2. Optimal constant in the Ohsawa-Takegoshi extension theorem (1987)
3. Mahler Conjecture (1938) from convex analysis

Suita Conjecture

Green function for bounded domain D in $\mathbb{C}$:

$$\begin{cases} \Delta G_D(\cdot, z) = 2\pi\delta_z \\ G_D(\cdot, z) = 0 \text{ on } \partial D \text{ (if } D \text{ is regular)} \end{cases}$$

$$c_D(z) := \exp \lim_{\zeta \to z} (G_D(\zeta, z) - \log|\zeta - z|)$$

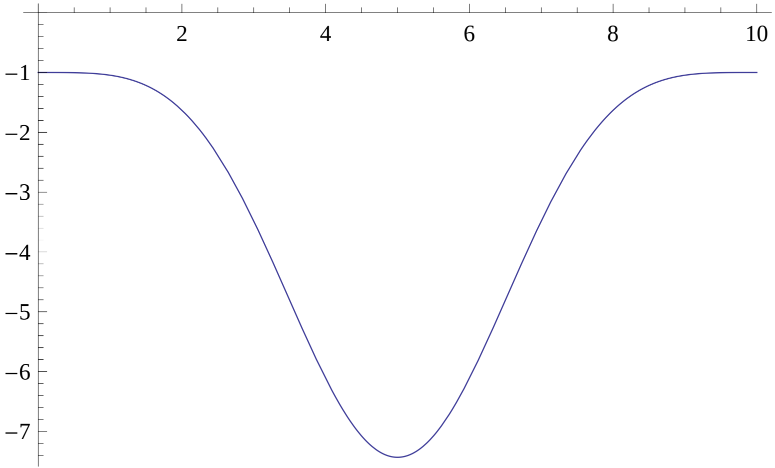
(logarithmic capacity of $\mathbb{C} \setminus D$ w.r.t. z)

$c_D|dz|$ is an invariant metric (Suita metric)

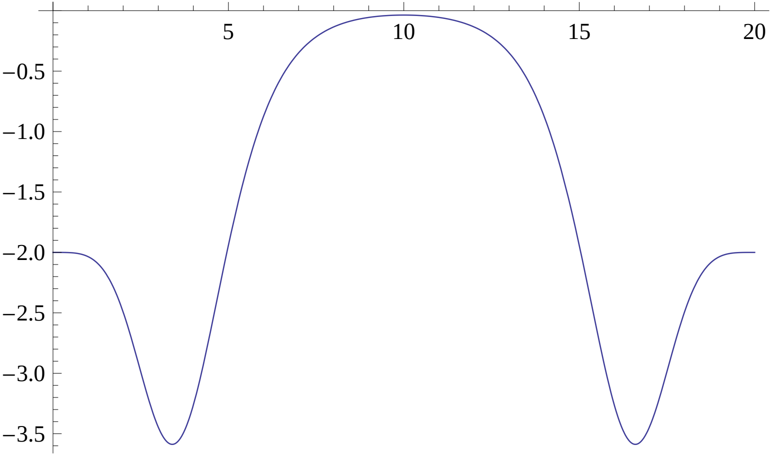
$$Curv_{c_D|dz|} = -\frac{(\log c_D)_{z\bar{z}}}{c_D^2}$$

Suita Conjecture (1972): $Curv_{c_D|dz|} \leq -1$

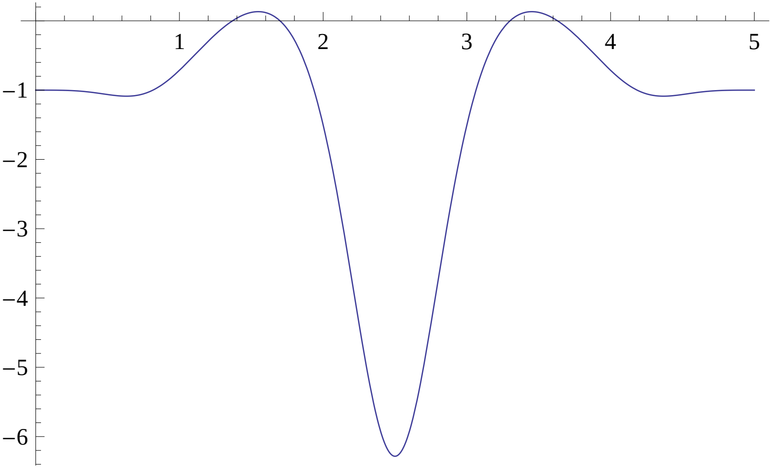
- "$=$" if D is simply connected
- "$<$" if D is an annulus (Suita)
- Enough to prove for D with smooth boundary
- "$=$" on ∂D if D has smooth boundary



