Bergman Kernel in the Annulus

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Formulas for the Bergman kernel in $P = \{r < |z| < 1\}$

I. Since $(z^j)_{j\in\mathbb{Z}}$ is an orthogonal system in P, we have

$$K_P(z,w) = \frac{h(\lambda)}{\pi\lambda},$$

where $\lambda = z\bar{w}$ and

$$h(\lambda) = \frac{1}{2\log(1/r)} + \sum_{j\in\mathbb{Z}} \frac{j\lambda^j}{1-r^{2j}}.$$

Zeros of K_P: If $r < e^{-4}$ then h has a zero in
$S := \{|\lambda| = r\} \cup \{\lambda \in \mathbb{R} : r^2 < |\lambda| < 1\}$
(Skwarczyński, 1969).

Proof: $h(\lambda) \in \mathbb{R}$ for $\lambda \in S$, $h > 0$ in $S \cap \mathbb{R}_+$ and
$h < 0$ in $S \cap \mathbb{R}_-$ near -1.

II. Weierstrass elliptic function $\mathcal{P}$:

$$\omega_1 = -\log r, \quad \omega_2 = \pi i, \quad \Lambda := \{2j\omega_1 + 2k\omega_2 : (j,k) \in \mathbb{Z}^2\}$$

$$\mathcal{P}(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda_*} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

$$h(\lambda) = \mathcal{P}(\log \lambda) + \frac{\eta_1}{\omega_1}, \quad \text{(Zarankiewicz, 1934)}$$

where $\eta_1 = \zeta(\omega_1)$ and the Weierstrass elliptic function ζ is determined by

$$\zeta' = -\mathcal{P}, \quad \zeta(z) = \frac{1}{z} + O(|z|).$$

Zeros of K_P: h has two zeros in $\{r^2 < |\lambda| < 1\}$ (for every r) (Rosenthal, 1969).

Proof: $\mathcal{P}$ attains every value of $\bar{\mathbb{C}}$ twice in $\{2t\omega_1 + 2s\omega_2 : s, t \in [0,1)\}$.

III. For any bounded $\Omega \subset \mathbb{C}$

$$K_\Omega = \frac{2}{\pi} \frac{\partial^2 G_\Omega}{\partial z \partial \bar{w}}, \quad \text{(Schiffer)},$$

where $G_\Omega(\cdot, w)$ is the (negative) Green function.

If $p : \Delta \to \Omega$ is a covering then for any $\lambda_0 \in p^{-1}(w)$

$$G_\Omega(z,w) = \sum_{\mu \in p^{-1}(z)} G_\Delta(\lambda_0, \mu) \quad \text{(Myrberg, 1933)}.$$

If U is a neighb. of w and V_j are s.th. $p^{-1}(U) = \bigcup V_j$ and $p|_{V_j} \to U$ are biholomorphic, then for $\varphi_j := (p|_{V_j})^{-1}$

$$G_\Omega(z,w) = \sum_j \log \left| \frac{\varphi_j(z) - \varphi_0(w)}{1 - \varphi_j(z)\overline{\varphi_0(w)}} \right|.$$

$$K_\Omega(z,w) = \pi \sum_j \frac{\varphi_j'(z)\overline{\varphi_0'(w)}}{(1 - \varphi_j(z)\overline{\varphi_0(w)})^2}.$$

For $\Omega = P$

$$p(\zeta) = \exp\left(\frac{\log r}{\pi i}\mathrm{Log}\left(i\frac{1+\zeta}{1-\zeta}\right)\right),$$

$$\varphi_j(z) = \frac{e^{\pi i(\mathrm{Log}\, z + 2j\pi i)/\log r} - i}{e^{\pi i(\mathrm{Log}\, z + 2j\pi i)/\log r} + i}, \quad j \in \mathbb{Z}.$$

Therefore

$$h(\lambda) = -\frac{\pi^2}{\log^2 r}\sum_{j\in\mathbb{Z}}\frac{f_j(\lambda)}{(1-f_j(\lambda))^2},$$

where

$$f_j(z) = \exp\frac{\pi i(\mathrm{Log}\, z + 2j\pi i)}{\log r}.$$

We will get

$$h(-r) = \frac{\pi^2}{\log^2 r} \sum_{j\in\mathbb{Z}} \frac{q^{2j+1}}{(1+q^{2j+1})^2} > 0,$$

$$h(-r^2) = h(-1) = -\frac{\pi^2}{\log^2 r} \sum_{j\in\mathbb{Z}} \frac{q^{2j+1}}{(1-q^{2j+1})^2} < 0$$

where $q = e^{\pi^2/\log r} < 1$.

