THE BERGMAN METRIC
AND THE PLURICOMPLEX GREEN FUNCTION

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ABSTRACT. We improve a lower bound for the Bergman distance in smooth pseudoconvex domains due to Diederich and Ohsawa. As the main tool we use the pluricomplex Green function and an L^2-estimate for the $\overline{\partial}$-operator of Donnelly and Fefferman.

1. INTRODUCTION

Diederich and Ohsawa [14] have shown that if Ω is a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, then the following lower bound for the Bergman distance in Ω holds: for a fixed $w_0 \in \Omega$ and $w \in \Omega$ close to the boundary, one has

$$(1.1) \qquad \operatorname{dist}_\Omega(w, w_0) \geq \frac{1}{C} \log \log(1/\delta_\Omega(w)),$$

where $\delta_\Omega(w)$ denotes the euclidean distance of w to $\partial\Omega$ and C is a constant depending only on Ω. They also asked if (1.1) could be improved to

$$(1.2) \qquad \operatorname{dist}_\Omega(w, w_0) \geq \frac{1}{C} \log(1/\delta_\Omega(w))$$

which is known to be the best estimate for strongly pseudoconvex domains.

The main goal of this paper is to show that one can improve (1.1) to

$$(1.3) \qquad \operatorname{dist}_\Omega(w, w_0) \geq \frac{\log(1/\delta_\Omega(w))}{C \log \log(1/\delta_\Omega(w))}$$

for C^2 smooth bounded pseudoconvex Ω in $\mathbb{C}^n$. Our main tool will be the pluricomplex Green function. We recall that for a bounded domain Ω in $\mathbb{C}^n$ and a pole $w \in \Omega$ it is defined by

$$g_{\Omega,w} := \sup\{u \in PSH(\Omega) : u < 0, \limsup_{z \to w}(u(z) - \log|z - w|) < \infty\}.$$

We refer to [10] or [20] for basic properties of g_Ω. The direct relation between the Bergman metric and the Green function has been explored quite extensively in recent years (see for example [6, 7, 13, 16]). In [14] a certain technical function similar but different from g_Ω was used. Here however, unlike in [14], we are able to apply the Green function directly. The main relation for us with the Bergman metric will be the following quite general result (it is a special case of Theorem 4.4 below).

Theorem 1.1. *There exists a positive constant c_n, depending only on n, such that if Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and $w, \widetilde{w} \in \Omega$ are such that $\{g_{\Omega,w} \leq -1\} \cap \{g_{\Omega,\widetilde{w}} < -1\} = \emptyset$, then*

$$(1.4) \qquad\qquad\qquad \operatorname{dist}_\Omega(w, \widetilde{w}) \geq c_n.$$

The main ingredients of the proof of Theorem 1.1 are the Kobayashi lower bound for the Bergman distance [21] and an L^2-estimate for the $\overline{\partial}$ operator essentially due to Donnelly and Fefferman [15]. As shown by Berndtsson [1], [3], it is in fact a simple consequence of the original Hörmander theory [18]. It should be pointed out that in many papers (see for example [7], [8], [14]) much more complicated L^2-estimates for the $\overline{\partial}$ operator were used.

Therefore, thanks to Theorem 1.1, in order to get a lower bound for the Bergman distance, it is enough to estimate the pluricomplex Green function from below in order to study the behavior of its sublevel sets. We do it in Section 5 following two basic ideas due to Herbort [17] and Diederich-Herbort [13]. The first is to use an inequality for the complex Monge-Ampère operator from [4] to estimate $|g_{\Omega,w}(\widetilde{\zeta})|$ from above in terms of $|g_{\Omega,\zeta}(w)|$ for some $\widetilde{\zeta}$ close to ζ. Then one estimates the modulus of continuity of $g_{\Omega,w}$ which is known to be continuous precisely when Ω is hyperconvex, that is, when it admits a bounded plurisubharmonic exhaustion function (see [10]). As a result, we improve some estimates from [17] and [13], by the way simplifying the part of Herbort's argument involving the estimate for the modulus of continuity of $g_{\Omega,w}$ [17, Main Lemma]). In particular, we get the following result (see Theorem 5.2 below with $a = b$).

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with diameter R for which there exists $v \in PSH(\Omega)$ and positive constants A and a such that in Ω we have*

$$(1.5) \qquad\qquad\qquad \frac{1}{A}\delta_\Omega^a \leq |v| \leq A\delta_\Omega^a.$$

Then there exist positive constants C_1, C_2 depending only on n, A, a and R such that if $w \in \Omega$ is such that $r := \delta_\Omega(w) \leq e^{-2}$, then

$$\{g_{\Omega,w} \leq -1\} \subset \{C_1^{-1} r(\log(1/r))^{-1/a} \leq \delta_\Omega \leq C_2 r(\log(1/r))^{n/a}\}.$$