$Curv_{c_D|dz|}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $t = -2\log|z|$



$Curv_{K_D|dz|^2}$ for $D = \{e^{-10} < |z| < 1\}$ as a function of $t = -2\log|z|$



$Curv_{(\log K_D)_{z\bar{z}}|dz|^2}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $t = -2\log|z|$

$$\frac{\partial^2}{\partial z \partial \bar{z}}(\log c_D) = \pi K_D \quad \text{(Suita)}$$

where K_D is the Bergman kernel on the diagonal:

$$K_D(z) := \sup\{|f(z)|^2 : f \in \mathcal{O}(D),\ \int_D |f|^2 d\lambda \leq 1\}.$$

Therefore the Suita conjecture is equivalent to

$$c_D^2 \leq \pi K_D.$$

It is thus an extension problem: for $z \in D$ find holomorphic f in D such that $f(z) = 1$ and

$$\int_D |f|^2 d\lambda \leq \frac{\pi}{(c_D(z))^2}.$$

Ohsawa (1995), using the methods of the Ohsawa-Takegoshi extension theorem, showed the estimate

$$c_D^2 \leq C\pi K_D$$

with $C = 750$.

$C = 2$ (B., 2007)
$C = 1.95388\ldots$ (Guan-Zhou-Zhu, 2011)

Ohsawa-Takegoshi Extension Theorem (1987)

Ω - bounded pseudoconvex domain in $\mathbb{C}^n$, φ - psh in Ω

H - complex affine subspace of $\mathbb{C}^n$

f - holomorphic in $\Omega' := \Omega \cap H$

Then there exists a holomorphic extension F of f to Ω such that

$$\int_{\Omega} |F|^2 e^{-\varphi} d\lambda \leq C \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda',$$

where C depends only on n and the diameter of Ω.

Siu / Berndtsson (1996): If $\Omega \subset \mathbb{C}^{n-1} \times \{|z_n < 1\}$ and $H = \{z_n = 0\}$ then $C = 4\pi$.

Problem. Can we improve to $C = \pi$?

B.-Y. Chen (2011): Ohsawa-Takegoshi extension theorem can be proved using directly Hörmander's estimate for $\bar{\partial}$-equation!

Mahler Conjecture

K - convex symmetric body in $\mathbb{R}^n$

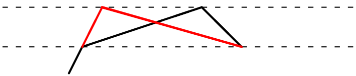
$$K' := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } x \in K\}$$

Mahler volume $:= \lambda(K)\lambda(K')$

Santaló Inequality (1949): Mahler volume is maximized by balls.

Mahler Conjecture (1938): Mahler volume is minimized by cubes.

True for $n = 2$:



Bourgain-Milman (1987): There exists $c > 0$ such that

$$\lambda(K)\lambda(K') \geq c^n \frac{4^n}{n!}.$$

Mahler Conjecture: $c = 1$

G. Kuperberg (2006): $c = \pi/4$

Equivalent SCV formulation (Nazarov, 2012)

For $u \in L^2(K')$ we have

$$|\widehat{u}(0)|^2 = \left|\int_{K'} u\, d\lambda\right|^2 \le \lambda(K')||u||^2_{L^2(K')} = (2\pi)^{-n}\lambda(K')||\widehat{u}||^2_{L^2(\mathbb{R}^n)}$$

with equality for $u = \chi_{K'}$. Therefore

$$\lambda(K') = (2\pi)^n \sup_{f\in\mathcal{P}} \frac{|f(0)|^2}{||f||^2_{L^2(\mathbb{R}^n)}},$$

where $\mathcal{P} = \{\widehat{u} : u \in L^2(K')\} \subset \mathcal{O}(\mathbb{C}^n)$. By Paley-Wiener thm the Mahler Conjecture is equivalent to the following SCV problem: find $f \in \mathcal{O}(\mathbb{C}^n)$ with exponential growth ($|f(z)| \le Ce^{C|z|}$) s.th. $f(0) = 1$,

$$|f(iy)| \le Ce^{q_K(y)}, \qquad (q_K \text{ is Minkowski function for } K),$$

and

$$\int_{\mathbb{R}^n} |f(x)|^2 d\lambda(x) \le n!\left(\frac{\pi}{2}\right)^n \lambda(K).$$

Nazarov: One can show the Bourgain-Milman inequality with $c = (\pi/4)^3$ using Hörmander's estimate.

