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On the Suita Conjecture for Some Convex Ellipsoids in $\mathbb{C}^2$

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CONTENTS

It was recently shown that for a convex domain Ω in $\mathbb{C}^n$ and $w \in \Omega$, the function $F_\Omega(w) := \left(K_\Omega(w)\lambda(I_\Omega(w))\right)^{1/n}$, where K_Ω is the Bergman kernel on the diagonal and $I_\Omega(w)$ the Kobayashi indicatrix, satisfies $1 \le F_\Omega \le 4$. While the lower bound is optimal, not much more is known about the upper bound. In general, it is quite difficult to compute F_Ω even numerically, and the largest value of it obtained so far is $1.010182\ldots$. In this article, we present precise, although rather complicated, formulas for the ellipsoids $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ (with $m \ge 1/2$) and all w, as well as for $\Omega = \{|z_1| + |z_2| < 1\}$ and w on the diagonal. The Bergman kernel for those ellipsoids was already known; the main point is to compute the volume of the Kobayashi indicatrix. It turns out that in the second case, the function $\lambda(I_\Omega(w))$ is not $C^{3,1}$.

1. INTRODUCTION

For a convex domain Ω in $\mathbb{C}^n$ and $w \in \Omega$, the following estimates were recently established:

$$\frac{1}{\lambda(I_\Omega(w))} \le K_\Omega(w) \le \frac{4^n}{\lambda(I_\Omega(w))}. \qquad (1–1)$$

Here

$$K_\Omega(w) = \sup\left\{|f(w)|^2 \colon f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2\, d\lambda \le 1\right\}$$

is the Bergman kernel on the diagonal, and

$$I_\Omega(w) = \left\{\varphi'(0) \colon \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\right\}$$

is the Kobayashi indicatrix, where Δ denotes the unit disk. The first inequality in (1–1) was proved in [Błocki 15], using L^2-estimates for $\bar\partial$ and Lempert's theory [Lempert 81]. It is optimal, for example, if Ω is balanced with respect to w (that is, every intersection of Ω with a complex line containing w is a disk). Then we have equality. It can be viewed as a multidimensional version of the Suita conjecture [Suita 72] proved in [Błocki 13] (see also [Guan and Zhou 15] for the precise characterization when equality holds).

The second equality in (1–1) was proved in [Blocki and Zwonek 15] using rather elementary methods. It was also

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shown that the constant 4 can be replaced by $16/\pi^2 = 1.6211\ldots$ if Ω is, in addition, symmetric with respect to w. We can write (1–1) as

$$1 \le F_\Omega(w) \le 4,$$

where $F_\Omega(w) := \left(K_\Omega(w)\lambda(I_\Omega(w))\right)^{1/n}$ is a biholomorphically invariant function in Ω. It is not clear what the optimal upper bound should be. It was, in fact, quite difficult to prove that one can have $F_\Omega > 1$ at all. That was done in [Blocki and Zwonek 15] for ellipsoids of the form $\{|z_1| + |z_2|^{2m} + \cdots + |z_n|^{2m} < 1\}$, where $m \ge 1/2$ and $w = (b, 0, \ldots, 0)$. The function F_Ω was also computed numerically for the ellipsoid $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$, $m \ge 1/2$, based on an implicit formula for the Kobayashi function from [Blank et al. 92]. Our first result is the precise formula in this case:

Theorem 1.1. *For $m \ge 1/2$, define*

$$\Omega_m = \left\{z \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 < 1\right\}.$$

Then for $m \ne 2/3$, $m \ne 2$, and b with $0 \le b < 1$, we have

$$\begin{aligned}\lambda(I_{\Omega_m}((b,0))) &= \pi^2 \Bigg[-\frac{m-1}{2m(3m-2)(3m-1)} b^{6m+2} \\ &\quad - \frac{3(m-1)}{2m(m-2)(m+1)} b^{2m+2} \\ &\quad + \frac{m}{2(m-2)(3m-2)} b^6 + \frac{3m}{3m-1} b^4 \\ &\quad - \frac{4m-1}{2m} b^2 + \frac{m}{m+1}\Bigg].\end{aligned}$$