Theorem. h has exactly two zeros in $\{r^2 < |\lambda| < 1\}$: one on the interval $(-1, -r)$ and one on $(-r, -r^2)$.

Suita Conjecture (1972): For $\Omega \subset\subset \mathbb{C}$ we have

$$e^{2\psi(z)} \le \pi K_\Omega(z,z),$$

where $\psi(w) := \lim_{z\to w}(G_\Omega(z,w) - \log|z-w|)$ (Robin f.)

Another formulation: since

$$K_\Omega(z,z) = \frac{1}{\pi}\psi_{z\bar z} \quad \text{(Suita, 1972)},$$

we have

$$\text{Suita Conjecture} \;\Leftrightarrow\; e^{2\psi} \le \psi_{z\bar z} \;\Leftrightarrow\; K_{e^\psi|dz|} \le -1.$$

Slightly more general statement would be:

$K_{e^\psi|dz|}$ satisfies the maximum principle.

Ohsawa, 1995: $e^{2\psi} \le 750\psi_{z\bar z}$

B., 2007: $e^{2\psi} \le 2\psi_{z\bar z}$

Guan-Zhou-Zhu, 2011: $e^{2\psi} \le 1.954\ldots\psi_{z\bar z}$

Suita, 1972: $e^{2\psi} < \psi_{z\bar{z}}$ for $\Omega = P$

Sketch of proof:

One can show that $\psi(z) = \gamma(t)$, where $t = -2\log|z|$,

$$\gamma(t) = \frac{c}{2}t^2 + \frac{t}{2} - \log\sigma(t),$$

$c = \eta_1/\omega_1$, and the Weierstrass elliptic function σ is determined by $\sigma'/\sigma = \zeta$, $\sigma(z) = z + O(|z|^2)$. Then

$$\psi_{z\bar{z}}(z) = e^t\gamma''(t) = e^t(\mathcal{P}(t) + c).$$

Set

$$\begin{aligned} F := \log(-K_{e^{\psi}|dz|}) &= \log\frac{\psi_{z\bar{z}}}{e^{2\psi}} \\ &= \log(\mathcal{P} + c) + 2\log\sigma - ct^2. \end{aligned}$$

Have to show that $F > 0$ on $(0, 2\omega_1)$.

$$F = \log(\mathcal{P} + c) + 2\log\sigma - ct^2.$$

We have $F(2\omega_1 - t) = F(t)$, $F(0) = F'(0) = F'(\omega_1) = 0$ and $F(\omega_1) > 0$. Key: differential equation for $\mathcal{P}$

$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3,$$

where

$$g_2 = 60 \sum_{\omega\in\Lambda_*} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega\in\Lambda_*} \frac{1}{\omega^6}.$$

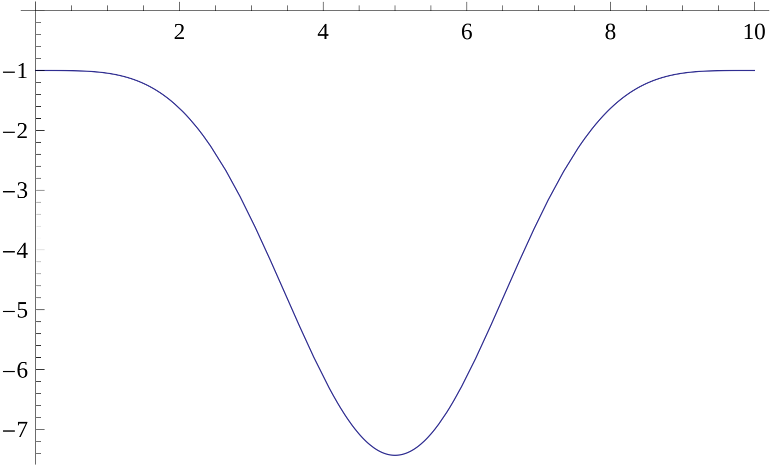
Therefore

$$F'' = \frac{b(\mathcal{P} + c) - a}{(\mathcal{P} + c)^2},$$

where

$$a = -4c^3 + cg_2 - g_3 > 0, \quad b = \frac{g_2}{2} - 6c^2 > 0,$$

and F'' vanishes exactly once in $(0, \omega_1)$ and thus $F > 0$.



