International Journal of Mathematics
Vol. 18, No. 10 (2007) 1113–1132

A BERGMAN KERNEL IN $\mathbb{R}^n$, HOLOMORPHIC FUNCTIONS, AND RELATED NON-ASSOCIATIVE ALGEBRAS

ZBIGNIEW BŁOCKI
Jagiellonian University, Institute of Mathematics
Reymonta 4, 30-059 Kraków, Poland
Zbigniew.Blocki@im.uj.edu.pl

Received 13 November 2006

For domains in $\mathbb{R}^n$ we construct the Bergman kernel on the diagonal using solutions of the Dirichlet problem. Starting from this, in a natural way we obtain an algebra $\mathbb{A}_n$ of dimension $n(n-1)/2+1$ over $\mathbb{R}$ and a class of holomorphic functions valued in $\mathbb{A}_n$. Of course $\mathbb{A}_2$ is the field of complex numbers, and it turns out that $\mathbb{A}_3$ is the algebra of quaternions, whereas for $n \geq 4$, $\mathbb{A}_n$ is non-associative. Holomorphic functions can be written as $f+\omega$, where f is a (real-valued) function and ω a differential 2-form such that $d^*\omega = df$ and $d\omega = 0$. We investigate the main properties of the obtained objects, especially from the analytic point of view.

Keywords: Bergman kernel; holomorphic functions; differential forms.

Mathematics Subject Classification 2000: 32A30, 31B05

1. Introduction

If Ω is a bounded, smoothly bounded domain in $\mathbb{C}$ then the Bergman kernel can be defined in terms of a solution of the Dirichlet problem: assuming for simplicity that Ω contains the origin, one has

$$K_\Omega(\cdot, 0) = \frac{\partial v}{\partial z}, \tag{1.1}$$

where v is a complex-valued harmonic function in Ω, smooth on $\overline{\Omega}$, such that $v(z) = 1/(\pi\overline{z})$ on $\partial\Omega$ (see e.g. [1, p. 97]). This may be viewed as another construction of the Bergman kernel and we first translate it into a purely real setting. For this write $v = a + bi$, then

$$2v_z = a_x + b_y + i(b_x - a_y)$$

and we get that

$$\operatorname{Re} K_\Omega(\cdot, 0) = \operatorname{div} \mathcal{V},$$

where $\mathcal{V} = (v^1, v^2)$ is the harmonic vector field in Ω, smooth on $\overline{\Omega}$, such that $v^1 = E_x$, $v^2 = E_y$ on $\partial\Omega$, and

$$E(z) = \frac{1}{2\pi} \log |z|$$

is the fundamental solution for the Laplacian, that is $\Delta E = \delta_0$.

The point is that the above construction makes perfect sense in $\mathbb{R}^n$ and one can define in this manner the Bergman kernel, at least on the diagonal of $\Omega \times \Omega$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Namely, assuming again that Ω contains the origin, let $\mathcal{V} : \Omega \to \mathbb{R}^n$ be the harmonic vector field in Ω, smooth on $\overline{\Omega}$, such that $\mathcal{V} = \nabla E$ on $\partial\Omega$, where

$$E(x) = \begin{cases} \dfrac{1}{2\pi} \log |x|, & n = 2, \\ -\dfrac{1}{(n-2)s_n} |x|^{2-n}, & n \geq 3, \end{cases} \tag{1.2}$$

is the fundamental solution for the Laplacian in $\mathbb{R}^n$ (s_n is the area of the unit sphere in $\mathbb{R}^n$). We may thus define

$$K_\Omega(0,0) := \operatorname{div} \mathcal{V}(0). \tag{1.3}$$

The goal of this paper is to look closer at this construction. In a rather natural way, we will obtain an algebra $\mathbb{A}_n$ and a class of holomorphic functions $\Omega \to \mathbb{A}_n$ such that the expression defined by (1.3) will be equal to the Bergman kernel for this family on the diagonal of $\Omega \times \Omega$. It will turn out that holomorphic functions may be identified with expressions of the form $f + \omega$, where f is a function and ω is a 2-form satisfying the following Cauchy–Riemann equations

$$\begin{cases} d^*\omega = df, \\ d\omega = 0, \end{cases} \tag{1.4}$$

where d^* is the formal adjoint to d in $\mathbb{R}^n$. It is clear that solutions to (1.4) have to be harmonic and therefore (1.4) is an example of a *generalized Cauchy–Riemann system* (see [9, p. 231]). More general Cauchy–Riemann systems of differential forms than (1.4) have been studied in [4].

The algebra $\mathbb{A}_n$ is equal to $\mathbb{R} \oplus \bigwedge^2((\mathbb{R}^n)^*)$, that is $\mathbb{A}_n$ is of real dimension $n(n-1)/2 + 1$. The multiplication in $\mathbb{A}_n$ is derived in Sec. 2 and it turns out that $\mathbb{A}_3$ is precisely the algebra of quaternions, in particular the multiplication is then not commutative. For $n \geq 4$, this multiplication is not even associative. We believe that this multiplication is perhaps the most interesting (and possibly new) element obtained by our construction.

With the above notions of holomorphic functions and the multiplication in $\mathbb{A}_n$, one can define the Bergman kernel in an arbitrary domain $\Omega \subset \mathbb{R}^n$, so that

$$F(y) = \int_\Omega F(x)\overline{K_\Omega(x,y)}d\lambda(x)$$

for every $y \in \Omega$ and for each square-integrable holomorphic function F. One can then show that this definition coincides with (1.3), one gets in particular

$$K_\Omega(y,y) = \sup\left\{\frac{f(y)^2}{\|f\|^2 + \|\omega\|^2} : \ f, \omega \text{ satisfy } (1.4),\ (f,\omega) \not\equiv 0\right\}, \quad y \in \Omega,$$

where $\|\cdot\|$ denotes the L^2-norm.

As an application of such a notion of holomorphic functions in $\mathbb{R}^n$, we will show the following characterization of closed polar sets in $\mathbb{R}^n$ (it is of course well-known for $n = 2$).

Theorem 1.1. *Let P be a closed subset of $\mathbb{R}^n$. Then P is non-polar if and only if there exist a nonconstant, square-integrable function f and a 2-form ω in the complement $\mathbb{R}^n \backslash P$ satisfying* (1.4).

The theory of holomorphic functions in $\mathbb{R}^n$ valued in Clifford algebras is well-developed, see e.g. [2]. It is therefore quite likely that there is an overlap with the notions and results presented here for $n = 3$. The author is unaware however of any literature where a notion of a holomorphic function valued in a non-associative algebra is considered. The novelty of our approach might be that the considered objects (especially the multiplicative structure of the algebra $\mathbb{A}_n$) are obtained in a natural way, starting from a PDE definition of the classical Bergman kernel.

The paper is organized as follows. In Sec. 2, we recall the proof of (1.1) and then perform similar integration by parts in higher dimensions. We stress again that the obtained conditions and assumptions are derived in a very natural way. One of the important features of these arguments is the formula for multiplication in $\mathbb{A}_n$. Mostly algebraic properties of the algebra $\mathbb{A}_n$ are analyzed in Sec. 3. In Sec. 4, the conditions obtained in Sec. 2 are translated into the language of differential forms, we get in particular the Cauchy–Riemann equations (1.4). In Sec. 5, we introduce the definition of the Bergman kernel and show its equivalence with (1.3). We also prove Theorem 1.1. Finally, in Sec. 6, we show the Cauchy formula and discuss the problem of producing holomorphic functions from continuous boundary data.