We will now explain how Theorems 1.1 and 1.2 imply the estimate (1.3) for C^2 smooth pseudoconvex domains in $\mathbb{C}^n$. By [11] such domains satisfy the assumption of Theorem 1.2 and therefore (1.4) holds provided that

$$(1.6) \qquad\qquad \delta_\Omega(\widetilde{w}) \geq \delta_\Omega(w)(\log(1/\delta_\Omega(w)))^C,$$

where $C > 1$ depends only on Ω and $\delta_\Omega(\widetilde{w}) \leq e^{-C}$. Assume that $r := \delta_\Omega(w) \leq \min\{e^{-C}, \delta_\Omega(w_0)/2\} =: r_0$. The function $\gamma(\rho) := \rho(\log(1/\rho))^C$ is increasing on the interval $(0, r_0]$. We can find an integer k such that $\gamma^{k-1}(r) \leq r_0 \leq \gamma^k(r)$, where $\gamma^k = \gamma \circ \cdots \circ \gamma$. Any curve joining w with w_0 intersects the level sets $\{\delta_\Omega = \gamma^j(r)\}$, $j = 1, \ldots, k-1$, and thus by (1.6) and Theorem 1.1 its Bergman length can be estimated from below by $(k-1)c_n$. One can show inductively that if $r(\log(1/r))^{(k-1)C} \leq r_0$, then $\gamma^k(r) \leq r(\log(1/r))^{kC}$. It follows that $r_0 \leq r(\log(1/r))^{kC}$ which easily implies the estimate (1.3).

In [14] the jump in the Bergman distance was obtained for $w, \widetilde{w}$ with

$$\delta_\Omega(\widetilde{w}) \geq \delta_\Omega(w)^{1/C}$$

and then (1.1) immediately followed. On the other hand, a slightly weaker condition than (1.5) was assumed in [14]. To obtain (1.2) one would need to improve (1.6) to

$$\delta_\Omega(\widetilde{w}) \geq C\delta_\Omega(w).$$

This we are able to prove in arbitrary bounded convex domains with the constant $C = (e+1)^2/(e-1)^2$ (Theorem 5.4 below). Thus, using the localization principle for the Bergman metric (see for example [12] or [22]), we obtain the following.

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{C}^n$ such that for every $z_0 \in \partial\Omega$ there exists an open neighborhood U of z_0, open V in $\mathbb{C}^n$ and a biholomorphism $F : U \to V$ such that $F(\Omega \cap U)$ is convex. Then there exists a positive constant C depending only on Ω such that (1.2) holds for every $w_0 \in \Omega$ and every $w \in \Omega$ sufficiently close to $\partial\Omega$.*

Note that again no assumption is made on the regularity of Ω (of course in the latter case the boundary must always be Lipschitz continuous).

2. The Kobayashi construction

In this section we will briefly sketch the construction of Kobayashi [21] and discuss some of its consequences. We assume that Ω is a bounded domain in $\mathbb{C}^n$. By $H^2(\Omega)$ we denote the Hilbert space of square integrable holomorphic functions in Ω and $K_\Omega(z, w)$ is the Bergman kernel of Ω (holomorphic in z, antiholomorphic in w). We define the immersion of Ω into the (infinitely dimensional) projective space $\mathbb{P}(H^2(\Omega))$ as follows:

$$\tau : \Omega \ni w \longmapsto [K_\Omega(\cdot, w)] \in \mathbb{P}(H^2(\Omega)).$$

One can show that the Bergman metric in Ω is precisely the pull-back of the Fubini-Study metric in $\mathbb{P}(H^2(\Omega))$. Therefore

$$\mathrm{dist}_\Omega(w, \widetilde{w}) \geq \mathrm{dist}_{\mathbb{P}(H^2(\Omega))}(\tau(w), \tau(\widetilde{w})), \quad w, \widetilde{w} \in \Omega.$$

Moreover, $\mathbb{P}(H^2(\Omega))$ (with the Fubini-Study metric) is complete and

$$\mathrm{dist}_{\mathbb{P}(H^2(\Omega))}([f], [g]) = \arccos \frac{|\langle f, g \rangle|}{||f|| \, ||g||}, \quad f, g \in H^2(\Omega) \setminus \{0\}.$$

We can now easily deduce the following two results.

Proposition 2.1. *For a bounded domain Ω in $\mathbb{C}^n$ we have*

$$\mathrm{dist}_\Omega(w, \widetilde{w}) \geq \arccos \frac{|K_\Omega(w, \widetilde{w})|}{\sqrt{K_\Omega(w, w)K_\Omega(\widetilde{w}, \widetilde{w})}}, \quad w, \widetilde{w} \in \Omega. \qquad \square$$

Proposition 2.2. *If a bounded domain Ω in $\mathbb{C}^n$ satisfies*

$$(2.1) \qquad \limsup_{w \to \partial\Omega} \frac{|f(w)|}{\sqrt{K_\Omega(w, w)}} < ||f||_{L^2(\Omega)}, \quad f \in H^2(\Omega) \setminus \{0\},$$

then it is Bergman complete.

Proof. Let w_j be a Cauchy sequence with respect to dist_Ω. Then let $\tau(w_j)$ be a Cauchy sequence with respect to $\mathrm{dist}_{\mathbb{P}(H^2(\Omega))}$. Since $\mathbb{P}(H^2(\Omega))$ is complete, we can find $f \in H^2(\Omega) \setminus \{0\}$ such that $\tau(w_j) = [K_\Omega(\cdot, w_j)] \to [f]$. In particular,

$$\frac{|f(w_j)|}{||f|| \sqrt{K_\Omega(w_j, w_j)}} = |\langle \frac{f}{||f||}, \frac{K_\Omega(\cdot, w_j)}{||K_\Omega(\cdot, w_j)||} \rangle| \to 1,$$

which by assumption means that w_j has no accumulation point on $\partial\Omega$. But this of course means that w_j is also a Cauchy sequence with respect to the euclidean metric. $\square$

Zwonek [24] constructed a bounded, Bergman complete domain in $\mathbb{C}$ not satisfying

$$\limsup_{w \to \partial\Omega} \frac{|f(w)|}{\sqrt{K_\Omega(w,w)}} = 0, \quad f \in H^2(\Omega),$$

which was the criterion for Bergman completeness formulated in [21]. It remains an open problem to construct a Bergman complete domain Ω in $\mathbb{C}^n$ such that the (possibly) weaker condition (2.1) does not hold.

