

## SOME ESTIMATES FOR THE BERGMAN KERNEL AND METRIC IN TERMS OF LOGARITHMIC CAPACITY

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**Abstract.** For a bounded domain  $\Omega$  on the plane we show the inequality  $c_\Omega(z)^2 \leq 2\pi K_\Omega(z)$ ,  $z \in \Omega$ , where  $c_\Omega(z)$  is the logarithmic capacity of the complement  $\mathbb{C} \setminus \Omega$  with respect to  $z$  and  $K_\Omega$  is the Bergman kernel. We thus improve a constant in an estimate due to T. Ohsawa but fall short of the inequality  $c_\Omega(z)^2 \leq \pi K_\Omega(z)$  conjectured by N. Suita. The main tool we use is a comparison, due to B. Berndtsson, of the kernels for the weighted complex Laplacian and the Green function. We also show a similar estimate for the Bergman metric and analogous results in several variables.

### §1. Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{C}$ . Suita [S] conjectured that

$$(1.1) \quad c_\Omega(z)^2 \leq \pi K_\Omega(z), \quad z \in \Omega,$$

where

$$K_\Omega(z) = \sup \left\{ \frac{|f(z)|^2}{\int_\Omega |f|^2} : f \text{ holomorphic in } \Omega, f \not\equiv 0 \right\}$$

is the Bergman kernel and  $c_\Omega(z)$  the logarithmic capacity of the complement  $\mathbb{C} \setminus \Omega$  with respect to  $z$ , that is

$$c_\Omega(z) = \exp \lim_{\zeta \rightarrow z} (G_\Omega(\zeta, z) - \log |\zeta - z|),$$

where  $G_\Omega$  is the (negative) Green function. If true, this estimate would be optimal, since for simply connected  $\Omega$  we have equality in (1.1). Ohsawa [O1], using the methods of the  $\bar{\partial}$ -equation, showed that

$$(1.2) \quad c_\Omega^2 \leq 750\pi K_\Omega.$$

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In fact, as noticed in [O1] and explored in [O2], the Suita conjecture seems to be closely related to the Ohsawa-Takegoshi theorem on extension of  $L^2$  holomorphic functions [OT]. In [O2] Ohsawa proved a general result which covered in particular the extension theorem, as well as the estimate (1.2) (with the constant  $2^8\pi$ ). Berndtsson [B3], using the methods of his proof of the Ohsawa-Takegoshi theorem from [B2] improved the constant in the Ohsawa estimate to  $6\pi$ . The author has also recently found the paper by B.-Y. Chen [C], where he shows the estimate with the constant  $\alpha\pi$ , where  $\alpha = 2(1 + \sqrt{5})e^{a+1-\sqrt{5}}$  and  $a$  is the solution of  $a + \log a = 0$  (then  $\alpha \approx 3.3155$ ).

One of the goals of this note is to show an estimate (see (2.4) below) from which it follows in particular that

$$(1.3) \quad c_\Omega^2 \leq 2\pi K_\Omega.$$

We do not aim at merely improving the constant in the Ohsawa estimate but also to present a slightly modified approach to the problem, where we more or less precisely construct a holomorphic function in  $\Omega$  with specified value at a given point and an appropriate bound for the  $L^2$  norm. We will use the kernel for the weighted complex Laplacian and the main tool will be a bound for this kernel in terms of the Green function due to Berndtsson [B1].

Our method yields also the following inequality for the Bergman metric

$$(1.4) \quad c_\Omega^4 \leq \pi K_\Omega B_\Omega,$$

where

$$B_\Omega = \frac{\partial^2}{\partial z \partial \bar{z}} \log K_\Omega.$$

The methods we use can be also applied in the same way for arbitrary Riemann surface which admits a Green function and estimates (1.3) and (1.4) are also valid. We also show that from one-dimensional case and the extension theorem of Ohsawa-Takegoshi [OT] one can easily deduce corresponding estimates in several complex variables.