2. Motivation

We first want to recall the proof of (1.1) because we are going to make similar arguments in higher dimensions. The idea is to show that $\partial v/\partial z$ reproduces $f(0)$ for f holomorphic in Ω, smooth up to the boundary. Then the result will follow because the space of such functions is dense in $H^2(\Omega)$ (see e.g. [1]). We have

$$\int_{\partial\Omega} f\,\frac{1}{z}\,dz = \pi \int_{\partial\Omega} f\overline{v}\,dz.$$

By the Green formula, since $\partial/\partial\overline{z}(1/z) = \pi\delta_0$,

$$\int_{\partial\Omega} f\,\frac{1}{z}\,dz = -\int_\Omega \frac{\partial}{\partial\overline{z}}\left(f\,\frac{1}{z}\right)\,dz \wedge d\overline{z} = 2\pi i f(0)$$

and

$$\int_{\partial\Omega} f\overline{v}\,dz = 2i\int_\Omega f\overline{\left(\frac{\partial v}{\partial z}\right)}\,d\lambda,$$

where $d\lambda$ is the Lebesgue measure. We thus get (1.1).

We now want to perform similar integration by parts in higher dimensions. Let Ω be a bounded, smoothly bounded domain in $\mathbb{R}^n$ and let $\mathcal{V} = (v^1, \dots, v^n)$ be the harmonic vector field in Ω such that $\mathcal{V} = \nabla E$ on $\partial\Omega$. For an arbitrary real-valued $f \in C^1(\overline{\Omega})$ and $i, j = 1, \dots, n$ we have

$$0 = \int_{\partial\Omega} f(E_j - v^j)\,dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n, \tag{2.1}$$

where we denote $E_j = \partial E/\partial x_j$. Therefore, by the Stokes theorem

$$\int_\Omega (fE_j)_i d\lambda = \int_\Omega (fv^j)_i d\lambda,$$

provided that the left-hand side makes sense. Take a matrix (f^{ij}) of functions from $C^1(\overline{\Omega})$ and sum over i, j. Then

$$\sum_{i,j}\int_\Omega \left(f^{ij}E_{ij} + f^{ij}_i E_j\right) d\lambda = \sum_{i,j}\int_\Omega \left(f^{ij}v^j_i + f^{ij}_i v^j\right) d\lambda. \tag{2.2}$$

On one hand, we would like the term $\sum_{i,j} f^{ij}E_{ij}$ to be of the form $f\Delta E$ (then in particular the left-hand side always makes sense). This will be the case provided that

$$f^{11} = \cdots = f^{nn} = f, \quad f^{ij} + f^{ji} = 0, \quad i \neq j. \tag{2.3}$$

On the other hand, the first order terms will disappear if we assume that

$$\sum_{i=1}^n f^{ij}_i = 0, \quad j = 1, \dots, n. \tag{2.4}$$

Then, assuming that (f^{ij}) satisfies (2.3) and (2.4), we will get

$$f(0) = \int_\Omega \left[f \operatorname{div}\mathcal{V} + \sum_{i<j} f^{ij}\left(v^j_i - v^i_j\right)\right] d\lambda. \tag{2.5}$$

In particular, if in (2.5) we take

$$(f^{ij}) = \begin{pmatrix} \operatorname{div}\mathcal{V} & & & & \\ & \cdot & & v^j_i - v^i_j & \\ & & \cdot & & \\ & v^i_j - v^j_i & & \cdot & \\ & & & & \operatorname{div}\mathcal{V} \end{pmatrix}$$

then (f^{ij}) satisfies (2.3), (2.4) (the latter we leave as an exercise to the reader) and

$$\operatorname{div}\mathcal{V}(0) = \int_\Omega \left[(\operatorname{div}\mathcal{V})^2 + \sum_{i<j} \left(v_i^j - v_j^i \right)^2 \right] d\lambda. \tag{2.6}$$

This implies in particular that $\operatorname{div}\mathcal{V}(0) \geq 0$. Note that (2.6) is a counterpart of the formula

$$\operatorname{Re} K_\Omega(0,0) = \int_\Omega |K_\Omega(\cdot, 0)|^2 d\lambda$$

for $n = 2$.

Take another, now arbitrary harmonic vector field $\mathcal{U} = (u^1, \ldots, u^n) : \Omega \to \mathbb{R}^n$, smooth up to the boundary. We want to find integral representations for $\operatorname{div}\mathcal{U}(0)$ and $u_i^j(0) - u_j^i(0)$, $i \neq j$ (since precisely terms of this kind appear in (2.6)). If in (2.5) we take

$$(f^{ij}) = \begin{pmatrix} \operatorname{div}\mathcal{U} & & & & \\ & \cdot & & u_i^j - u_j^i & \\ & & \cdot & & \\ & u_j^i - u_i^j & & \cdot & \\ & & & & \operatorname{div}\mathcal{U} \end{pmatrix},$$

we will get

$$\operatorname{div}\mathcal{U}(0) = \int_\Omega \left[\operatorname{div}\mathcal{U} \operatorname{div}\mathcal{V} + \sum_{i<j} \left(u_i^j - u_j^i \right) \left(v_i^j - v_j^i \right) \right] d\lambda.$$

To get a representation for $u_1^2(0) - u_2^1(0)$, it is convenient to take

$$(f^{ij}) = \begin{pmatrix} u_1^2 - u_2^1 & -\operatorname{div}\mathcal{U} & u_2^3 - u_3^2 & \ldots & u_2^n - u_n^2 \\ \operatorname{div}\mathcal{U} & u_1^2 - u_2^1 & u_3^1 - u_1^3 & \ldots & u_n^1 - u_1^n \\ u_3^2 - u_2^3 & u_1^3 - u_3^1 & u_1^2 - u_2^1 & & 0 \\ \vdots & \vdots & & \ddots & \\ u_n^2 - u_2^n & u_1^n - u_n^2 & 0 & & u_1^2 - u_2^1 \end{pmatrix}.$$

(Again, we leave it to the reader to check that (f^{ij}) so defined satisfies (2.4).) From (2.5), we will obtain

$$\begin{aligned} u_1^2(0) - u_2^1(0) = \int_\Omega \Bigg[& \left(u_1^2 - u_2^1\right) \operatorname{div}\mathcal{V} - \operatorname{div}\mathcal{U} \left(v_1^2 - v_2^1\right) \\ & + \sum_{i=3}^n \left[\left(u_2^i - u_i^2\right)\left(v_1^i - v_i^1\right) - \left(u_1^i - u_i^1\right)\left(v_2^i - v_i^2\right) \right] \Bigg] d\lambda. \end{aligned}$$