Proposition 2.1 shows that in order to estimate $\mathrm{dist}_\Omega(w, \widetilde{w})$ from below we need to estimate $|K_\Omega(w, \widetilde{w})|/\sqrt{K_\Omega(w,w)K_\Omega(\widetilde{w}, \widetilde{w})}$ from above. Similarly as in [14], we will see that it is enough to construct a right function from $H^2(\Omega)$.

Proposition 2.3. *Let Ω be a bounded domain in $\mathbb{C}^n$, $w, \widetilde{w} \in \Omega$. Suppose that $f \in H^2(\Omega)$ is such that $f(w) = K_\Omega(w, \widetilde{w})/\sqrt{K_\Omega(\widetilde{w}, \widetilde{w})}$ and $f(\widetilde{w}) = 0$. Then*

$$\frac{|K_\Omega(w, \widetilde{w})|}{\sqrt{K_\Omega(w,w)K_\Omega(\widetilde{w}, \widetilde{w})}} \leq \frac{||f||_{L^2(\Omega)}}{\sqrt{1 + ||f||_{L^2(\Omega)}^2}}$$

and

$$\mathrm{dist}_\Omega(w, \widetilde{w}) \geq \frac{\pi}{2} - \arctan ||f||_{L^2(\Omega)}.$$

Proof. We first note that the second estimate is a direct consequence of the first one and Proposition 2.1. We may assume that $f \neq 0$. Set $h := K_\Omega(\cdot, \widetilde{w})/\sqrt{K_\Omega(\widetilde{w}, \widetilde{w})}$. Then $\langle f, h \rangle = f(\widetilde{w})/\sqrt{K_\Omega(\widetilde{w}, \widetilde{w})} = 0$ and therefore we can find an orthonormal basis $\{\varphi_0, \varphi_1, \dots\}$ of $H^2(\Omega)$ such that $\varphi_0 = h$ and $\varphi_1 = f/||f||$. Then

$$K_\Omega(z, z) = \sum_{j=0}^\infty |\varphi_j(z)|^2 \geq |h(z)|^2 + \frac{|f(z)|^2}{||f||^2}, \quad z \in \Omega.$$

Applying it for $z = w$ we get the desired estimate. $\square$

3. The Hörmander-Donnelly-Fefferman-Berndtsson L^2-estimate for the $\overline{\partial}$ operator

Our main tool in constructing square integrable holomorphic functions will be the following estimate for the $\overline{\partial}$ operator, essentially due to Donnelly and Fefferman [15].

Theorem 3.1. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and let ψ be a plurisubharmonic function in Ω such that $-e^{-\psi}$ is also plurisubharmonic. Assume that $\alpha \in L^2_{loc,(0,1)}(\Omega)$ is such that $\overline{\partial}\alpha = 0$ and that*

$$(3.1) \qquad\qquad\qquad i\overline{\alpha} \wedge \alpha \leq H\, i\partial\overline{\partial}\psi$$

for some nonnegative, locally integrable function H in Ω. Then for every plurisubharmonic φ in Ω there exists $u \in L^2_{loc}(\Omega)$ with $\overline{\partial}u = \alpha$ and such that

$$\int_\Omega |u|^2 e^{-\varphi} \leq 16 \int_\Omega H e^{-\varphi}.$$

Theorem 3.1 is a direct consequence of the next result (with $r = 1/4$ and φ, ψ replaced with $\varphi + r\psi$, $r\psi$, respectively), which is a slight improvement of an estimate due to Berndtsson [1, Theorem 3.1].

Theorem 3.2. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and let ψ be a plurisubharmonic function in Ω such that for some fixed $r \in (0,1)$, the function $-e^{-\psi/r}$ is also plurisubharmonic. Then for any α, H and φ chosen as in Theorem 3.1 we can find $u \in L^2_{loc}(\Omega)$ with $\overline{\partial} u = \alpha$ and*

$$(3.2) \qquad \int_\Omega |u|^2 e^{\psi-\varphi} \le \frac{1}{(1-\sqrt{r})^2} \int_\Omega H e^{\psi-\varphi}.$$

Proof. We first assume that Ω is smooth and φ, ψ are smooth up to the boundary. We now follow the proof of [2, Lemma 2.2]. We have in particular $L^2(\Omega) = L^2(\Omega, e^{a\varphi+b\psi})$ for real a, b and $-e^{-\psi/r} \in PSH(\Omega)$ means precisely that

$$i\partial\psi \wedge \overline{\partial}\psi \le r\, i\partial\overline{\partial}\psi.$$

Let u be the solution to $\overline{\partial} u = \alpha$ which is minimal in the $L^2(\Omega, e^{-\varphi})$ norm. This means that

$$\int_\Omega u\overline{f}e^{-\varphi} = 0, \quad f \in H^2(\Omega).$$

Set $v := e^\psi u$. Then

$$\int_\Omega v\overline{f}e^{-\varphi-\psi} = 0, \quad f \in H^2(\Omega),$$

thus v is the minimal solution in the $L^2(\Omega, e^{-\varphi-\psi})$ norm to $\overline{\partial} v = \beta$, where

$$\beta = \overline{\partial}(e^\psi u) = e^\psi(\alpha + u\overline{\partial}\psi).$$