Hörmander's Estimate (1965)

Ω - pseudoconvex in $\mathbb{C}^n$, φ - smooth, strongly psh in Ω
$\alpha = \sum_j \alpha_j d\bar{z}_j \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
Then one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda.$$

Here $|\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k} \varphi^{j\bar{k}} \bar\alpha_j \alpha_k$, where $(\varphi^{j\bar{k}}) = (\partial^2\varphi/\partial z_j \partial\bar{z}_k)^{-1}$ is the length of α w.r.t. the Kähler metric $i\partial\bar\partial\varphi$.

The estimate also makes sense for non-smooth φ: instead of $|\alpha|^2_{i\partial\bar\partial\varphi}$ one has to take any nonnegative $H \in L^\infty_{loc}(\Omega)$ with

$$i\bar\alpha \wedge \alpha \le H\, i\partial\bar\partial\varphi$$

(B., 2005).

Donnelly-Fefferman (1982)
Ω, α, φ as before
ψ psh in Ω s.th. $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq 1$ (that is $i\partial\psi \wedge \bar\partial\psi \leq i\partial\bar\partial\psi$)
Then one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq C \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{-\varphi} d\lambda,$$

where C is an absolute constant.

Berndtsson (1996)
Ω, α, φ, ψ as before
Then, if $0 \leq \delta < 1$, one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{\delta\psi-\varphi} d\lambda \leq \frac{4}{(1-\delta)^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\delta\psi-\varphi} d\lambda.$$

The above constant was obtained in B. 2004 and is optimal (B. 2012).
Therefore $C = 4$ is optimal in Donnelly-Fefferman.

Berndtsson's estimate is not enough to obtain Ohsawa-Takegoshi (it would be if it were true for $\delta = 1$).

Berndtsson's Estimate

Ω - pseudoconvex
$\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
φ, ψ - psh, $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq 1$
Then, if $0 \leq \delta < 1$, one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{\delta\psi-\varphi} d\lambda \leq \frac{4}{(1-\delta)^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\delta\psi-\varphi} d\lambda.$$

Theorem. Ω, α, φ, ψ as above
Assume in addition that $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq \delta < 1$ on $\operatorname{supp}\alpha$.
Then there exists $u \in L^2_{loc}(\Omega)$ solving $\bar\partial u = \alpha$ with

$$\int_\Omega |u|^2 (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\psi}) e^{\psi-\varphi} d\lambda \leq \frac{1}{(1-\sqrt{\delta})^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\psi-\varphi} d\lambda.$$

From this estimate one can obtain Ohsawa-Takegoshi and Suita with $C = 1.95388\ldots$ (obtained earlier by Guan-Zhou-Zhu).

Theorem. Ω - pseudoconvex in $\mathbb{C}^n$, φ - psh in Ω
$\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
$\psi \in W^{1,2}_{loc}(\Omega)$ locally bounded from above, s.th.

$$|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \begin{cases} \leq 1 & \text{in } \Omega \\ \leq \delta < 1 & \text{on supp}\,\alpha. \end{cases}$$

Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2(1 - |\bar\partial\psi|^2_{i\partial\bar\partial\varphi})e^{2\psi-\varphi}d\lambda \leq \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}}\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi}e^{2\psi-\varphi}d\lambda.$$

Proof. (Some ideas going back to Berndtsson and B.-Y. Chen.)
By approximation we may assume that φ is smooth up to the boundary and strongly psh, and ψ is bounded.
u - minimal solution to $\bar\partial u = \alpha$ in $L^2(\Omega, e^{\psi-\varphi})$
$\Rightarrow u \perp \ker\bar\partial$ in $L^2(\Omega, e^{\psi-\varphi})$
$\Rightarrow v := ue^{\psi} \perp \ker\bar\partial$ in $L^2(\Omega, e^{-\varphi})$
$\Rightarrow v$ - minimal solution to $\bar\partial v = \beta := e^{\psi}(\alpha + u\bar\partial\psi)$ in $L^2(\Omega, e^{-\varphi})$
By Hörmander's estimate

$$\int_\Omega |v|^2 e^{-\varphi}d\lambda \leq \int_\Omega |\beta|^2_{i\partial\bar\partial\varphi}e^{-\varphi}d\lambda.$$