Similarly, for arbitrary $p \neq q$, we will get

$$u_p^q(0) - u_q^p(0) = \int_\Omega \Bigg[\left(u_p^q - u_q^p\right) \operatorname{div} \mathcal{V} - \operatorname{div} \mathcal{U} \left(v_p^q - v_q^p\right) + \sum_{i \neq p,q}^{n} \left[\left(u_q^i - u_i^q\right)\left(v_p^i - v_i^p\right) - \left(u_p^i - u_i^p\right)\left(v_q^i - v_i^q\right) \right] \Bigg] d\lambda.$$

By the way, notice that if we have an arbitrary matrix (f^{ij}) of C^1 functions for which (2.3) and (2.4) hold, then the matrix

$$\begin{pmatrix} f^{12} & -f & f^{23} & \dots & f^{2n} \\ f & f^{12} & f^{31} & \dots & f^{n1} \\ f^{32} & f^{13} & f^{12} & & 0 \\ \vdots & \vdots & & \ddots & \\ f^{n2} & f^{1n} & 0 & & f^{12} \end{pmatrix} \tag{2.7}$$

satisfies (2.4) provided that

$$f_k^{12} + f_1^{2k} + f_2^{k1} = 0, \quad k \geq 3.$$

Similarly we can argue for arbitrary $p \neq q$ and so it is natural to consider an extra condition

$$f_k^{ij} + f_i^{jk} + f_j^{ki} = 0, \quad \text{if } \#\{i, j, k\} = 3. \tag{2.8}$$

To summarize, we have just proved the following result:

Theorem 2.1. *Assume that Ω is a bounded, smoothly bounded domain in $\mathbb{R}^n$ containing the origin. Let $\mathcal{V} = (v^1, \dots, v^n)$ be the harmonic vector field in Ω, smooth on $\overline{\Omega}$, given by the condition $\mathcal{V} = \nabla E$ on $\partial\Omega$, where E is the fundamental solution for the Laplacian defined by* (1.2). *Set*

$$g := \operatorname{div} \mathcal{V}, \quad g^{ij} := v_i^j - v_j^i, \quad i, j = 1, \dots, n, \quad i \neq j.$$

Assume that $f^{ij} \in C^1(\overline{\Omega})$, $i, j = 1, \dots, n$, satisfy (2.3), (2.4) *and* (2.8). *Then*

$$f(0) = \int_\Omega \Bigg[fg + \sum_{i<j} f^{ij} g^{ij} \Bigg] d\lambda,$$

$$f^{pq}(0) = \int_\Omega \Bigg[f^{pq} g - f g^{pq} + \sum_{i \neq p,q} \left(f^{qi} g^{pi} - f^{pi} g^{qi} \right) \Bigg] d\lambda, \quad p \neq q.$$

In particular

$$\operatorname{div} \mathcal{V}(0) = \int_\Omega \Bigg[(\operatorname{div} \mathcal{V})^2 + \sum_{i<j} \left(v_i^j - v_j^i \right)^2 \Bigg] d\lambda,$$

$$v_p^q(0) - v_q^p(0) = 0, \quad p \neq q.$$

It does therefore seem natural to consider matrices of functions satisfying (2.3), (2.4) and (2.8) as holomorphic functions. We can treat them as functions $\Omega \to \mathbb{A}_n$, where Ω is open in $\mathbb{R}^n$ and $\mathbb{A}_n$ is the set of matrices from $\mathbb{R}^{n\times n}$ satisfying (2.3). Note that Theorem 2.1 gives the rule for a product of two elements from $\mathbb{A}_n$ and we are now going to analyze this.

3. The Algebra $\mathbb{A}_n$

Let $\mathbb{A}_n$ denote the set of matrices $A = (a^{ij}) \in \mathbb{R}^{n\times n}$ such that

$$a^{11} = \cdots = a^{nn} = a, \quad a^{ij} + a^{ji} = 0, \quad i \neq j. \tag{3.1}$$

We may write $\mathbb{A}_n = \mathbb{R} \oplus \bigwedge^{2,n}$, where by $\bigwedge^{p,n}$ we denote the exterior algebra $\bigwedge^p((\mathbb{R}^n)^*)$. The elements of $\mathbb{A}_n$ we will write in the form

$$A = a + w,$$

where $a \in \mathbb{R}$, we call the *real part* and $w \in \bigwedge^{2,n}$ the *imaginary part* of A, and denote $\operatorname{Re} A = a$, $\operatorname{Im} A = w$. Then

$$w = \sum_{i<j} a^{ij} e_{ij},$$

where for $i \neq j$ $e_{ij} = e_i^* \wedge e_j^*$ (here e_i are the standard basis vectors in $\mathbb{R}^n$ and e_i^* the dual covectors), that is the matrix $(e_{ij}^{kl}) \in \mathbb{A}_n$ is such that $e_{ij}^{ij} = -e_{ij}^{ji} = 1$, and $e_{ij}^{kl} = 0$ otherwise.

Theorem 2.1 provides a formula for the product $A\overline{B}$ in $\mathbb{A}_n$. Since of course we set

$$\overline{A} := A^T = a - w,$$

the product in $\mathbb{A}_n$ is defined as follows

$$\left(a + \sum_{i<j} a^{ij} e_{ij}\right)\left(b + \sum_{i<j} b^{ij} e_{ij}\right)$$
$$:= ab - \sum_{i<j} a^{ij} b^{ij} + \sum_{p<q}\left[a^{pq} b + a b^{pq} + \sum_{i\neq p,q}\left(a^{pi}b^{qi} - a^{qi}b^{pi}\right)\right] e_{pq}.$$

One can immediately see that $\mathbb{A}_n$ with this product has the structure of an algebra over $\mathbb{R}$: for $A, B, C \in \mathbb{A}_n$ and $\lambda \in \mathbb{R}$, we have

$$A(B + C) = AB + AC, \quad (A + B)C = AC + BC, \quad \lambda(AB) = (\lambda A)B = A(\lambda B).$$

It is easy to check that the product is determined by the following formulas (remembering that $e_{ij} = -e_{ji}$, $i \neq j$)

$$\begin{aligned} e_{ij}^2 &= -1, && i \neq j, \\ e_{ij}e_{jk} &= -e_{ik}, && \#\{i,j,k\} = 3, \\ e_{ij}e_{kl} &= 0, && \#\{i,j,k,l\} = 4. \end{aligned}$$

Note that if we associate a matrix (f^{ij}) satisfying (2.3) with $F = f + \sum_{i<j} f^{ij}e_{ij} \in \mathbb{A}_n$ then the matrix (2.7) we can associate with $-e_{12}F$.