For $m = 2/3$ and $m = 2$, we have

$$\lambda(I_{\Omega_{2/3}}((b,0))) = \frac{\pi^2}{80}\left(-65b^6 + 40b^6 \log b + 160b^4 - 27b^{10/3} - 100b^2 + 32\right)$$

and

$$\lambda(I_{\Omega_2}((b,0))) = \frac{\pi^2}{240}\left(-3b^{14} - 25b^6 - 120b^6 \log b + 288b^4 - 420b^2 + 160\right).$$

The general formula for the Kobayashi function for Ω_m is known, see [Blank et al. 92], but it is implicit in the sense that it requires solving a nonlinear equation that is polynomial of degree $2m$ if $2m$ is an integer. It turns out, however, that the volume of the Kobayashi indicatrix for Ω_m, that is, the set on which the Kobayashi function is at most 1, can be found explicitly. It would be interesting to check whether Theorem 1.1 also holds in the nonconvex case, that is, when $0 < m < 1/2$ (see [Pflug and Zwonek 96] for computations of the Kobayashi metric in this case).

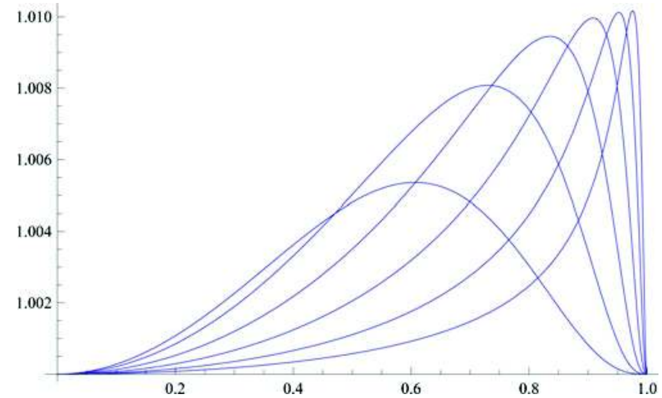

FIGURE 1. $F_{\Omega_m}((b,0))$ for $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ and $m = 4, 8, 16, 32, 64, 128$.

The formula for the Bergman kernel for this ellipsoid is well known (see, e.g., [Jarnicki and Pflug 13, Example 6.1.6]),

$$K_{\Omega_m}(w) = \frac{1}{\pi^2}\left(1 - |w_2|^2\right)^{1/m-2} \times \frac{(1/m+1)(1-|w_2|^2)^{1/m} + (1/m-1)|w_1|^2}{\left((1-|w_2|^2)^{1/m} - |w_1|^2\right)^3},$$

so that

$$K_{\Omega_m}((b,0)) = \frac{m+1+(1-m)b^2}{\pi^2 m(1-b^2)^3}.$$

The graphs of $F_{\Omega_m}((b,0))$ in Figure 1 are consistent with the graphs from [Blocki and Zwonek 15] obtained numerically using the implicit formula from [Blank et al. 92]. Note that for $t \in \mathbb{R}$ and $a \in \Delta$, the mapping

$$\Omega_m \ni z \mapsto \left(e^{it}\frac{(1-|a|^2)^{1/2m}}{(1-\bar{a}z_2)^{1/m}} z_1, \frac{z_2 - a}{1 - \bar{a}z_2}\right)$$

is a holomorphic automorphism of Ω_m, and therefore $F_{\Omega_m}((b,0))$, where $0 \le b < 1$, attains all values of F_{Ω_m} in Ω_m. One can show numerically that

$$\sup_{m \ge 1/2} \sup_{\Omega_m} F_{\Omega_m} = 1.010182\ldots,$$

which was already noticed in [Blocki and Zwonek 15]. This is the highest value of F_Ω (in arbitrary dimension) obtained so far.