## §2. Proofs of one-dimensional estimates

Without loss of generality we may assume that  $\Omega$  is smooth and bounded in  $\mathbb{C}$ , and  $0 \in \Omega$ . We will always denote  $G = G_\Omega(\cdot, 0)$ . We also use the notation (slightly different than the one from several variables)

$$\partial\alpha = \frac{\partial\alpha}{\partial z}, \quad \bar{\partial}\alpha = \frac{\partial\alpha}{\partial \bar{z}}.$$

If  $\varphi$  is smooth in  $\bar{\Omega}$  then the adjoint to  $\bar{\partial}$  with respect to the scalar product in  $L^2(\Omega, e^{-\varphi})$  is given by

$$\bar{\partial}^* \alpha = -e^\varphi \partial(e^{-\varphi} \alpha) = -\partial \alpha + \alpha \partial \varphi.$$

The complex Laplacian in  $L^2(\Omega, e^{-\varphi})$  is defined by

$$\square \alpha = -\bar{\partial} \bar{\partial}^* \alpha = \partial \bar{\partial} \alpha - \partial \varphi \bar{\partial} \alpha - \alpha \partial \bar{\partial} \varphi.$$

The basic relation to the standard Laplacian is given by the following formula of Berndtsson [B1]:

$$(2.1) \quad \partial \bar{\partial}(|\alpha|^2 e^{-\varphi}) = \left( 2 \operatorname{Re}(\bar{\alpha} \square \alpha) + |\bar{\partial} \alpha|^2 + |\bar{\partial}^* \alpha|^2 + |\alpha|^2 \partial \bar{\partial} \varphi \right) e^{-\varphi}.$$

If  $\varphi$  is subharmonic then in particular we can find  $N \in C^\infty(\bar{\Omega} \setminus \{0\}) \cap L^1(\Omega)$  such that

$$\square N = \frac{\pi}{2} e^{\varphi(0)} \delta_0, \quad N = 0 \quad \text{on} \quad \partial \Omega.$$

(The constant  $\pi/2$  is chosen so that  $N = G$  if  $\varphi \equiv 0$ .) The estimate of Berndtsson [B1] asserts that

$$(2.2) \quad |N|^2 \leq e^{\varphi + \varphi(0)} G^2.$$

*Remark.* Berndtsson in [B1] shows using (2.1) that for any  $C^2$  smooth  $\alpha$  and  $\varepsilon > 0$  one has

$$\partial \bar{\partial}(|\alpha|^2 e^{-\varphi} + \varepsilon)^{1/2} \geq -|\square \alpha| e^{-\varphi/2}.$$

Now by approximation one can easily deduce that

$$\partial \bar{\partial} \left( -|N| e^{-(\varphi + \varphi(0))/2} \right) \leq \frac{\pi}{2} \delta_0 = \partial \bar{\partial} G$$

from which (2.2) immediately follows. In a particular case when  $\Omega$  is simply connected and  $\varphi$  harmonic we have

$$N = e^{g + \overline{g(0)}} G,$$

where  $g$  is a holomorphic function in  $\Omega$  such that  $\operatorname{Re} g = \varphi/2$ . Therefore, in this case we have equality in (2.2).

As in [B3] we shall use the weight

$$(2.3) \quad \varphi := 2(\log |z| - G).$$

Note that  $\varphi$  is harmonic in  $\Omega$ , smooth on  $\bar{\Omega}$  and

$$e^{-\varphi(0)} = c_{\Omega}(0)^2.$$

For harmonic weights the operators  $\bar{\partial}$  and its adjoint commute

$$\square = -\bar{\partial}\bar{\partial}^* = -\bar{\partial}^*\bar{\partial}.$$

Therefore

$$\bar{\partial}(e^{-\varphi}\bar{\partial}\bar{N}) = \bar{\partial}(-e^{-\varphi(0)}\bar{\partial}^*N) = \frac{\pi}{2}\delta_0.$$

It follows that the functions

$$f := ze^{-\varphi}\bar{\partial}\bar{N}, \quad g := -ze^{-\varphi(0)}\bar{\partial}^*N$$

are holomorphic in  $\Omega$ , smooth on  $\bar{\Omega}$ , and  $f(0) = g(0) = 1/2$ .