Of course we have $\mathbb{A}_2 \simeq \mathbb{C}$. For $n \geq 3$, the algebra $\mathbb{A}_n$ is not commutative, for example,

$$e_{23} = e_{12}e_{13} \neq e_{13}e_{12} = -e_{23}.$$

Since

$$e_{12}^2 = e_{13}^2 = e_{23}^2 = e_{12}e_{13}e_{23} = -1,$$

we see that $\mathbb{A}_3 \simeq \mathbb{H}$, the algebra of quaternions. For $n \geq 4$, the algebra $\mathbb{A}_n$ is not associative, for example,

$$0 = (e_{12}e_{34})e_{34} \neq e_{12}(e_{34}e_{34}) = -e_{12}.$$

We have however the following weak commutativity and associativity:

Proposition 3.1. *For $A, B, C \in \mathbb{A}_n$, we have*

$$\mathrm{Re}(AB) = \mathrm{Re}(BA), \quad \mathrm{Re}((AB)C) = \mathrm{Re}(A(BC)).$$

Proof. The first formula is obvious. To show the second one, it is enough to consider only elements of the form $A = e_{ij}$, $B = e_{kl}$, $C = e_{pq}$, where $i \neq j$, $k \neq l$, $p \neq q$. Note that $\mathrm{Re}\left((e_{ij}e_{kl})e_{pq}\right) \neq 0$ only if $e_{ij}e_{kl} = \pm e_{pq}$. Then

$$\mathrm{Re}\left((e_{pj}e_{jq})e_{pq}\right) = 1 = \mathrm{Re}\left(e_{pj}(e_{jq}e_{pq})\right)$$

and similarly we check the other possibilities. □

We endow $\mathbb{A}_n$ with the euclidean norm

$$|A|^2 := a^2 + \sum_{i<j}(a^{ij})^2.$$

We can easily prove the following formulas

$$A\overline{A} = \overline{A}A = |A|^2, \quad \overline{AB} = \overline{B}\,\overline{A}.$$

We also have

$$|AB| \leq b_n|A||B|, \tag{3.2}$$

where

$$b_n = \max_{|A|=|B|=1} |AB| < \infty.$$

If $n = 2$ or $n = 3$, then in fact $|AB| = |A||B|$ for any $A, B \in \mathbb{A}_n$, so that $b_2 = b_3 = 1$. However,

$$(e_{12} + e_{34})\,(e_{13} - e_{24}) = 2\,(e_{14} + e_{23})$$

and we see that $b_n \geq \sqrt{2}$ for $n \geq 4$.

4. Holomorphic Functions as Differential Forms

Let $\Omega \subset \mathbb{R}^n$ be open. We will say that a C^1 function $F = (f^{ij}) : \Omega \to \mathbb{A}_n$ is *holomorphic* if it satisfies (2.4) and (2.8). The set of holomorphic functions will be denoted by $\mathcal{O}(\Omega)$. We may write

$$F = f + \omega,$$

where $f \in C^1(\Omega)$ and

$$\omega = \sum_{i<j} f^{ij} dx_i \wedge dx_j \in C^1_{(2)}(\Omega),$$

that is ω is a differential 2-form in Ω. Then (2.8) reads

$$d\omega = 0.$$

Recall that the Hodge $*$ operator

$$* : \bigwedge\nolimits^{p,n} \to \bigwedge\nolimits^{n-p,n}$$

is determined by

$$\alpha \wedge *\alpha = |\alpha|^2 d\lambda.$$

Then

$$*^2 = (-1)^{p(n-p)}$$

and the scalar product in $L^2_{(p)}(\Omega)$ can be written as

$$\langle\langle \alpha, \beta \rangle\rangle = \int_\Omega \alpha \wedge *\beta.$$

The formal adjoint of d

$$d^* : C^k_{(p)}(\Omega) \to C^{(k-1)}_{(p-1)}(\Omega)$$

is determined by

$$\langle\langle \alpha, d\beta \rangle\rangle = \langle\langle d^*\alpha, \beta \rangle\rangle, \quad \alpha \in C^1_{(p)}(\Omega), \quad \beta \in C^1_{0,(p-1)}(\Omega)$$

(that is β is compactly supported). Then

$$d^* = -(-1)^{n(p-1)} * d * .$$

If $u = \sum_j u^j dx_j \in C^k_{(1)}(\Omega)$ and $\omega = \sum_{i<j} f^{ij} dx_i \wedge dx_j \in C^k_{(2)}(\Omega)$ then

$$d^* u = -\sum_j u^j_j, \quad d^*\omega = -\sum_j \sum_{i \neq j} f^{ij}_i dx_j.$$

For any form $\alpha = \sum_{|I|=p} \alpha^I dx_I \in C^\infty_{(p)}(\Omega)$, we also have

$$dd^*\alpha + d^*d\alpha = -\sum_{|I|=p} \Delta\alpha^I dx_I.$$

The Cauchy–Riemann equations (2.4) are thus equivalent to

$$d^*\omega = df.$$

Examples of holomorphic functions appeared already in Sec. 2: for a harmonic vector field $\mathcal{U} = (u^1, \dots, u^n)$ the function

$$F = \begin{pmatrix} \operatorname{div}\mathcal{U} & & & \\ & \cdot & u^j_i - u^i_j & \\ & & \cdot & \\ & u^i_j - u^j_i & & \cdot & \\ & & & & \operatorname{div}\mathcal{U} \end{pmatrix}$$

is holomorphic. In other words, $F = -d^*u + du$, where u is a harmonic 1-form. In fact, at least locally every holomorphic function must be of this form.

Proposition 4.1. *Assume that a domain $\Omega \subset \mathbb{R}^n$ is such that the cohomology group $H^2(\Omega, \mathbb{R})$ vanishes. Then*

$$\mathcal{O}(\Omega) = \{-d^*u + du : u \in \mathcal{H}_{(1)}(\Omega)\},$$

where $\mathcal{H}_{(p)}(\Omega)$ denotes the set of harmonic p-forms in Ω.

Proof. The inclusion $\supset$ is clear. On the other hand, take $f + \omega \in \mathcal{O}(\Omega)$, that is $d^*\omega = df$ and $d\omega = 0$. Since $H^2(\Omega, \mathbb{R}) = 0$, it follows that there exists $v \in C^\infty_{(1)}(\Omega)$ with $dv = \omega$. We can also find $h \in C^\infty(\Omega)$ such that $\Delta h = f + d^*v$, that is $d^*dh = -f - d^*v$. Then $u := v + dh$ is a harmonic 1-form such that $-d^*u = f$ and $du = \omega$. $\square$

Remark. As observed by Ohsawa, one cannot remove the assumption $H^2(\Omega, \mathbb{R}) = 0$ in Proposition 4.1. For if Ω is bounded, smoothly bounded, and $H^2(\Omega, \mathbb{R}) \neq 0$ then by the Hodge decomposition theorem (see e.g. [8]), one can find a non d-exact form $\omega \in C^\infty_{(2)}(\overline{\Omega})$ with $d\omega = 0$, $d^*\omega = 0$, and such that the normal component of ω vanishes at the boundary. Therefore $F = \omega$ is holomorphic but cannot be written in the form $-d^*u + du$. In fact, by a more complicated approximation argument one can show that the assumption $H^2(\Omega, \mathbb{R}) = 0$ is necessary also for non-smooth Ω.

For $n = 2$ the assumption $H^2(\Omega, \mathbb{R}) = 0$ is always satisfied. Then Proposition 4.1 means that every holomorphic function must be globally of the form $\partial u/\partial z$ for some complex-valued harmonic function u. This follows also for example from the solution to the inhomogeneous Cauchy–Riemann equation (see e.g. [5, Theorem 1.4.4]).

One can also show that locally any harmonic function is a real part and every closed harmonic 2-form is an imaginary part of some holomorphic function.