Therefore

$$\int_\Omega |u|^2 e^{2\psi-\varphi} d\lambda \le \int_\Omega |\alpha + u\,\bar\partial\psi|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda$$
$$\le \int_\Omega \left(|\alpha|^2_{i\partial\bar\partial\varphi} + 2|u|\sqrt{H}|\alpha|_{i\partial\bar\partial\varphi} + |u|^2 H \right) e^{2\psi-\varphi} d\lambda,$$

where $H = |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}$. For $t > 0$ we will get

$$\int_\Omega |u|^2(1-H)e^{2\psi-\varphi} d\lambda$$
$$\le \int_\Omega \left[|\alpha|^2_{i\partial\bar\partial\varphi} \left(1 + t^{-1}\frac{H}{1-H} \right) + t|u|^2(1-H) \right] e^{2\psi-\varphi} d\lambda$$
$$\le \left(1 + t^{-1}\frac{\delta}{1-\delta} \right) \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda$$
$$+ t \int_\Omega |u|^2(1-H)e^{2\psi-\varphi} d\lambda.$$

We will obtain the required estimate if we take $t := 1/(\delta^{-1/2}+1)$.

Theorem. Ω - pseudoconvex in $\mathbb{C}^n$, φ - psh in Ω
$\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
$\psi \in W^{1,2}_{loc}(\Omega)$ locally bounded from above, s.th.

$$|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \begin{cases} \leq 1 & \text{in } \Omega \\ \leq \delta < 1 & \text{on } \operatorname{supp}\alpha. \end{cases}$$

Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2(1-|\bar\partial\psi|^2_{i\partial\bar\partial\varphi})e^{2\psi-\varphi}d\lambda \leq \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}}\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi}e^{2\psi-\varphi}d\lambda.$$

Remarks. 1. Setting $\psi \equiv 0$ we recover the Hörmander estimate.

2. This theorem implies Donnelly-Fefferman and Berndtsson's estimates with optimal constants: for psh φ, ψ with $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq 1$ and $\delta < 1$ set $\widetilde{\varphi} := \varphi + \psi$ and $\widetilde{\psi} = \frac{1+\delta}{2}\psi$.
Then $2\widetilde{\psi} - \widetilde{\varphi} = \delta\psi - \varphi$ and $|\bar\partial\widetilde{\psi}|^2_{i\partial\bar\partial\widetilde{\varphi}} \leq \frac{(1+\delta)^2}{4} =: \widetilde{\delta}$.
We will get Berndtsson's estimate with the constant

$$\frac{1+\sqrt{\widetilde{\delta}}}{(1-\sqrt{\widetilde{\delta}})(1-\widetilde{\delta})} = \frac{4}{(1-\delta)^2}.$$

Theorem (Ohsawa-Takegoshi with optimal constant)
Ω - pseudoconvex in $\mathbb{C}^{n-1} \times D$, where $0 \in D \subset \mathbb{C}$,
φ - psh in Ω, f - holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

(For $n = 1$ and $\varphi \equiv 0$ we obtain the Suita Conjecture.)

Sketch of proof. By approximation may assume that Ω is bounded, smooth, strongly pseudoconvex, φ is smooth up to the boundary, and f is holomorphic in a neighborhood of $\overline{\Omega'}$.

$\varepsilon > 0$

$$\alpha := \bar\partial\big(f(z')\chi(-2\log|z_n|)\big),$$

where $\chi(t) = 0$ for $t \leq -2\log\varepsilon$ and $\chi(\infty) = 1$.

$G := G_D(\cdot, 0)$
$\widetilde{\varphi} := \varphi + 2G + \eta(-2G)$
$\psi := \gamma(-2G)$

$F := f(z')\chi(-2\log|z_n|) - u$, where u is a solution of $\bar\partial u = \alpha$ given by the previous thm.

Crucial ODE Problem

Find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ such that $h' < 0$, $h'' > 0$,

$$\lim_{t\to\infty}(g(t) + \log t) = \lim_{t\to\infty}(h(t) + \log t) = 0$$

and

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1.$$

Crucial ODE Problem

Find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ such that $h' < 0$, $h'' > 0$,

$$\lim_{t\to\infty}(g(t) + \log t) = \lim_{t\to\infty}(h(t) + \log t) = 0$$

and

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1.$$

Solution:

$$h(t) := -\log(t + e^{-t} - 1)$$

$$g(t) := -\log(t + e^{-t} - 1) + \log(1 - e^{-t}).$$

Another approach: general lower bound for the Bergman kernel

$$K_\Omega(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 d\lambda \le 1\} \qquad \text{(Bergman kernel)}$$

$$G_\Omega(\cdot, w) = \sup\{v \in PSH^-(\Omega),\ \overline{\lim_{z\to w}}\,(v(z) - \log|z-w|) < \infty\}$$