In [Blocki and Zwonek 15], it was also shown that for $\Omega = \{|z_1| + |z_2| < 1\}$ and b with $0 < b < 1$, one has

$$\lambda(I_\Omega((b,0))) = \frac{\pi^2}{6}(1-b)^4\left((1-b)^4 + 8b\right),$$

so that in particular, similarly as in Theorem 1.1, it is an analytic function on this part of Ω. This raises the question whether $\lambda(I_\Omega(w))$ is smooth in general. In [Blocki and Zwonek 15], it was also predicted that the highest value of F_Ω for convex Ω in $\mathbb{C}^2$ should be attained for $\Omega = \{|z_1| + |z_2| < 1\}$

$$
\begin{aligned}
\lambda(I_\Omega((b,b))) &= \frac{2\pi^2 b(1-2b)^3\left(-2b^3+3b^2-6b+4\right)}{3(1-b)^2} \\
&+ \frac{\pi\left(30b^{10}-124b^9+238b^8-176b^7-260b^6+424b^5-76b^4-144b^3+89b^2-18b+1\right)}{6(1-b)^2}\arccos\left(-1+\frac{4b-1}{2b^2}\right) \\
&+ \frac{\pi(1-2b)\left(-180b^7+444b^6-554b^5+754b^4-1214b^3+922b^2-305b+37\right)}{72(1-b)}\sqrt{4b-1} \\
&+ \frac{4\pi b(1-2b)^4\left(7b^2+2b-2\right)}{3(1-b)^2}\arctan\sqrt{4b-1} \\
&+ \frac{4\pi b^2(1-2b)^4(2-b)}{(1-b)^2}\arctan\frac{1-3b}{(1-b)\sqrt{4b-1}}.
\end{aligned}
$$

FIGURE 2. $\lambda(I_\Omega((b,b)))$ when $1/4 \le b < 1/2$.

on the diagonal. The following result will answer both of these questions in the negative.

Theorem 1.2. *Let* $\Omega = \{z \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$. *Then for* b *with* $0 \le b \le 1/4$*, we have*

$$
\lambda(I_\Omega((b,b))) = \frac{\pi^2}{6}\left(30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1\right), \qquad (1\text{–}2)
$$

and when $1/4 \le b < 1/2$, $\lambda(I_\Omega((b,b)))$ *is as given in Figure 2 (where it was placed because the columns of this journal are too small to contain it).*

The function

$$
b \mapsto \lambda(I_\Omega((b,b)))
$$

is C^3 *on the interval* $(0, 1/2)$ *but not* $C^{3,1}$ *at* $1/4$.

Again, the formula for the Bergman metric for this ellipsoid is known; see [Hahn and Pflug 88] or [Jarnicki and Pflug 13, Example 6.1.9]:

$$
K_\Omega(w) = \frac{2}{\pi^2}\frac{3\left(1-|w|^2\right)^2\left(1+|w|^2\right) + 4|w_1|^2|w_2|^2\left(5-3|w|^2\right)}{\left(\left(1-|w|^2\right)^2 - 4|w_1|^2|w_2|^2\right)^3},
$$

so that

$$
K_\Omega((b,b)) = \frac{2\left(3-6b^2+8b^4\right)}{\pi^2\left(1-4b^2\right)^3}. \qquad (1\text{–}3)
$$

The first part of Theorem 1.2, formula (1–2) on the interval $(0, 1/4)$, is easier to prove than the second. Combining it with (1–3), one can obtain the graph of $F_\Omega((b,b))$ for $b \in (0, 1/4)$. It is shown in Figure 3.

One can show that its analytic continuation to $(0, 1/2)$ attains values below 1, and thus it follows already from (1–1) that F_Ω cannot be analytic; see Figure 4. To conclude that it is in fact not $C^{3,1}$, one has to prove the much harder formula of Figure 2. One can check that the maximal value of $F_\Omega((b,b))$ for $b \in (0, 1/2)$ is $1.008902\ldots$.

All pictures and numerical computations in this paper, as well as many of the symbolic computations used in the proofs of Theorems 1.1 and 1.2, were done using Mathematica.