Proposition 4.2. (i) *Assume that* $H^{n-1}(\Omega, \mathbb{R}) = 0$ *and that* f *is a harmonic function in* Ω*. Then there exists* $F \in \mathcal{O}(\Omega)$ *such that* $f = Re F$*.*

(ii) *Assume that* $H^1(\Omega, \mathbb{R}) = 0$ *and that* ω *is a harmonic* 2*-form in* Ω*. Then there exists* $F \in \mathcal{O}(\Omega)$ *such that* $\omega = Im F$*.*

Proof. (i) We can find $\widetilde{\omega} \in C^\infty_{(2)}(\Omega)$ with $d^*\widetilde{\omega} = df$ (because $H^{n-1}(\Omega, \mathbb{R}) = 0$ and since f is harmonic) and $\gamma \in C^\infty_{(2)}(\Omega)$ such that $\widetilde{\omega} = dd^*\gamma + d^*d\gamma$. Then $F = f + dd^*\gamma \in \mathcal{O}(\Omega)$.

(ii) Follows directly from the definition. □

On the other hand, for example, the product of two holomorphic functions need not be holomorphic for $n \geq 3$.

Many elementary properties of holomorphic functions follow from the fact that both their real and imaginary parts have to be harmonic. In particular, we have the following.

Proposition 4.3. *If* F *is holomorphic then* $|F|^2$ *is subharmonic.*

Remark. Using only the fact that $|F|^2$ is a finite sum of squares of harmonic functions, one can easily show that $|F|^p$ is subharmonic for every $p \geq 1$. Using in addition the fact that F is a solution to a generalized Cauchy–Riemann system, it follows that there exists $p_0 < 1$ such that $|F|^p$ is subharmonic for every $p \geq p_0$ (see [9, p. 233, Theorem 4.9]). It would perhaps be interesting to determine the best possible p_0 for $n \geq 3$. Since

$$F := (d - d^*)(E dx_1) = E_1 - \sum_{j \geq 2} E_j dx_1 \wedge dx_j \in \mathcal{O}(\mathbb{R}^n \backslash \{0\})$$

and $|F|^p$ is not subharmonic for $p < (n-2)/(n-1)$ (see also [9, p. 234]), we see that $p_0 \geq (n-2)/(n-1)$.

5. The Bergman Kernel

In this section, we will define the Bergman kernel in a standard way, that is by the Hilbert space approach. For bounded, smoothly bounded domains this notion will coincide with the one obtained in Theorem 2.1. For open $\Omega \subset \mathbb{R}^n$ set

$$H^2(\Omega) := \mathcal{O}(\Omega) \cap L^2(\Omega, \mathbb{A}_n).$$

We have

$$H^2(\Omega) = \{F = f + \omega \in L^2(\Omega, \mathbb{A}_n) : d^*\omega = df,\ d\omega = 0 \text{ (in the weak sense)}\}$$

and it is clear that it is closed in $L^2(\Omega, \mathbb{A}_n)$ (because convergence in L^2 implies weak convergence). In $H^2(\Omega)$, we have the generalized scalar product

$$\langle F, G\rangle = \int_\Omega F\overline{G}\, d\lambda \in \mathbb{A}_n.$$

Here we used (3.2), more precisely we have the counterpart of the Schwarz inequality

$$|\langle F, G\rangle| \le b_n \|F\|\,\|G\|, \quad F, G \in H^2(\Omega),$$

where

$$\|F\| := \left(\int_\Omega |F|^2 d\lambda\right)^{1/2}.$$

The space $H^2(\Omega)$ with the product $\operatorname{Re}\langle\cdot,\cdot\rangle$ is a Hilbert space over $\mathbb{R}$. We have the following representation theorem for bounded linear functionals on $H^2(\Omega)$.

Proposition 5.1. *Let $\mathcal{F} : H^2(\Omega) \to \mathbb{A}_n$ be a bounded, $\mathbb{R}$-linear functional such that*

$$\operatorname{Re}(\mathcal{F}(AF)) = \operatorname{Re}(A\mathcal{F}(F)), \quad A \in \mathbb{A}_n, \quad F \in H^2(\Omega).$$

Then there exists a unique $G \in H^2(\Omega)$ such that

$$\mathcal{F}(F) = \langle F, G\rangle, \quad F \in H^2(\Omega).$$

Proof. By the classical case there exists a unique $G \in H^2(\Omega)$ with

$$\operatorname{Re}\mathcal{F}(F) = \operatorname{Re}\langle F, G\rangle, \quad F \in H^2(\Omega).$$

For $p \ne q$, by Proposition 3.1, we have

$$\operatorname{Re}(e_{pq}\mathcal{F}(F)) = \operatorname{Re}(\mathcal{F}(e_{pq}F)) = \operatorname{Re}\langle e_{pq}F, G\rangle = \operatorname{Re}(e_{pq}\langle F, G\rangle).$$

From the definition of multiplication in $\mathbb{A}_n$, it follows that each component of $\mathcal{F}(F)$ coincides with the corresponding component of $\langle F, G\rangle$; i.e. $\mathcal{F}(F) = \langle F, G\rangle$. □

From Proposition 4.3, we easily deduce that for a fixed $y \in \Omega$ the functional

$$H^2(\Omega) \ni F \mapsto F(y) \in \mathbb{A}_n$$

is bounded. Therefore, there is a unique $K_\Omega(\cdot, y) \in H^2(\Omega)$ such that

$$F(y) = \int_\Omega F(x)\overline{K_\Omega(x,y)}\, d\lambda(x), \quad F \in H^2(\Omega), \quad y \in \Omega. \tag{5.1}$$

Substituting $F = K_\Omega(\cdot, x)$, we will get

$$K_\Omega(y,x) = \overline{\int_\Omega K_\Omega(w,y)\overline{K_\Omega(w,x)}\, d\lambda(w)} = \overline{K_\Omega(x,y)}.$$

It follows that $K_\Omega(x,y)$ is holomorphic in x and $\overline{K_\Omega(x,y)}$ is holomorphic in y. Thus every coordinate function of K_Ω is in particular separately harmonic and by a result of Lelong [7], we get that it is harmonic in $\Omega\times\Omega$.

We have

$$K_\Omega(y,y) = \|K_\Omega(\cdot,y)\|^2, \quad y\in\Omega,$$

thus

$$|F(y)| \le b_n\|F\|\sqrt{K_\Omega(y,y)}, \quad F\in H^2(\Omega), \quad y\in\Omega,$$

and

$$|K_\Omega(x,y)| \le b_n\sqrt{K_\Omega(x,x)K_\Omega(y,y)}, \quad x,y\in\Omega.$$

We also have

$$|\operatorname{Re} F(y)| \le \|F\|\sqrt{K_\Omega(y,y)}, \quad F\in H^2(\Omega), \quad y\in\Omega,$$

hence

$$K_\Omega(y,y) = \sup\left\{\frac{|\operatorname{Re} F(y)|^2}{\|F\|^2} : F\in H^2(\Omega)\backslash\{0\}\right\}, \quad y\in\Omega.$$

It follows in particular that $K_\Omega(y,y)>0$, $y\in\Omega$, for bounded Ω. If $\Omega'\subset\Omega$ then

$$K_\Omega(y,y) \le K_{\Omega'}(y,y), \quad y\in\Omega'.$$

The Bergman kernel coincides with the one obtained in Sec. 2.