(pluricomplex Green function)

Theorem. Assume Ω is pseudoconvex in $\mathbb{C}^n$. Then for $a \ge 0$ and $w \in \Omega$

$$K_\Omega(w) \ge \frac{1}{e^{2na}\lambda(\{G_\Omega(\cdot, w) < -a\})}.$$

For $n = 1$ letting $a \to \infty$ this gives the Suita Conjecture:

$$K_\Omega(w) \ge \frac{c_\Omega(w)^2}{\pi}.$$

Theorem. Assume Ω is pseudoconvex in $\mathbb{C}^n$. Then for $a \geq 0$ and $w \in \Omega$

$$K_\Omega(w) \geq \frac{1}{e^{2na}\lambda(\{G_\Omega(\cdot, w) < -a\})}.$$

Proof. May assume that Ω is bounded, smooth and strongly pseudoconvex. $G := G_{\Omega,w}$. Will use Donnelly-Fefferman with

$$\varphi := 2nG, \quad \psi := -\log(-G),$$

$$\alpha := \bar\partial(\chi \circ G) = \chi' \circ G\, \bar\partial G,$$

(χ will be determined later).

$$i\bar\alpha \wedge \alpha \leq (\chi' \circ G)^2 i\partial G \circ \bar\partial G \leq G^2(\chi' \circ G)^2 i\partial\bar\partial\psi$$

We will find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 d\lambda \leq \int_\Omega |u|^2 e^{-\varphi} d\lambda \leq C \int_\Omega G^2(\chi' \circ G)^2 e^{-2nG} d\lambda.$$

With $\chi(t) := \begin{cases} 0 & t \geq -a, \\ \int_a^{-t} \frac{e^{-ns}}{s} ds & t < -a, \end{cases}$ we thus get

$$\int_\Omega |u|^2 d\lambda \leq C\, \lambda(\{G < -a\}).$$

$f := \chi \circ G - u \in \mathcal{O}(\Omega)$ satisfies

$$f(w) = \chi(-\infty) = \int_{na}^{\infty} \frac{e^{-s}}{s}\, ds = \mathrm{Ei}(na)$$

(because $e^{-\varphi}$ is not integrable near w). Also

$$||f|| \le ||\chi \circ G|| + ||u|| \le (\chi(-\infty) + \sqrt{C})\sqrt{\lambda(\{G < -a\})}.$$

Therefore

$$K_\Omega(w) \ge \frac{|f(w)|^2}{||f||^2} \ge \frac{c_{n,a}}{\lambda(\{G < -a\})},$$

where

$$c_{n,a} = \frac{\mathrm{Ei}(na)^2}{(\mathrm{Ei}(na) + \sqrt{C})^2}.$$

Tensor power trick. $\widetilde{\Omega} := \Omega^m \subset \mathbb{C}^{nm}$, $\widetilde{w} := (w, \dots, w)$, $m \gg 0$

$$K_{\widetilde{\Omega}}(\widetilde{w}) = (K_\Omega(w))^m, \quad \lambda_{2nm}(\{G_{\widetilde{\Omega},\widetilde{w}} < -a\}) = (\lambda_{2n}(\{G < -a\})^m.$$

$$(K_\Omega(w))^m \ge \frac{c_{nm,a}}{(\lambda_{2n}(\{G < -a\}))^m}$$

but

$$\lim_{m\to\infty} c_{nm,a}^{1/m} = e^{-2na}.$$

Application to the Bourgain-Milman Inequality

K - convex symmetric body in $\mathbb{R}^n$
Nazarov: consider the tube domain $T_K := \mathrm{int}K + i\mathbb{R}^n \subset \mathbb{C}^n$. Then

(1) $$\left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda_n(K))^2} \le K_{T_K}(0) \le \frac{n!}{\pi^n} \frac{\lambda_n(K')}{\lambda_n(K)}.$$

Therefore

$$\lambda_n(K)\lambda_n(K') \ge \left(\frac{\pi}{4}\right)^{3n} \frac{4^n}{n!}.$$

To show the lower bound in (1) we can use the previous estimate:

$$K_\Omega(w) \ge \frac{1}{e^{2na}\lambda_{2n}(\{G_\Omega(\cdot, w) < -a\})}, \quad w \in \Omega, \ a \ge 0.$$

By Lempert's theorem we will get as $a \to \infty$

Theorem. If Ω is a convex domain in $\mathbb{C}^n$ then for $w \in \Omega$

$$K_\Omega(w) \ge \frac{1}{\lambda_{2n}(I_\Omega(w))},$$

where $I_\Omega(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega), \ \varphi(0) = w\}$ (Kobayashi indicatrix).