For every $t > 0$ we have

$$i\overline{\beta} \wedge \beta \le e^{2\psi}[(1+t^{-1})i\overline{\alpha} \wedge \alpha + (1+t)|u|^2 i\partial\psi \wedge \overline{\partial}\psi]$$
$$\le e^{2\psi}[(1+t^{-1})H + (1+t)r|u|^2]i\partial\overline{\partial}\psi$$
$$\le e^{2\psi}[(1+t^{-1})H + (1+t)r|u|^2]i\partial\overline{\partial}(\varphi + \psi).$$

Therefore by [18, Lemma 4.4.1] (which is also true with the constant 2 replaced by 1) we get

$$\int_\Omega |u|^2 e^{\psi-\varphi} = \int_\Omega |v|^2 e^{-\varphi-\psi} \le (1+t^{-1})\int_\Omega H e^{\psi-\varphi} + (1+t)r \int_\Omega |u|^2 e^{\psi-\varphi}.$$

For $t = r^{-1/2} - 1$ we obtain the required result when φ, ψ are smooth.

Now assume that Ω, φ are arbitrary and ψ is strongly plurisubharmonic but otherwise arbitrary (possibly even not locally bounded). By the Radon-Nikodym theorem there exists $\beta = \sum_{j,k} \beta_{jk} i dz_j \wedge d\overline{z}_k \in L^1_{loc,(1,1)}(\Omega)$ such that $0 < \beta \le i\partial\overline{\partial}\psi$ and $i\overline{\alpha} \wedge \alpha \le H\beta$. For $\varepsilon > 0$ let $\Omega_\varepsilon \subset \Omega$ be a smooth pseudoconvex domain such that $\varphi_\varepsilon := \varphi * \rho_\varepsilon$, $\psi_\varepsilon := \psi * \rho_\varepsilon$, the standard regularizations, are defined in a neighborhood of $\overline{\Omega}_\varepsilon$. If (ψ_ε^{jk}) denotes the inverse matrix of $(\partial^2\psi_\varepsilon/\partial z_j \partial\overline{z}_k)$, then $H_\varepsilon := \sum_{j,k} \psi_\varepsilon^{jk} \overline{\alpha}_j \alpha_k$ is the least function satisfying $i\overline{\alpha} \wedge \alpha \le H_\varepsilon i\partial\overline{\partial}\psi_\varepsilon$. One can easily check that

$$i\partial\psi_\varepsilon \wedge \overline{\partial}\psi_\varepsilon \le (i\partial\psi \wedge \overline{\partial}\psi) * \rho_\varepsilon \le r i\partial\overline{\partial}\psi_\varepsilon$$

and thus $-e^{-\psi_\varepsilon/r}$ is also plurisubharmonic. By the previous part we can thus find $u_\varepsilon \in L^2_{loc}(\Omega_\varepsilon)$ such that $\overline\partial u_\varepsilon = \alpha$ in Ω_ε and

$$\int_{\Omega_\varepsilon} |u_\varepsilon|^2 e^{\psi_\varepsilon - \varphi_\varepsilon} \le \int_{\Omega_\varepsilon} H_\varepsilon e^{\psi_\varepsilon - \varphi_\varepsilon} \le \int_{\Omega_\varepsilon} H_\varepsilon e^{\psi_\varepsilon - \varphi}.$$

We have $\beta_\varepsilon := \beta * \rho_\varepsilon \le i\partial\overline\partial\psi_\varepsilon$ and there is a sequence $\varepsilon_l \downarrow 0$ such that the coefficients of β_{ε_l} converge pointwise almost everywhere to the respective coefficients of β. Therefore

$$\varlimsup_{l\to\infty} H_{\varepsilon_l} \le \varlimsup_{l\to\infty} \sum_{j,k} \beta_{\varepsilon_l}^{jk} \overline\alpha_j \alpha_k = \sum_{j,k} \beta^{jk} \overline\alpha_j \alpha_k \le H,$$

where (β^{jk}) and (β_ε^{jk}) denote the inverse matrices of (β_{jk}) and $(\beta_{jk} * \rho_\varepsilon)$, respectively. If we now assume that the left-hand side of (3.2) is equal to 1, by the Fatou lemma we have

$$\varlimsup_{l\to\infty} \int_{\Omega_{\varepsilon_l}} |u_{\varepsilon_l}|^2 e^{\psi_{\varepsilon_l} - \varphi_{\varepsilon_l}} \le 1.$$

Since φ_{ε_l} is a decreasing sequence and ψ is locally bounded from above, we see that the L^2 norm of u_{ε_l} over Ω_ε is bounded for every fixed ε. Therefore, replacing ε_l with its subsequence if necessary, we see that u_{ε_l} converges weakly in Ω_ε for every ε to $u \in L^2_{loc}(\Omega)$. We can now show that u satisfies (3.2), which completes the proof for strongly plurisubharmonic ψ.