Theorem 5.2. *Let Ω be a bounded, regular domain in $\mathbb{R}^n$ containing the origin and let v be the harmonic* 1*-form in Ω, continuous on $\overline{\Omega}$, such that $v = dE$ on $\partial\Omega$. Then*

$$K_\Omega(\cdot,0) = -d^*v + dv.$$

Theorem 5.2 will be a consequence of Theorem 2.1 and the proof of the following approximation theorem which is the same as in the classical case (see, e.g. [6, p. 180]).

Theorem 5.3. *If Ω_j is a sequence of domains in $\mathbb{R}^n$ increasing to Ω then K_{Ω_j} tends to K_Ω locally uniformly in $\Omega\times\Omega$.*

Proof. Suppose $\Omega'\Subset\Omega$. Then for j sufficiently big

$$|K_{\Omega_j}(x,y)| \le b_n\sqrt{K_{\Omega_j}(x,x)K_{\Omega_j}(y,y)} \le b_n\sqrt{K_{\Omega'}(x,x)K_{\Omega'}(y,y)}, \quad x,y\in\Omega',$$

and thus K_{Ω_j} is locally bounded in $\Omega\times\Omega$. Since the coordinate functions of K_{Ω_j} are harmonic, we can find a subsequence converging locally uniformly. To finish the proof we thus have to show that if $K_{\Omega_j} \to K$ locally uniformly in $\Omega \times \Omega$ then $K = K_\Omega$. Note that for $y \in \Omega$

$$\begin{aligned}\int_{\Omega'} |K(\cdot, y)|^2 d\lambda &= \lim_{j\to\infty} \int_{\Omega'} |K_{\Omega_j}(\cdot, y)|^2 d\lambda \\ &\le \liminf_{j\to\infty} \int_{\Omega_j} |K_{\Omega_j}(\cdot, y)|^2 d\lambda \\ &= \liminf_{j\to\infty} K_{\Omega_j}(y, y) \\ &= K(y, y),\end{aligned}$$

hence $K(\cdot, y) \in H^2(\Omega)$. For $F \in H^2(\Omega)$ and j big enough, we have

$$F(y) = \int_{\Omega_j} F(x)\overline{K_{\Omega_j}(x, y)} d\lambda(x)$$

and

$$\begin{aligned}F(y) - \int_\Omega F(x)\overline{K(x, y)} d\lambda(x) = &\int_{\Omega'} F(x) \left(\overline{K_{\Omega_j}(x, y)} - \overline{K(x, y)}\right) d\lambda(x) \\ &+ \int_{\Omega_j \setminus \Omega'} F(x)\overline{K_{\Omega_j}(x, y)} d\lambda(x) \\ &- \int_{\Omega \setminus \Omega'} F(x)\overline{K(x, y)} d\lambda(x).\end{aligned}$$

We now easily conclude that the norm of each of the three integrals is arbitrarily small if Ω' is sufficiently close to Ω and j is big enough. We conclude that K satisfies (5.1) and thus $K = K_\Omega$. □

Proof of Theorem 5.2. Let Ω_j be a sequence of smoothly bounded domains, relatively compact in Ω, containing the origin, increasing to Ω. Let v_j be the harmonic 1-form on $\overline{\Omega}_j$ such that $v_j = dE$ on $\partial\Omega_j$. It follows that $v_j \to v$ locally uniformly in Ω and thus also

$$K_j := -d^* v_j + dv_j \to -d^* v + dv =: K$$

locally uniformly in Ω. For $F \in H^2(\Omega)$, Theorem 2.1 gives

$$F(0) = \int_{\Omega_j} F\overline{K}_j \, d\lambda.$$

Arguing similarly as in the last part of the proof of Theorem 5.3, we will obtain

$$F(0) = \int_\Omega F\overline{K} \, d\lambda,$$

that is $K_\Omega(\cdot, 0) = K$. □

As in the classical case, the Bergman kernel may be expressed in terms of the Green function. By the approximation theorem, it will be no loss of generality to assume that Ω is bounded and smoothly bounded. It will be convenient to consider the following definition of the Green function (which will differ from the classical one by a negative constant): for a given pole $y \in \Omega$ the function $G_\Omega(\cdot, y)$ is the solution of the Dirichlet problem

$$\Delta G_\Omega(\cdot, y) = \delta_y, \quad G_\Omega(x, y) = 0 \ \text{ if } x \in \partial\Omega.$$

Or equivalently,

$$G_\Omega(x, y) = E(x - y) + \Psi(x, y), \quad x, y \in \overline{\Omega}, \tag{5.2}$$

where $\Psi \in C^\infty(\overline{\Omega} \times \overline{\Omega} \backslash D_{\partial\Omega})$ ($D_{\partial\Omega}$ is the diagonal of $\partial\Omega \times \partial\Omega$) is such that $\Psi(\cdot, y)$ is harmonic in Ω for $y \in \Omega$ and $\Psi(x, y) = -E(x - y)$ for $x \in \partial\Omega$ and $y \in \Omega$. For not necessarily smoothly bounded Ω, we have

$$G_\Omega(\cdot, y) = \sup \mathcal{B}_y, \quad y \in \Omega,$$

where

$$\mathcal{B}_y = \sup\{v \in SH(\Omega) : v < 0, \limsup_{x \to y}(v(x) - E(x - y)) < \infty\}.$$

Then $G_\Omega(\cdot, y) \in \mathcal{B}_y$ and $\Delta G_\Omega(\cdot, y) = \delta_y$ if either $n \geq 3$ or $\mathbb{R}^2 \backslash \Omega$ is not polar.

Assume $0 \in \Omega$ and that Ω is bounded and smoothly bounded. Let $\mathcal{U} = (u^1, \ldots, u^n)$ be the harmonic vector field on $\overline{\Omega}$ defined by

$$u^j(x) = \frac{\partial \Psi}{\partial y_j}(x, 0) = \frac{\partial G_\Omega}{\partial y_j}(x, 0) + \frac{\partial E}{\partial x_j}(x), \quad x \in \overline{\Omega}, \quad j = 1, \ldots, n.$$

We see that $\mathcal{U} = \nabla E$ on $\partial\Omega$, that is $\mathcal{U} = \mathcal{V}$, where $\mathcal{V}$ is defined in Theorem 2.1. We have thus obtained the following result.

Theorem 5.4. *In $\Omega \times \Omega$ away from the diagonal we have*

$$K_\Omega = \begin{pmatrix} \sum_k \frac{\partial^2 G_\Omega}{\partial x_k \partial y_k} & & & \\ & \cdot & \frac{\partial^2 G_\Omega}{\partial x_i \partial y_j} - \frac{\partial^2 G_\Omega}{\partial x_j \partial y_i} & \\ & & \cdot & \\ & \frac{\partial^2 G_\Omega}{\partial x_j \partial y_i} - \frac{\partial^2 G_\Omega}{\partial x_i \partial y_j} & & \cdot \\ & & & & \sum_k \frac{\partial^2 G_\Omega}{\partial x_k \partial y_k} \end{pmatrix}.$$

On the diagonal

$$K_\Omega(x, x) = \Delta\psi(x),$$

where $\psi(x) = \Psi(x, x)$ is given by (5.2).