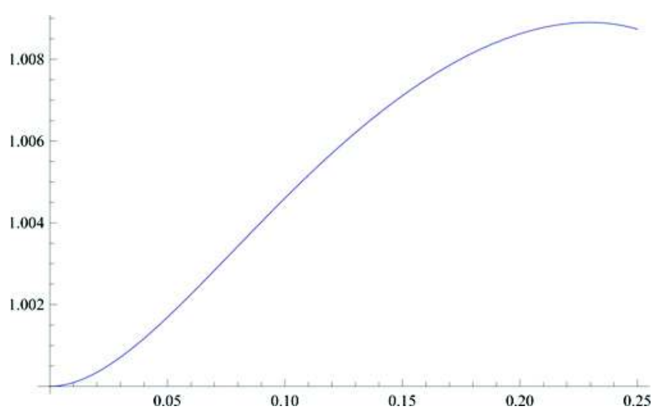

FIGURE 3. $F_\Omega((b,b))$ for $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in (0, 1/4)$.

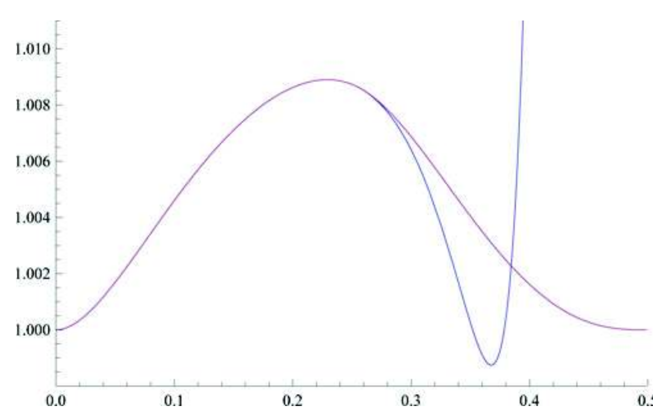

FIGURE 4. The continuation of $F_\Omega((b,b))$ from Figure 2 to (0,1/2) and the actual graph there.

2. GENERAL FORMULA FOR GEODESICS IN CONVEX COMPLEX ELLIPSOIDS

The boundary of the Kobayashi indicatrix of a convex domain Ω at w consists of the vectors $\varphi'(0)$, where $\varphi \in \mathcal{O}(\Delta, \Omega)$ is a geodesic of Ω satisfying $\varphi(0) = w$. Theorems 1.1 and 1.2 will be proved using a general formula for geodesics in convex complex ellipsoids from [Jarnicki et al. 93] based on Lempert's theory [Lempert 81] describing geodesics of smooth strongly convex domains.

For $p = (p_1, \ldots, p_n)$ with $p_j \geq 1/2$, set

$$\mathcal{E}(p) = \left\{z \in \mathbb{C}^n : |z_1|^{2p_1} + \cdots + |z_n|^{2p_n} < 1\right\},$$

and for $A \subset \{1, \ldots, n\}$, define

$$\varphi_j(\zeta) = \begin{cases} a_j \dfrac{\zeta - \alpha_j}{1 - \bar{\alpha}_j \zeta} \left(\dfrac{1 - \bar{\alpha}_j \zeta}{1 - \bar{\alpha}_0 \zeta}\right)^{1/p_j}, & j \in A, \\ a_j \left(\dfrac{1 - \bar{\alpha}_j \zeta}{1 - \bar{\alpha}_0 \zeta}\right)^{1/p_j}, & j \notin A, \end{cases}$$

where $a_j \in \mathbb{C}_*$, $\alpha_0, \alpha_j \in \Delta$ for $j \in A$, $\alpha_j \in \bar{\Delta}$ for $j \notin A$,

$$\alpha_0 = |a_1|^{2p_1} \alpha_1 + \cdots + |a_n|^{2p_n} \alpha_n, \tag{2–1}$$

and

$$1 + |\alpha_0|^2 = |a_1|^{2p_1} \left(1 + |\alpha_1|^2\right) + \cdots + |a_n|^{2p_n} \left(1 + |\alpha_n|^2\right). \tag{2–2}$$