Using the fact that both the function  $|N|^2e^{-\varphi}$  and its derivative vanish on  $\partial\Omega$ , integration by parts and (2.1) give

$$\begin{aligned} \int_{\Omega} |N|^2e^{-\varphi}\bar{\partial}\bar{\partial}(|z|^2e^{-\varphi}) &= \int_{\Omega} |z|^2(|\bar{\partial}N|^2 + |\bar{\partial}^*N|^2)e^{-2\varphi} \\ &= \int_{\Omega} |f|^2 + e^{2\varphi(0)} \int_{\Omega} |g|^2e^{-2\varphi}. \end{aligned}$$

On the other hand, we have  $|z|^2e^{-\varphi} = e^{2G}$  and by (2.2)

$$\int_{\Omega} |N|^2e^{-\varphi}\bar{\partial}\bar{\partial}(|z|^2e^{-\varphi}) \leq e^{\varphi(0)} \int_{\Omega} G^2\bar{\partial}\bar{\partial}e^{2G}.$$

We need the following simple lemma.

LEMMA. *For every summable  $\gamma : (-\infty, 0) \rightarrow \mathbb{R}$  we have*

$$\int_{\Omega} \gamma \circ G |\nabla G|^2 = 2\pi \int_{-\infty}^0 \gamma(t) dt.$$

*Proof.* Let  $\chi : (-\infty, 0) \rightarrow \mathbb{R}$  be such that  $\chi' = \gamma$  and  $\chi(-\infty) = 0$ . Then

$$\int_{\Omega} \gamma \circ G |\nabla G|^2 = \int_{\Omega} \langle \nabla(\chi \circ G), \nabla G \rangle = \int_{\partial\Omega} \chi(0) \frac{\partial G}{\partial n} = 2\pi\chi(0). \quad \square$$

It follows that

$$\int_{\Omega} G^2 \partial \bar{\partial} e^{2G} = \int_{\Omega} G^2 e^{2G} |\nabla G|^2 = \frac{\pi}{2}$$

and

$$\int_{\Omega} |f|^2 + e^{2\varphi(0)} \int_{\Omega} |g|^2 e^{-2\varphi} \leq \frac{\pi}{2} e^{\varphi(0)}.$$

We conclude that

$$(2.4) \quad \frac{1}{K_{\Omega}(0)} + \frac{1}{c_{\Omega}(0)^4 K_{\Omega}^{2\varphi}(0)} \leq \frac{2\pi}{c_{\Omega}(0)^2},$$

where  $\varphi$  is given by (2.3) and

$$K_{\Omega}^{2\varphi}(z) = \sup \left\{ \frac{|f(z)|^2}{\int_{\Omega} |f|^2 e^{-2\varphi}} : f \text{ holomorphic in } \Omega, f \not\equiv 0 \right\}$$

is the weighted Bergman kernel. In particular, we get (1.3).

To show (1.4) we first recall a well known formula for the Bergman metric

$$B_{\Omega}(z) = \frac{1}{K_{\Omega}(z)} \sup \left\{ \frac{|f'(z)|^2}{\int_{\Omega} |f|^2} : f \text{ holomorphic in } \Omega, f(z) = 0, f \not\equiv 0 \right\}.$$

We now proceed the same way as before, we only choose the weight

$$\varphi := 4(\log |z| - G),$$

so that

$$e^{-\varphi(0)} = c_{\Omega}(0)^4,$$

and the functions

$$f := z^2 e^{-\varphi} \partial \bar{N}, \quad g := -z^2 e^{-\varphi(0)} \bar{\partial}^* N,$$

so that they are holomorphic in  $\Omega$ ,  $f(0) = g(0) = 0$ , and  $f'(0) = g'(0) = 1/2$ .

We will get

$$\int_{\Omega} |f|^2 + e^{2\varphi(0)} \int_{\Omega} |g|^2 e^{-2\varphi} \leq e^{\varphi(0)} \int_{\Omega} G^2 \partial \bar{\partial} e^{4G} = \frac{\pi}{4} e^{\varphi(0)}$$

and (1.4) follows.

*Remark.* Similarly as for the Suita conjecture, it would be interesting to improve the constant in (1.4) to  $\pi/2$  – it would then be optimal. It is also interesting whether a reverse to the Ohsawa estimate

$$K_\Omega \leq Cc_\Omega^2$$

holds for some constant  $C$ . This would have far reaching consequences: for example it would give another potential theoretic characterization of Bergman exhaustive domains (compare [Z2]). It would also provide a quantitative version of the following well known result (see e.g. [Co]): an (unbounded) domain  $\Omega$  in  $\mathbb{C}$  contains a non-vanishing square-integrable holomorphic function if and only if  $\mathbb{C} \setminus \Omega$  is not polar. Another consequence would be an estimate

$$c_\Omega^2 \leq CB_\Omega,$$

and, in higher dimensions, an estimate from below of the Bergman metric in terms of the Azukawa metric (see below).