Remark. The second part of Theorem 5.4 (for $n = 2$, it was proved in [10]) follows from the first one in an elementary way.

If $B_R = B(0,R)$ is the ball centered at the origin with radius R, then

$$G_{B_R}(x,y) = E(x-y) - E\left(\frac{|y|}{R}x - \frac{R}{|y|}y\right).$$

We may thus compute K_{B_R}, we will get in particular

$$K_{B_R}(y,y) = R^{n-2}\frac{(n-2)|y|^2 + nR^2}{s_n(R^2-|y|^2)^n}.$$

Note that although $G_{\mathbb{R}^n}(x,y) = E(x-y) \neq 0$ for $n \geq 3$, we nevertheless still have $K_{\mathbb{R}^n} \equiv 0$.

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. If P is polar then $G_{\mathbb{R}^n\setminus P} = G_{\mathbb{R}^n}$ and by Theorem 5.4, we have $K_{\mathbb{R}^n\setminus P} \equiv 0$, that is $H^2(\mathbb{R}^n\setminus P) = \{0\}$. On the other hand, let P be non-polar. Then there exists compact, non-polar $K \subset P$. We may of course assume that $n \geq 3$ (the proof is then in fact slightly simpler than for the well known case $n = 2$, see [3, pp. 73–74]). We can then find a non-constant harmonic function h in $\mathbb{R}^n\setminus K$ such that the gradient ∇h is square integrable (see [3]). Then

$$F := (d - d^*)(h\,dx_1) = h_1 - \sum_{j\geq 2} h_j e_{1j}$$

is a non-zero function in $H^2(\mathbb{R}^n\setminus K)$. □

6. The Cauchy Formula

For $F = f + \omega \in C^1(\Omega, \mathbb{A}_n)$, we set

$$\overline{\partial} F := df - d^*\omega$$

which is a 1-form in Ω. In this section it will be convenient to identify (in a natural way) covectors with vectors in $\mathbb{R}^n$, so that we get the mapping

$$\overline{\partial} : C^1(\Omega, \mathbb{A}_n) \to C(\Omega, \mathbb{R}^n).$$

We want to prove a counterpart of the Cauchy–Green formula in $\mathbb{R}^n$, that is given $F \in C^1(\overline{\Omega}, \mathbb{A}_n)$, where Ω is bounded and smoothly bounded, we would like to express the values of F in Ω in terms of F on $\partial\Omega$ and $\overline{\partial} F$ in Ω. In order to do that we will perform a similar integration by parts as in Sec. 2. By Stokes' theorem

$$\sum_{i,j}\int_{\partial\Omega} f^{ij} E_j * dx_i = \int_{\partial\Omega}(f * dE + dE \wedge *\omega) = f(0) + \langle df - d^*\omega, dE\rangle. \qquad (6.1)$$

To write the left-hand side of (6.1) in a simplified form, it is convenient to introduce some notation. For $u = (u^1, \dots, u^n)$, $v = (v^1, \dots, v^n) \in \mathbb{R}^n$ set

$$u \cdot v := \begin{pmatrix} \sum_k u^k v^k & & & \\ & \cdot & u^j v^i - u^i v^j & \\ & & \cdot & \\ & u^i v^j - u^j v^i & \cdot & \\ & & & \sum_k u^k v^k \end{pmatrix} \in \mathbb{A}_n.$$

On $\partial\Omega$ we also have

$$*dx_i = n_\Omega^i d\sigma, \quad i = 1, \dots, n,$$

where $n_\Omega = (n_\Omega^1, \dots, n_\Omega^n)$ denotes the unit outer normal vector to $\partial\Omega$ and $d\sigma$ is the area measure on $\partial\Omega$.

We have the following Cauchy–Green formula.

Theorem 6.1. *Assume that* $F = f + \omega \in C^1(\overline{\Omega}, \mathbb{A}_n)$, *where* Ω *is a bounded, smoothly bounded domain in* $\mathbb{R}^n$. *Then for* $y \in \Omega$

$$\mathrm{Re} F(y) = \mathrm{Re}\left(\int_{\partial\Omega} F\left(\nabla E(\cdot - y) \cdot n_\Omega\right) d\sigma - \int_\Omega \overline{\partial} F \cdot \nabla E(\cdot - y)\, d\lambda \right).$$

If in addition $d\omega = 0$, *then*

$$F(y) = \int_{\partial\Omega} F\left(\nabla E(\cdot - y) \cdot n_\Omega\right) d\sigma - \int_\Omega \overline{\partial} F \cdot \nabla E(\cdot - y)\, d\lambda.$$

Proof. The first part is precisely (6.1). Note that $f^{ij} = -\mathrm{Re}\,(e_{ij}F)$, $i \neq j$, and the second part follows from (6.1), Proposition 3.1 and the following result:

Proposition 6.2. *For* $A \in \mathbb{A}_n$, $F = f + \omega \in C^1(\Omega, \mathbb{A}_n)$ *with* $d\omega = 0$, *and* $u \in \mathbb{R}^n$ *we have*

$$\overline{\partial}(AF) \cdot u = A\left(\overline{\partial} F \cdot u\right).$$

Proof of Proposition 6.2. Without loss of generality we may assume that $A = e_{pq}$. If we write $\overline{\partial} F = (v^1, \dots, v^n)$, then $\overline{\partial}(e_{pq}F) = -v^q e_p + v^p e_q$. From this we will easily get $\overline{\partial}(e_{pq}F) \cdot u = e_{pq}(\overline{\partial} F \cdot u)$. □

This completes the proof of Theorem 6.1. □

As an immediate consequence of Theorem 6.1, we obtain the Cauchy formula for holomorphic functions:

Corollary 6.3. *Assume that* Ω *is a bounded, smoothly bounded domain in* $\mathbb{R}^n$ *and* $F \in C^1(\overline{\Omega}, \mathbb{A}_n) \cap \mathcal{O}(\Omega)$. *Then*

$$F(y) = \int_{\partial\Omega} F\left(\nabla E(\cdot - y) \cdot n_\Omega\right) d\sigma, \quad y \in \Omega.$$

Since $n^i_\Omega d\sigma = *dx_i$, we may also write (with some abuse of notation)

$$\int_{\partial\Omega} F\left(\nabla E(\cdot - y)\cdot n_\Omega\right) d\sigma = \int_{\partial\Omega} F(\nabla E(\cdot - y)\cdot \widehat{dx}),$$

where $\widehat{dx} = (*dx_1, \dots, *dx_n)$. For $n = 2$, this becomes (as in the standard Cauchy formula)

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{F(\zeta)}{\zeta - y} d\zeta.$$

On the other hand, given continuous $G : \partial\Omega \to \mathbb{A}_n$, one can ask whether the formula in Corollary 6.3 could be used to produce a holomorphic function in Ω. It turns out to be the case only for $n = 2$ and $n = 3$. The reason is that Part (ii) of the following proposition does not hold for $n \geq 4$.