If ψ is not necessarily strongly plurisubharmonic, then we may approximate it by functions of the form $\psi + \varepsilon|z|^2$. Note that $i\overline\alpha \wedge \alpha \le H\, i\partial\overline\partial(\psi + \varepsilon|z|^2)$ and the general case easily follows along the same lines as before. $\square$

Note that the assumption on ψ in Theorem 3.1 means precisely that ψ is of the form

$$\psi = -\log(-v),$$

where v is a negative plurisubharmonic function in Ω. We shall usually use Theorem 3.1 for α given by

$$(3.3) \qquad\qquad \alpha = -f\overline\partial(\chi(\log(-v))),$$

where f is holomorphic and $\chi \in C^{0,1}(\mathbb{R})$. If v is locally bounded near $\partial\Omega$, then by [9] $v \in W^{1,2}_{loc}(\Omega)$, and thus $\alpha \in L^2_{(0,1),loc}(\Omega)$. The inequality (3.1) is then satisfied with

$$H = |f\chi'(\log(-v))|^2.$$

Theorem 3.1 thus gives the following result.

Theorem 3.3. *Let f be a holomorphic function in a pseudoconvex domain Ω in $\mathbb{C}^n$. Assume that φ, v are plurisubharmonic in Ω such that $v < 0$ and v is locally bounded near $\partial\Omega$. Let moreover $\chi \in C^{0,1}(\mathbb{R})$. Then one can find a holomorphic function F in Ω satisfying the following estimate:*

$$\int_\Omega |F - f\chi(\log(-v))|^2 e^{-\varphi} \le 16 \int_\Omega |f\chi'(\log(-v))|^2 e^{-\varphi}.$$

4. Applications to the Bergman kernel and metric

Theorems 3.1 and 3.3 seem to be quite universal tools in obtaining various estimates related to the Bergman kernel. First, we get the following estimate due to Herbort [16] (with a different constant though, depending also on the diameter of Ω).

Theorem 4.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $f \in H^2(\Omega)$. Then*
$$\frac{|f(w)|}{\sqrt{K_\Omega(w,w)}} \leq \left(1 + \frac{4}{\eta(n)}\right) ||f||_{L^2(\{g_{\Omega,w} \leq -1\})}, \quad w \in \Omega,$$
where
$$\eta(y) := \int_y^\infty \frac{dx}{xe^x}.$$

Proof. We apply Theorem 3.3 with $\varphi := 2ng_{\Omega,w}$, $v := g_{\Omega,w}$ and
$$\chi(t) := \begin{cases} 0, & t \leq 0, \\ \int_0^t e^{-ne^s} ds, & t > 0. \end{cases}$$
We can find a holomorphic F in Ω such that
$$\begin{aligned} ||F||_{L^2(\Omega)} &\leq ||f\chi(\log(-v))||_{L^2(\Omega)} + ||F - f\chi(\log(-v))||_{L^2(\Omega)} \\ &\leq \chi(\infty)||f||_{L^2(\{g_{\Omega,w} \leq -1\})} + ||F - f\chi(\log(-v))||_{L^2(\Omega, e^{-\varphi})} \\ &\leq \eta(n)||f||_{L^2(\{g_{\Omega,w} \leq -1\})} + 4||f\chi'(\log(-v))||_{L^2(\Omega, e^{-\varphi})} \\ &\leq (\eta(n) + 4)||f||_{L^2(\{g_{\Omega,w} \leq -1\})}. \end{aligned}$$
Since $e^{-\varphi}$ is not integrable near w, from Theorem 3.3 it also follows that $F(w) = \chi(\infty)f(w) = \eta(n)f(w)$. Hence
$$\frac{|f(w)|}{\sqrt{K_\Omega(w,w)}} = \frac{|F(w)|}{\eta(n)\sqrt{K_\Omega(w,w)}} \leq \frac{||F||_{L^2(\Omega)}}{\eta(n)}.$$
$\square$

Theorem 3.3 together implies in particular, thanks to Proposition 2.2, that if Ω is bounded pseudoconvex in $\mathbb{C}^n$ and
$$\lim_{w \to \partial\Omega} \text{vol}(\{g_{\Omega,w} \leq -1\}) = 0,$$
then Ω must be Bergman complete.

Next, we generalize results of Chen [8] to several variables.

Theorem 4.2. *Let Ω and U be bounded domains in $\mathbb{C}^n$ such that $\Omega \cup U$ is pseudoconvex with diameter R. Assume that $U \subset B(z_0, r)$. Then for every $f \in H^2(\Omega)$ there exists $F \in H^2(\Omega \cup U)$ such that for every $\lambda > 1$ we have*
$$||F - f||_{L^2(\Omega)} \leq \left(1 + \frac{4}{\log\lambda}\right) ||f||_{L^2(\Omega \cap \overline{B}(z_0, (r/R)^{1/\lambda}))}.$$

Proof. We set $v(z) := \log|z - z_0|/R$, $\varphi := 0$ and, for $\rho > r$,
$$\chi(t) := \begin{cases} 1, & t \leq \log\log(R/\rho) - \log\lambda, \\ \frac{\log\log(R/\rho)-t}{\log\lambda}, & \log\log(R/\rho) - \log\lambda < t \leq \log\log(R/\rho), \\ 0, & t > \log\log(R/\rho). \end{cases}$$

Note that (3.3) defines $\overline{\partial}$-closed $\alpha \in L^2_{(0,1),loc}(\Omega \cup U)$. We obtain a holomorphic F with

$$||F - f||_{L^2(\Omega)} \leq ||f(1 - \chi(\log(-v)))||_{L^2(\Omega)} + 4||f\chi'(\log(-v))||_{L^2(\Omega)}$$

and the desired estimate will follow if we let ρ tend to r. $\qquad\square$

Corollary 4.3. *Assume that Ω is a bounded domain in $\mathbb{C}^n$ satisfying the following property: for every $z_0 \in \partial\Omega$ there exists a neighborhood basis U_j of z_0 such that $\Omega \cup U_j$ is pseudoconvex for every j. Then, if*

$$\lim_{w \to \partial\Omega} K_\Omega(w, w) = \infty,$$

it follows that Ω is Bergman complete.