### §3. Several dimensional analogues

Let now  $\Omega$  be a bounded domain in  $\mathbb{C}^n$ . The pluricomplex Green function is defined by

$$G_\Omega(\cdot, w) = \sup \left\{ u \in PSH(\Omega) : u < 0, \overline{\lim}_{z \rightarrow w} (u(\zeta) - \log |z - w|) < \infty \right\}, \quad w \in \Omega.$$

The multidimensional logarithmic capacity can be defined as

$$c_\Omega(w) = \exp \overline{\lim}_{z \rightarrow w} (G_\Omega(z, w) - \log |z - w|), \quad w \in \mathbb{C}^n.$$

The Azukawa metric can be regarded as a directional logarithmic capacity

$$A_\Omega(w; X) = \exp \overline{\lim}_{\lambda \rightarrow 0} (G_\Omega(w + \lambda X, w) - \log |\lambda|), \quad w \in \Omega, X \in \mathbb{C}^n.$$

For basic properties of  $A_\Omega$  we refer to [Z1]. Recall also that the Bergman metric is defined as

$$B_\Omega(z; X) = \frac{\partial^2}{\partial \lambda \partial \bar{\lambda}} \log K_\Omega(z + \lambda X) \Big|_{\lambda=0}, \quad z \in \Omega, X \in \mathbb{C}^n.$$

We are now in position to formulate multidimensional analogues of the previous estimates.

THEOREM. For a bounded pseudoconvex domain  $\Omega$  in  $\mathbb{C}^n$  we have

$$c_\Omega(w)^2 \leq CK_\Omega(w), \quad A_\Omega(w; X)^4 \leq C'K_\Omega(w)B_\Omega(w; X), \quad w \in \Omega, X \in \mathbb{C}^n,$$

where  $C, C'$  depend only on  $n$  and the diameter of  $\Omega$ .

*Proof.* Since all the considered functions behave well under approximation, without loss of generality we may assume that  $\Omega$  is sufficiently regular, even smooth, then  $G_\Omega(\cdot, w)$  is continuous on  $\overline{\Omega} \setminus \{w\}$ . By [Z1]  $A_\Omega$  is continuous (as a function on  $\Omega \times \mathbb{C}^n$ ),  $\overline{\lim}$  in the definition of  $A_\Omega$  can be replaced with  $\lim$ , and

$$c_\Omega(w) = A_\Omega(w; X)$$

for some  $X \in \mathbb{C}^n$ . Let

$$D := \{\lambda \in \mathbb{C} : w + \lambda X \in \Omega\}.$$

Then

$$G_\Omega(w + \lambda X, w) \leq G_D(\lambda, 0), \quad \lambda \in D,$$

and thus

$$A_\Omega(w; X) \leq c_D(0).$$

On the other hand, by the Ohsawa-Takegoshi extension theorem [OT] (see also [B2]),

$$K_D(\lambda) \leq CK_\Omega(w + \lambda X), \quad \lambda \in D,$$

and the first inequality follows from the one-dimensional Ohsawa estimate.

The proof of the second inequality is similar. We have

$$\begin{aligned} & B_\Omega(w; X) \\ &= \frac{1}{K_\Omega(w)} \sup \left\{ \frac{|D_X f(w)|^2}{\int_\Omega |f|^2} : f \text{ holomorphic in } \Omega, f(w) = 0, f \not\equiv 0 \right\}, \end{aligned}$$

where

$$D_X f(w) = \sum_{j=1}^n X_j \frac{\partial f}{\partial z_j}(w).$$

Therefore, by the Ohsawa-Takegoshi theorem,

$$K_D(\lambda)B_D(\lambda) \leq CK_\Omega(w + \lambda X)B_\Omega(w + \lambda X; X), \quad \lambda \in D,$$

and it is enough to use (1.4).  $\square$

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