Proposition 6.4. *Let h be a C^1 function, $A \in \mathbb{A}_n$ and $u = (u^1, \dots, u^n) \in \mathbb{R}^n$. Set $F := A(\nabla h \cdot u)$ and $f := \operatorname{Re} F$, $\omega := \operatorname{Im} F$. Then*

(i) *If h is harmonic then $d^*\omega = df$.*
(ii) *If $n = 3$ and h is harmonic then $d\omega = 0$.*
(iii) *If $\operatorname{Im} A \wedge (u^1 dx_1 + \cdots + u^n dx_n) = 0$, then $d\omega = 0$.*

Proof. Write $A = (a^{ij})$, assuming that (3.1) holds. One may compute that

$$\begin{aligned} A(\nabla h \cdot u) = \sum_{i,j} a^{ij}u^i h_j + \sum_{p<q}\Bigg[\sum_i \left(a^{iq}u^i h_p - a^{ip}u^i h_q\right) \\ + \sum_{i\neq p,q}\left(a^{pq}u^i + a^{ip}u^q + a^{qi}u^p\right) h_i\Bigg] e_{pq}. \end{aligned}$$

If $\omega = \sum_{i<j} f^{ij} dx_i \wedge dx_j$ then

$$df - d^*\omega = \sum_{p,q} f^{pq}_p dx_q$$

and for a fixed q

$$\begin{aligned} \sum_p f^{pq}_p &= \sum_i \left[\sum_j a^{ij}h_{jq} + \sum_{p\neq q}\left(a^{iq}h_{pp} - a^{ip}h_{pq}\right)\right] u^i \\ &\quad + \sum_{i,p:\ \#\{i,p,q\}=3} \left(a^{pq}u^i + a^{ip}u^q + a^{qi}u^p\right) h_{ip} \\ &= \sum_i a^{iq}u^i \Delta h \\ &= 0. \end{aligned}$$

On the other hand,

$$d\omega = \sum_{p<q<r}\left(f^{pq}_r + f^{qr}_p + f^{rp}_q\right) dx_p \wedge dx_q \wedge dx_r$$

and, if $\#\{p,q,r\} = 3$,

$$\begin{aligned} f_r^{pq} + f_p^{qr} + f_q^{rp} = {} & (a^{pq}u^r + a^{rp}u^q + a^{qr}u^p)\,(h_{pp} + h_{qq} + h_{rr}) \\ & + \sum_{i \neq p,q,r} \big[(a^{pq}u^i + a^{ip}u^q + a^{qi}u^p)h_{ir} \\ & + \left(a^{rp}u^i + a^{ir}u^p + a^{pi}u^r\right) h_{iq} \\ & + \left(a^{qr}u^i + a^{iq}u^r + a^{ri}u^q\right) h_{ip}\big], \end{aligned}$$

thus we get (ii) and (iii). □

Corollary 6.5. *Let M be a smooth, compact, oriented hypersurface in $\mathbb{R}^n$ (with boundary or not) with continuous normal vector field n. For a continuous $G : M \to \mathbb{A}_n$ set*

$$F(y) = \int_M G\,(\nabla E(\cdot - y)\cdot n)\,d\sigma, \quad y \in \mathbb{R}^n \backslash M,$$

and $f := \mathrm{Re}F$, $\omega := \mathrm{Ie}F$. By T denote the $n-1$-current supported on M such that $T = d\sigma$ on M. Then

(1) $d^*\omega = df$.
(2) *If either $n \leq 3$ or $\mathrm{Im}\,G \wedge *T = 0$ (that is the tangential component of $\mathrm{Im}\,G$ vanishes at M), then $d\omega = 0$, that is F is holomorphic in $\mathbb{R}^n \backslash M$.*

For $n \geq 4$, one could obtain a holomorphic function in $\mathbb{R}^n\backslash M$ from $F = f + \omega$ given by Corollary 6.5 (for arbitrary M and G) by projecting orthogonally ω in $L^2_{(2)}(\mathbb{R}^n\backslash M)$ into the subspace of closed forms (for $n \geq 3$, if F is as in Corollary 6.5, we will always have $F \in L^2(\mathbb{R}^n \setminus M, \mathbb{A}_n)$).

On the other hand, the Cauchy–Green formula allows one to solve the inhomogeneous $\overline{\partial}$-equation for forms with compact support.

Proposition 6.6. *For $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ set*

$$F(y) := \int_{\mathbb{R}^n} u \cdot \nabla E(\cdot - y)\,d\lambda, \quad y \in \mathbb{R}^n,$$

and $f := \mathrm{Re}\,F$, $\omega := \mathrm{Im}\,F$. Then $d^*\omega - df = u$ and $d\omega = 0$.

Proof. After a change of variables, we get

$$F(y) = \int u(\cdot + y)\cdot \nabla E\,d\lambda.$$

For a fixed $y \in \mathbb{R}^n$ set $v := u(\cdot + y)$. Then for $j = 1, \ldots, n$

$$\sum_i f_i^{ij}(y) = \sum_i \int v_j^i E_i d\lambda + \sum_{i \neq j} \int (v_i^j E_i - v_i^i E_j) d\lambda.$$

Integrating by parts

$$-\sum_i f_i^{ij}(y) = \sum_i \int v^i E_{ij} d\lambda + \sum_{i \neq j} \int \left(v^j E_{ii} - v^i E_{ij}\right) d\lambda$$
$$= v^j(0) = u^j(y)$$

and thus $d^*\omega - df = u$. For different p, q, r we similarly have

$$\left(f_r^{pq} + f_q^{rp} + f_p^{qr}\right)(y) = \int \left(v_r^q E_p - v_r^p E_q + v_q^p E_r - v_q^r E_p + v_p^r E_q - v_p^q E_r\right) d\lambda = 0,$$

hence $d\omega = 0$. $\square$

Acknowledgments

The author is indebted to Professor Takeo Ohsawa for pointing out that the assumption $H^2(\Omega, \mathbb{R}) = 0$ in Proposition 4.1 is necessary.

Partially supported by KBN Grant #2 P03A 03726.

References

[1] S. R. Bell, *The Cauchy Transform, Potential Theory, and Conformal Mapping*, Studies in Advanced Mathematics (CRC Press 1992).
[2] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Research Notes in Mathematics, Vol. 76 (Pitman, 1982).
[3] L. Carleson, *Selected Problems on Exceptional Sets* (Van Nostrand, 1967).
[4] A. Cialdea, On the theory of self-conjugate differential forms, *Atti Sem. Mat. Fis. Univ. Modena* **46**(suppl.) (1998) 595–620.
[5] L. Hörmander, *An Introduction to Complex Analysis in Several Variables* (North-Holland, 1989).
[6] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis* (Walter de Gruyter, 1993).
[7] P. Lelong Functions plurisousharmoniques et fonctions analytiques de variables reélles, *Ann. Inst. Fourier* **11** (1961) 515–562.
[8] G. Schwarz, *Hodge Decomposition — A Method for Solving Boundary Value Problems*, Lecture Notes in Mathematics, Vol. 1607 (Springer-Verlag, 1995).
[9] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces* (Princeton University Press, 1971)
[10] N. Suita, Capacities and kernels on Riemann surfaces, *Arch. Ration. Mech. Anal.* **46** (1972) 212–217.