A component φ_j has a zero in Δ if and only if $j \in A$. We have

$$\varphi_j(0) = \begin{cases} -a_j \alpha_j, & j \in A, \\ a_j, & j \notin A, \end{cases} \tag{2–3}$$

and

$$\varphi_j'(0) = \begin{cases} a_j \left(1 + \left(\frac{1}{p_j} - 1\right) |\alpha_j|^2 - \frac{\alpha_j \bar{\alpha}_0}{p_j}\right), & j \in A, \\ a_j \frac{\bar{\alpha}_0 - \bar{\alpha}_j}{p_j}, & j \notin A. \end{cases} \tag{2–4}$$

For $w \in \mathcal{E}(p)$, the set of vectors $\varphi'(0)$ where $\varphi(0) = w$ forms a subset of $\partial I^K_{\mathcal{E}(p)}(w)$ of full measure. The geodesics in $\mathcal{E}(p)$ are uniquely determined: for a given $w \in \mathcal{E}(p)$ and $X \in (\mathbb{C}^n)_*$, there exists a unique geodesic $\varphi \in \mathcal{O}(\Delta, \mathcal{E}(p))$ such that $\varphi(0) = w$ and $\varphi'(0) = X$.

3. PROOF OF THEOREM 1.1

First note that the formulas for $m = 2/3$ and $m = 2$ follow easily from the first one by approximation. For $\Omega_m = \mathcal{E}(m, 1)$ and $w = (b, 0)$, there are two possibilities for a geodesic φ: either φ crosses the axis $\{z_1 = 0\}$ or it does not. Let I_{12} and I_2 denote the respective parts of $I_{\Omega_m}(w)$. In the first case, φ must be of the form

$$\varphi(\zeta) = \left(a_1 \frac{\zeta - \alpha_1}{1 - \bar{\alpha}_1 \zeta} \left(\frac{1 - \bar{\alpha}_1 \zeta}{1 - \bar{\alpha}_0 \zeta}\right)^{1/m}, a_2 \frac{\zeta - \alpha_2}{1 - \bar{\alpha}_0 \zeta}\right),$$

where $a_1, a_2 \in \mathbb{C}_*$ and $\alpha_0, \alpha_1, \alpha_2 \in \Delta$ satisfy (2–1) and (2–2). By (2–3) and since $\varphi(0) = (b, 0)$, we have $a_1 = -b/\alpha_1$, $\alpha_2 = 0$, and by (2–1), $\alpha_0 = b^{2m} \alpha_1 / |\alpha_1|^{2m}$. By (2–2), we have

$$1 + b^{4m} |\alpha_1|^{2-4m} = b^{2m} |\alpha_1|^{-2m} \left(1 + |\alpha_1|^2\right) + |a_2|^2,$$

that is,

$$|a_2|^2 = \left(1 - b^{2m} |\alpha_1|^{-2m}\right) \left(1 - b^{2m} |\alpha_1|^{2-2m}\right). \tag{3–1}$$

Since $\alpha_0, \alpha_1 \in \Delta_*$, it follows that $b < |\alpha_1| < 1$. Write $\alpha_1 = -re^{-it}$, $a_2 = \rho e^{is}$. Then by (2–4) and (3–1), with $b < r < 1$, we have

$$\varphi'(0) = \left(\left(\frac{b}{r} + b\left(\frac{1}{m} - 1\right) r - \frac{b^{2m+1} r^{1-2m}}{m}\right) e^{it}, \sqrt{(1 - b^{2m} r^{-2m})(1 - b^{2m} r^{2-2m})} e^{is}\right) =: \left(\gamma_1(r) e^{it}, \gamma_2(r) e^{is}\right).$$

The mapping

$$\Delta \times [0, 2\pi) \times (b, 1) \ni (\zeta, t, r) \mapsto \zeta \left(\gamma_1(r) e^{it}, \gamma_2(r)\right) \tag{3–2}$$

parameterizes I_{12}. We will need a lemma.