Proof. Let $\Omega \ni w_k \to z_0 \in \partial\Omega$ and $f \in H^2(\Omega)$. By Theorem 4.2 there exists a sequence $F_j \in H^2(\Omega \cup U_j)$ such that $||F_j - f||_{L^2(\Omega)} \to 0$. We have

$$\frac{|f(w_k)|}{\sqrt{K_\Omega(w_k, w_k)}} \leq \frac{|F_j(w_k)|}{\sqrt{K_\Omega(w_k, w_k)}} + ||F_j - f||_{L^2(\Omega)}.$$

For every fixed j, since F_j is holomorphic in a neighborhood of z_0, the sequence $|F_j(w_k)|$ is bounded. Thus, if we first let $k \to \infty$ and then $j \to \infty$, the corollary follows from Proposition 2.2. $\qquad\square$

Note that the assumption on Ω in Corollary 4.3 is always true if $n = 1$ but not if $n > 1$: it is not satisfied for the Hartogs triangle $\Omega = \{(z, w) \in \mathbb{C}^2 : |z| < |w| < 1\}$.

Theorem 4.4. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that $w, \widetilde{w} \in \Omega$ and $\alpha, \widetilde{\alpha} > 0$ are such that $\{g_{\Omega,w} \leq -\alpha\} \cap \{g_{\Omega,\widetilde{w}} < -\widetilde{\alpha}\} = \emptyset$. Then*

$$\mathrm{dist}_\Omega(w, \widetilde{w}) \geq \frac{\pi}{2} - \arctan\left(1 + 4\frac{e^{n\widetilde{\alpha}}}{\eta(n\alpha)}\right),$$

where η is as in Theorem 4.1.

Proof. Let $h := K_\Omega(\cdot, \widetilde{w})/\sqrt{K_\Omega(\widetilde{w}, \widetilde{w})} \in H^2(\Omega)$ so that $||h||_{L^2(\Omega)} = 1$. Set $\varphi := 2n(g_{\Omega,w} + g_{\Omega,\widetilde{w}})$, $v := g_{\Omega,w}$ and

$$\chi(t) := \begin{cases} 0, & t \leq \log\alpha, \\ \displaystyle\int_{\log\alpha}^t e^{-ne^s} ds, & t > \log\alpha. \end{cases}$$

Then in particular $\chi(\log(-v)) = \chi(\infty) = \eta(n\alpha)$ at w and $\chi(\log(-v)) = 0$ at $\widetilde{w}$. By Theorem 3.3 there exists $f \in H^2(\Omega)$ such that

$$\begin{aligned} ||f||_{L^2(\Omega)} &\leq ||h\chi(\log(-v))||_{L^2(\Omega)} + ||f - h\chi(\log(-v))||_{L^2(\Omega)} \\ &\leq 1 + ||f - h\chi(\log(-v))||_{L^2(\Omega, e^{-\varphi})} \\ &\leq 1 + 4||h\chi'(\log(-v))||_{L^2(\Omega, e^{-\varphi})} \\ &\leq 1 + 4\frac{e^{n\widetilde{\alpha}}}{\eta(n\alpha)}. \end{aligned}$$

Since $e^{-\varphi}$ is not integrable near w and $\widetilde{w}$, it also follows that $f(w) = h(w)$ and $f(\widetilde{w}) = 0$. It is now sufficient to apply Proposition 2.3. $\qquad\square$

5. Estimates for the pluricomplex Green function

The following theorem will be the main step in estimating the Green function. The main idea of the proof comes from [17] (see also [13]).

Theorem 5.1. *Assume that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ with the diameter R. Let $\zeta, w \in \Omega$ and $0 < \varepsilon < \min\{r/2, |\zeta - w|/2\}$, where $r := \delta_\Omega(w)$. Then*

$$|g_{\Omega,w}(\zeta)| \leq \frac{\log(R/\varepsilon)}{\log(r/(2\varepsilon))} \left(\sup_{\{\delta_\Omega = \varepsilon\}} |g_{\Omega,w}| + (n!)^{1/n} (\log(R/\varepsilon))^{1-1/n} |g_{\Omega,\zeta}(w)|^{1/n} \right).$$

Proof. By [10] (see also [20]) the function $g_{\Omega,w}$ is continuous on $\overline{\Omega} \setminus \{w\}$ (with $g_{\Omega,w} = 0$ on $\partial\Omega$). Let $\alpha := \log(R/\varepsilon)$. By [4] and since $(dd^c g_{\Omega,w})^n = (2\pi)^n \delta_w$,

$$\int_\Omega |g_{\Omega,w}|^n (dd^c \max\{g_{\Omega,\zeta}, -\alpha\})^n \leq n!(2\pi)^n \alpha^{n-1} |g_{\Omega,\zeta}(w)|.$$