Proposition (Nazarov). $I_{T_K}(0) \subset \frac{4}{\pi}(K+iK)$

Sketch of proof. For $y \in K'$ consider

$$F(z) = \Phi(z \cdot t) \in \mathcal{O}(\Omega, \Delta),$$

where $\Phi : \{|\mathrm{Re}\,\zeta| < 1\} \to \Delta$ is conformal with $\Phi(0) = 0$. By the Schwarz lemma we will get

$$I_{T_K}(0) \subset \frac{4}{\pi}\{z \in \mathbb{C}^n : |z \cdot y| \leq 1 \text{ for every } y \in K'\}.$$

Corollary. $\lambda_{2n}(I_{T_K}(0)) \leq \left(\frac{4}{\pi}\right)^{2n} (\lambda_n(K))^2$

Conjecture. $\lambda_{2n}(I_{T_K}(0)) \leq \left(\frac{4}{\pi}\right)^{n} (\lambda_n(K))^2$

$$K_{T_K}(0) \geq \left(\frac{\pi}{4}\right)^{n} \frac{1}{(\lambda_n(K))^2}. \quad \text{(equality for cubes)}$$

Lempert (1981)

Ω - bounded strongly convex domain in $\mathbb{C}^n$ with smooth boundary

$\varphi \in \mathcal{O}(\Delta, \Omega) \cap C(\bar{\Delta}, \bar{\Omega})$ is a geodesic if and only if $\varphi(\partial\Delta) \subset \partial\Omega$ and there exists $h \in \mathcal{O}(\Delta, \mathbb{C}^n) \cap C(\bar{\Delta}, \mathbb{C}^n)$ s.th. the vector $e^{it}\overline{h(e^{it})}$ is outer normal to $\partial\Omega$ at $\varphi(e^{it})$ for every $t \in \mathbb{R}$.

There exists $F \in \mathcal{O}(\Omega, \Delta)$, a left-inverse to φ (i.e. $F \circ \varphi = id_\Delta$) s.th.

$$(z - \varphi(F(z))) \cdot h(F(z)) = 0, \quad z \in \Omega.$$

Lempert's Theory for Tube Domains (S. Zając)

$\Omega = T_K = \mathrm{int}K + i\mathbb{R}^n$, where K is smooth and strongly convex in $\mathbb{R}^n$

Since $\mathrm{Im}\,(e^{it}\overline{h(e^{it})}) = 0$, h must be of the form

$$h(\zeta) = \bar{w} + \zeta b + \zeta^2 w$$

for some $w \in \mathbb{C}^n$ and $b \in \mathbb{R}^n$. Therefore

$$\mathrm{Re}\,\varphi(e^{it}) = \nu^{-1}\left(\frac{b + 2\mathrm{Re}\,(e^{it}w)}{|b + 2\mathrm{Re}\,(e^{it}w)|}\right),$$

where $\nu : \partial K \to S^{n-1}$ is the Gauss map.

By the Schwarz formula

$$\varphi(\zeta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+\zeta}{e^{it}-\zeta}\,\nu^{-1}\left(\frac{b+2\mathrm{Re}\,(e^{it}w)}{|b+2\mathrm{Re}\,(e^{it}w)|}\right)dt+i\mathrm{Im}\,\varphi(0).$$

If K is in addition symmetric then all geodesics in T_K with $\varphi(0)=0$ are of the form

$$\varphi(\zeta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+\zeta}{e^{it}-\zeta}\,\nu^{-1}\left(\frac{\mathrm{Re}\,(e^{it}w)}{|\mathrm{Re}\,(e^{it}w)|}\right)dt$$

for some $w\in(\mathbb{C}^n)_*$. Then

$$\varphi'(0)=\frac{1}{\pi}\int_0^{2\pi}e^{it}\,\nu^{-1}\left(\frac{\mathrm{Re}\,(e^{it}\bar{w})}{|\mathrm{Re}\,(e^{it}\bar{w})|}\right)dt$$

parametrizes $\partial I_{T_K}(0)$ for $w\in S^{2n-1}$.

Conjecture $\lambda_{2n}(I_{T_K}(0))\leq\left(\frac{4}{\pi}\right)^n(\lambda_n(K))^2$