Lemma 3.1. *Let $F(\zeta, z) = \zeta(f(z), g(z))$ be a function of two complex variables, where f and g are C^1. Then the real Jacobian of F is equal to $|\zeta|^2 H(z)$, where*

$$H = |f|^2(|g_{\bar{z}}|^2 - |g_z|^2) + |g|^2(|f_{\bar{z}}|^2 - |f_z|^2) + 2\,\mathrm{Re}\left(f \bar{g}(\overline{f_z} g_z - \overline{f_{\bar{z}}} g_{\bar{z}})\right).$$

The proof is left to the reader. For the mapping (3–2), we can compute that

$$\begin{aligned} H &= \gamma_1 \gamma_2 (\gamma_1 \gamma_2' - \gamma_1' \gamma_2) \\ &= -\frac{b^2}{m^2} r^{-6m-3} \left[b^{2m}\left(-mr^2 + m - 1\right) + r^{2m}\right] \\ &\quad \times \left[r^{2m}\left((m-1)r^2 + m\right) - (2m-1) r^2 b^{2m}\right] \\ &\quad \times \left[r^2 b^{2m} + r^{2m}\left((m-1)r^2 - m\right)\right]. \end{aligned}$$

Since

$$\int_\Delta |\zeta|^2 d\lambda(\zeta) = \frac{\pi}{2}, \tag{3–3}$$

we obtain

$$\begin{aligned}\lambda(I_{12}) &= \pi^2\int_b^1 |H|\,dr \qquad (3\text{–}4)\\ &= \pi^2\Big(\frac{(1-2m)^2}{m^2(3m-1)(3m-2)}b^{6m+2}\\ &\quad - \frac{3}{m^2(m+1)(m-2)}b^{2m+2} - \frac{3}{2m^2}b^{4m+2}\\ &\quad + \frac{m}{2(m-2)(3m-2)}b^6 + \frac{3m}{3m-1}b^4\\ &\quad - \frac{4m^2-m+1}{2m^2}b^2 + \frac{m}{m+1}\Big).\end{aligned}$$

To compute the volume of I_2, we consider geodesics of the form

$$\varphi(\zeta) = \left(a_1\left(\frac{1-\bar\alpha_1\zeta}{1-\bar\alpha_0\zeta}\right)^{1/m}, a_2\frac{\zeta-\alpha_2}{1-\bar\alpha_0\zeta}\right),$$

where $a_1, a_2 \in \mathbb{C}_*$, $\alpha_0, \alpha_2 \in \Delta$, $\alpha_1 \in \bar\Delta$ satisfy (2–1), (2–2). By (2–3) and since $\varphi(0) = (b, 0)$, we have $a_1 = b$, $\alpha_2 = 0$, and by (2–1), we have $\alpha_0 = b^{2m}\alpha_1$. By (2–2), we have

$$1 + b^{4m}|\alpha_1|^2 = b^{2m}\left(1+|\alpha_1|^2\right) + |a_2|^2,$$

that is,

$$|a_2|^2 = \left(1-b^{2m}\right)\left(1-b^{2m}|\alpha_1|^2\right).$$

This means that every $\alpha_1 \in \Delta$ is allowed, and by (2–4),

$$\begin{aligned}\varphi'(0) &= \left(\frac{b(b^{2m}-1)}{m}\bar\alpha_1, a_2\right)\\ &= \left(\frac{b(1-b^{2m})r}{m}e^{it}, \sqrt{(1-b^{2m})(1-b^{2m}r^2)}e^{is}\right),\end{aligned}$$

where $\alpha_1 = -re^{-it}$, $a_2 = \rho e^{is}$. Similarly as before, we have

$$H = -\frac{b^2\left(1-b^{2m}\right)^3 r}{m^2}$$

and

$$\lambda(I_2) = \pi^2\int_0^1 |H|\,dr = \frac{\pi^2 b^2(1-b^{2m})^3}{2m^2}.$$

This combined with (3–4) finishes the proof of Theorem 1.1.