$K_{e^{\psi}|dz|}$ for $r = e^{-5}$

Curvature of the "Bergman" metric $K_P(z,z)|dz|^2$

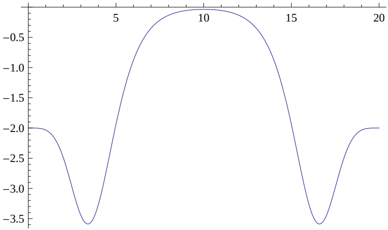
$$R_1 = 2K_{\psi_{z\bar z}|dz|^2} = -\frac{(\log \psi_{z\bar z})_{z\bar z}}{\psi_{z\bar z}} = -\frac{Q \circ \mathcal{P}}{(\mathcal{P}+c)^3},$$

where

$$Q(x) = 2(x+c)^3 + b(x+c) - a.$$

Therefore

$$R_1' = \frac{2b(\mathcal{P}+c) - 3a}{(\mathcal{P}+c)^4}\mathcal{P}'.$$



$r = e^{-10}$

Curvature of the Bergman metric $(\log K_P(z,z))_{z\bar{z}}|dz|^2$

$$R_2 := 2\,K_{(\log \psi_{z\bar{z}})_{z\bar{z}}|dz|^2} = -\frac{(\log(\log \psi_{z\bar{z}})_{z\bar{z}})_{z\bar{z}}}{(\log \psi_{z\bar{z}})_{z\bar{z}}}$$

Since $\psi_{z\bar{z}} = (\mathcal{P}+c)e^t$, we can compute that

$$R_2 = 2 - (\mathcal{P}+c)^3 \frac{S\circ\mathcal{P}}{(Q\circ\mathcal{P})^3},$$

where

$$S(y-c) = 24y^6 + 60by^4 - 96(a+bc)y^3 + 6(36ac-b^2)y^2 + 24aby - 12a^2 + b^3 + 12abc.$$

Then

$$R_2' = (\mathcal{P}+c)^3 \frac{W\circ\mathcal{P}}{(Q\circ\mathcal{P})^4}\mathcal{P}',$$

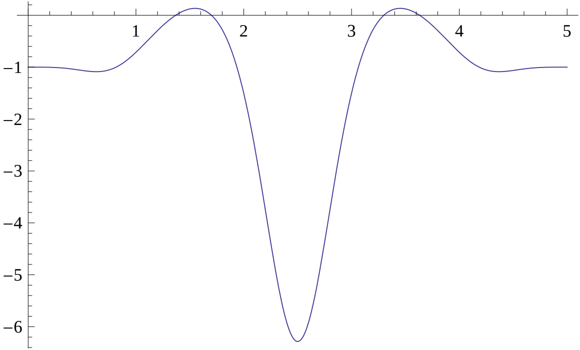
where W is a polynomial of degree 7.

$$
\begin{aligned}
W(y-c) = &- 36a^3 + 3ab^3 + 36a^2bc + 96a^2by - 54ab^2y^2 \\
&+ 1080a^2cy^2 - 720a^2y^3 + 24b^3y^3 - 864abcy^3 \\
&+ 948aby^4 + 288b^2cy^4 - 288b^2y^5 + 1728acy^5 \\
&- 360ay^6 - 576bcy^6 + 96by^7
\end{aligned}
$$

Proposition. $W(\mathcal{P}(\omega_1)) > 0$

Conjecture. $W(\mathcal{P}(\omega_1/2)) < 0$

$$R_2' = (\mathcal{P} + c)^3 \frac{W \circ \mathcal{P}}{(Q \circ \mathcal{P})^4} \mathcal{P}',$$



$$r = e^{-2.5}$$