The measure $(dd^c \max\{g_{\Omega,\zeta}, -\alpha\})^n$ is supported on the set $\{g_{\Omega,\zeta} = -\alpha\} \subset \overline{B}(\zeta, \varepsilon)$ and its total mass is equal to $(2\pi)^n$. Therefore, there exists $\widetilde{\zeta} \in \overline{B}(\zeta, \varepsilon)$ such that

$$(5.1) \qquad |g_{\Omega,w}(\widetilde{\zeta})|^n \leq n!(\log(R/\varepsilon))^{n-1} |g_{\Omega,\zeta}(w)|.$$

By u denote the relative extremal function of the ball $\overline{B}(w, \varepsilon)$, that is,

$$u = \sup\{v \in PSH(\Omega) : v|_\Omega < 0, v|_{\overline{B}(w,\varepsilon)} \leq -1\}.$$

By [23] (see also [5]) the function u is continuous on $\overline{\Omega}$ (with $u = 0$ on $\partial\Omega$ and $u = -1$ on $\overline{B}(w, \varepsilon)$). One can easily check that

$$(5.2) \qquad \log(R/\varepsilon)\, u \leq g_{\Omega,w} \leq \log(r/\varepsilon)\, u \quad \text{on } \Omega \setminus B(w, \varepsilon)$$

(since the functions are maximal in $\Omega \setminus \overline{B}(w, \varepsilon)$ and vanish on $\partial\Omega$, it is enough to show these inequalities on $\partial B(w, \varepsilon)$). In particular,

$$|u(z)| \leq \frac{\displaystyle\sup_{\{\delta_\Omega = \varepsilon\}} |g_{\Omega,w}|}{\log(r/\varepsilon)} =: \delta, \quad \text{if } \delta_\Omega(z) \leq \varepsilon.$$

Set $\widetilde{\Omega} := \{z \in \Omega : z + \widetilde{\zeta} - \zeta \in \Omega\}$ and for $\delta' > \delta$

$$h(z) := \begin{cases} \max\{u(z), u(z + \widetilde{\zeta} - \zeta) - \delta'\}, & z \in \widetilde{\Omega}, \\ u(z), & z \in \Omega \setminus \widetilde{\Omega}. \end{cases}$$

We claim that h is a negative plurisubharmonic function in Ω. Indeed, for $z \in \Omega \cap \partial\widetilde{\Omega}$ we have $\delta_\Omega(z) \leq \varepsilon$ and $u(z) \geq -\delta > u(z + \widetilde{\zeta} - \zeta) - \delta'$, hence $h = u$ in a neighborhood of $\Omega \cap \partial\widetilde{\Omega}$ and it follows that h is plurisubharmonic. We have

$$u(z) \leq \frac{\log(|z - w|/r)}{\log(r/\varepsilon)}, \quad z \in \Omega \setminus B(w, \varepsilon)$$

(because both functions are maximal and the inequality holds on the boundary), and

$$z + \widetilde{\zeta} - \zeta \in \overline{B}(w, 2\varepsilon) \subset B(w, r), \quad z \in \overline{B}(w, \varepsilon) \subset \widetilde{\Omega}.$$

Therefore

$$u(z + \widetilde{\zeta} - \zeta) \leq -\beta, \quad z \in \overline{B}(w, \varepsilon),$$

and

$$h \leq \max\{-1, -\beta - \delta'\} \leq -\beta \quad \text{on } \overline{B}(w, \varepsilon),$$

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where

$$\beta := \frac{\log(r/(2\varepsilon))}{\log(r/\varepsilon)}.$$

From the definition of u it thus follows that $h(z)/\beta \le u(z)$ for $z \in \Omega$. For $z = \zeta \in \widetilde{\Omega}$ letting δ' tend to δ we get

$$u(\widetilde{\zeta}) - \delta \le h(\zeta) \le \beta u(\zeta),$$

which together with (5.1) and (5.2) gives the required inequality. $\qquad\square$

Remark. For $n = 1$ the theorem recovers the symmetry of g_Ω: it is enough to let $\varepsilon \to 0$. $\qquad\square$

Theorem 5.2. *Let Ω be a bounded domain in $\mathbb{C}^n$, where we can find $v \in PSH(\Omega)$ and positive constants A, B, a, b such that in Ω the following estimate holds:*

$$(5.3) \qquad \frac{1}{A}\delta_\Omega^a \le |v| \le B\delta_\Omega^b.$$

Then there exist positive constants $C, \widetilde{C}$ depending only on n, A, B, a, b and R, the diameter of Ω such that for $\zeta, w \in \Omega$ with $r := \delta_\Omega(w) \le e^{-2}$ and $\rho := \delta_\Omega(\zeta) \le e^{-2}$ we have

$$(5.4) \qquad |g_{\Omega,w}(\zeta)| \le \begin{cases} C\dfrac{\rho^b}{r^a}\log(1/r), & \text{if } \rho \le r/2, \\[2ex] \widetilde{C}\dfrac{r^{b/n}}{\rho^{a/n}}(\log(1/r))^{1-1/n}(\log(1/\rho))^{1/n}, & \text{if } \rho \ge 2r. \end{cases}$$

In particular,

$$\{g_{\Omega,w} \le -1\} \subset \{C^{-1/b}r^{a/b}(\log(1/r))^{-1/b} \le \delta_\Omega \le \widetilde{C}^{n/a}r^{b/a}(\log(1/r))^{n/a}\}.$$

Proof. Assume first that $\rho \le r/2$. We have

$$g_{\Omega,w}(z) \ge \log(|z-w|/R), \quad z \in \Omega.$$

Therefore

$$g_{\Omega,w} \ge \frac{\log(2R/r)}{\inf\limits_{\overline{B}(w,r/2)} |v|} v \quad \text{in} \ \Omega \setminus B(w,r/2)$$

(because the inequality holds on the boundary) and (5.3) now gives

$$|g_{\Omega,w}(\zeta)| \le 2^a AB \frac{\rho^b}{r^a}\log(2R/r)$$

which implies the first inequality in (5.4).