4. PROOF OF THEOREM 1.2

For $\Omega = \mathcal{E}(1/2, 1/2)$ and $w = (b, b)$, where $0 < b < 1/2$, we have by (2–3),

$$a_j = \begin{cases} -\frac{b}{\alpha_j}, & j \in A,\\ b, & j \notin A,\end{cases} \qquad (4\text{–}1)$$

and by (2–4),

$$\varphi_j'(0) = \begin{cases} 2b\bar\alpha_0 - b\left(\bar\alpha_j + \frac{1}{\alpha_j}\right), & j \in A,\\ 2b(\bar\alpha_0 - \bar\alpha_j), & j \notin A.\end{cases} \qquad (4\text{–}2)$$

There are four possibilities for the set A: $\emptyset$, $\{1\}$, $\{2\}$, and $\{1, 2\}$. Denote the corresponding parts of $I_\Omega(w)$ by I_0, I_1, I_2, and I_{12}, respectively, so that

$$\begin{aligned}\lambda(I_\Omega(w)) &= \lambda(I_0) + \lambda(I_1) + \lambda(I_2) + \lambda(I_{12}) \qquad (4\text{–}3)\\ &= \lambda(I_0) + 2\lambda(I_1) + \lambda(I_{12}).\end{aligned}$$

4.1. The Case $A = \{1, 2\}$

By (2–1), (2–2), and (4–1), we have

$$\begin{aligned}&\left(\frac{1}{b}+2b\right)|\alpha_1|\,|\alpha_2| + 2b\,\mathrm{Re}(\alpha_1\bar\alpha_2) \qquad (4\text{–}4)\\ &\quad = \left(1+|\alpha_1|^2\right)|\alpha_2| + \left(1+|\alpha_2|^2\right)|\alpha_1|.\end{aligned}$$

Since the set of $\alpha \in \Delta^2$ satisfying (4–4) is S^1-invariant, let us consider only those α with $\alpha_2 > 0$. If we then replace α_1 with $\bar\alpha_1$, then (4–4) will still be valid, and $\varphi'(0)$ will be replaced by $\overline{\varphi'(0)}$. We thus consider

$$\alpha_1 = re^{it} \quad \text{and} \quad \alpha_2 = \rho, \quad r, \rho \in (0, 1), \quad t \in (0, \pi).$$

To get $\lambda(I_{12})$, we will have to multiply the volume obtained by 2. The condition (4–4) transforms to

$$\frac{1}{b} + 2b(1+\cos t) = r + \frac{1}{r} + \rho + \frac{1}{\rho}. \qquad (4\text{–}5)$$

It will be convenient to substitute $x = r + 1/r$, $y = t$, and consider the domain

$$\begin{aligned}U := \Big\{&(x, y) \in \left(2, \frac{1}{b}+4b-2\right)\times(0,\pi) : \qquad (4\text{–}6)\\ &x < \frac{1}{b} + 2b(1+\cos y) - 2\Big\}.\end{aligned}$$

We have

$$\alpha_0 = b\left(\frac{\alpha_1}{|\alpha_1|} + \frac{\alpha_2}{|\alpha_2|}\right) = b(e^{it}+1),$$

and thus by (4–2) and (4–5),

$$\begin{aligned}\varphi'(0) &= b\left(2\bar\alpha_0 - \bar\alpha_1 - \frac{1}{\alpha_1}, 2\bar\alpha_0 - \bar\alpha_2 - \frac{1}{\alpha_2}\right) \qquad (4\text{–}7)\\ &= \Bigg(2b^2\left(e^{-it}+1\right) - b\left(r+\frac{1}{r}\right)e^{-it},\\ &\qquad 2b^2\left(e^{-it}+1\right) - b\left(\rho+\frac{1}{\rho}\right)\Bigg)\\ &= \left(2b^2 + b\,(2b-x)\,e^{-iy}, bx - 1 - 2b^2 i\sin y\right)\\ &=: (f(z), g(z)).\end{aligned}$$