Now assume that $\rho \ge 2r$. By $C_1, C_2, \dots$ we will denote positive constants depending only on n, A, B, a, b and R. If $0 < \varepsilon < r/2$, then by the first inequality

$$\sup_{\{\delta_\Omega = \varepsilon\}} |g_{\Omega,w}| \le C_1 \frac{\varepsilon^b}{r^a}\log(1/r)$$

and

$$|g_{\Omega,\zeta}(w)| \le C_1 \frac{r^b}{\rho^a}\log(1/\rho),$$

if r is sufficiently small. Therefore, by Theorem 5.1
$$(5.5)$$
$$|g_{\Omega,w}(\zeta)| \le C_2 \frac{\log(1/\varepsilon)}{\log(r/(2\varepsilon))}\left(\frac{\varepsilon^b}{r^a}\log(1/r) + \frac{r^{b/n}}{\rho^{a/n}}(\log(1/\varepsilon))^{1-1/n}(\log(1/\rho))^{1/n}\right).$$

We set

$$\varepsilon := r^\alpha (\log(1/r))^{-\frac{n-1}{bn}},$$

where

$$\alpha := \frac{a(n-1)}{bn} + \frac{1}{n} + 1 \geq 2,$$

since $a \geq b$. Then

$$
\begin{aligned}
\frac{\varepsilon^b}{r^a} \log(1/r) &\leq r^{(b-a)/n} (\log(1/r))^{1/n} \\
(5.6) \qquad &\leq r^{(b-a)/n} (\log(1/\varepsilon))^{1-1/n} (\log(1/r))^{1/n} \\
&\leq C_3 \frac{r^{b/n}}{\rho^{a/n}} (\log(1/\varepsilon))^{1-1/n} (\log(1/\rho))^{1/n}
\end{aligned}
$$

(recall that $\rho \geq 2r$). We also have

$$(5.7) \qquad\qquad \frac{\log(1/\varepsilon)}{\log(r/(2\varepsilon))} \leq C_4$$

and

$$(5.8) \qquad\qquad \log(1/\varepsilon) \leq C_5 \log(1/r).$$

Combining (5.5)-(5.8) we arrive at the second inequality in (5.4). $\qquad\square$

Theorem 5.2 immediately gives the following result which slightly generalizes the main result from [17].

Corollary 5.3. *Let Ω be as in Theorem 5.2. Then for every compact subset K of Ω we have*

$$\lim_{w \to \partial\Omega} \sup_K |g_{\Omega,w}| = 0.$$

Remark. It remains an open problem if Corollary 5.3 holds for arbitrary bounded hyperconvex Ω. Note that then we know from [6] that

$$\lim_{w \to \partial\Omega} ||g_{\Omega,w}||_{L^p(\Omega)} = 0$$

for every $p < \infty$. $\qquad\square$

Theorem 5.2 can be improved and its proof simplified when Ω is convex.

Theorem 5.4. *Let Ω be a bounded convex domain in $\mathbb{C}^n$. For given $\zeta, w \in \Omega$ set $\rho := \delta_\Omega(\zeta)$, $r := \delta_\Omega(w)$. Then*

$$g_{\Omega,w}(\zeta) \geq \log \frac{|\rho - r|}{\rho + r}.$$

In particular,

$$\{g_{\Omega,w} \leq -1\} \subset \{\frac{e-1}{e+1} r \leq \delta_\Omega \leq \frac{e+1}{e-1} r\}.$$

Proof. By the Lempert theorem (see [19]) $g_{\Omega,w}(\zeta)$ is symmetric in w and ζ and thus we way assume that $\rho > r$. Let H be a real hyperplane in $\mathbb{C}^n$ with $H \cap \Omega = \emptyset$ and $\delta_\Omega(w) = \text{dist}(w, H)$. After an orthonormal change of variables we may assume that $H = \{Re\, z_1 = 0\}$, $\Omega \subset \{Re\, z_1 > 0\}$, $w = (r, 0, \ldots, 0)$ and $\widetilde{\rho} := \text{dist}(\zeta, H) = Re\, \zeta_1 \geq \rho$. Then

$$g_{\Omega,w}(\zeta) \geq \log \frac{|\zeta_1 - r|}{|\zeta_1 + r|} \geq \log \frac{\widetilde{\rho} - r}{\widetilde{\rho} + r} \geq \log \frac{\rho - r}{\rho + r}. \qquad\qquad \square$$

Remark. Actually, one can avoid the use of the Lempert theorem in the proof of Theorem 5.4. Namely, for $\rho < r$ one has to repeat the same argument but with the hyperplane $\widetilde{H}$ such that $\widetilde{H} \cap \Omega = \emptyset$ and $\delta_\Omega(\zeta) = \mathrm{dist}(\zeta, \widetilde{H})$. $\square$

Acknowledgements

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