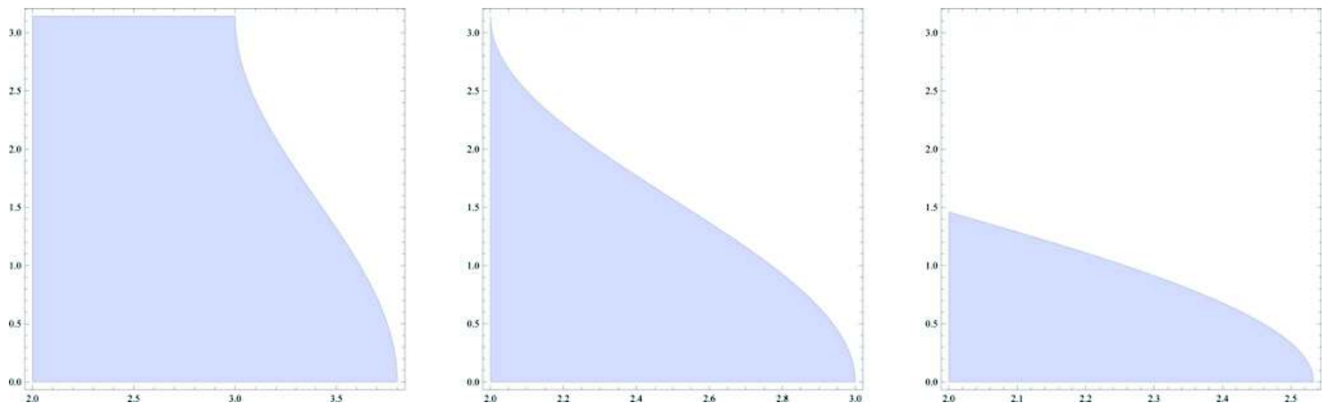

FIGURE 5. The region U for $b = 0.2$, $b = 0.25$, and $b = 0.3$.

The mapping

$$\Delta \times U \ni (\zeta, z) \mapsto \zeta(f(z), g(z))$$

parameterizes I_{12}. From Lemma 3.1 and (3–3), it follows that

$$\lambda(I_{12}) = \pi \iint_U |H|\, d\lambda,$$

where f, g are given by (4–7), U by (4–6) (recall that again we had to multiply by 2), and we can compute that

$$\begin{aligned} H = b^2\left[1 - 2b^2(\cos y + 1)\right] \\ \times \left[-bx^2 + (1 + 2b^2(\cos y + 1))(x - 2b)\right. \\ \left. - 2b(b^2\cos(2y) + 1)\right]. \end{aligned}$$

One can check that $H > 0$ in U. The region U may look like that shown in Figure 5.

We set

$$y_0 := \begin{cases} \pi & b \le 1/4, \\ \arccos\left(-1 + \frac{4b-1}{2b^2}\right) & b > 1/4. \end{cases}$$

Then

$$\lambda(I_{12}) = \pi \int_0^{y_0} \int_2^{1/b + 2b(1+\cos y) - 2} H\, dx\, dy.$$

For $b \le 1/4$, we obtain

$$\begin{aligned} \lambda(I_{12}) = \frac{\pi^2}{6}\big(1 - 32b^2 + 80b^3 - 12b^4 - 112b^5 \\ + 176b^6 - 192b^7 + 110b^8\big), \end{aligned} \tag{4–8}$$

and for $b > 1/4$,

$$\begin{aligned} \lambda(I_{12}) = \frac{\pi}{72}\big(37 - 140b + 270b^2 - 528b^3 + 530b^4 \\ - 712b^5 + 660b^6\big)(1 - 2b)\sqrt{4b - 1} \\ + \frac{\pi}{6}\big(1 - 32b^2 + 80b^3 - 12b^4 - 112b^5 \\ + 176b^6 - 192b^7 + 110b^8\big) \\ \times \arccos\left(-1 + \frac{4b - 1}{2b^2}\right). \end{aligned} \tag{4–9}$$

4.2. The Case $A = \{1\}$

By (4–1), $a_1 = -b/\alpha_1$, $a_2 = b$, and by (2–1), $\alpha_0 = b(\alpha_1/|\alpha_1| + \alpha_2)$. From (2–2), we get

$$\begin{aligned} 1 + b^2\left(1 + \frac{2\,\mathrm{Re}(\alpha_1\bar{\alpha}_2)}{|\alpha_1|} + |\alpha_2|^2\right) \\ = \frac{b}{|\alpha_1|}\left(1 + |\alpha_1|^2\right) + b\left(1 + |\alpha_2|^2\right). \end{aligned} \tag{4–10}$$

We may assume that $\alpha_1 > 0$. Then (4–10) has a solution $\alpha_1 \in (0, 1)$ if and only if $T > 2$, where

$$\begin{aligned} T &= \frac{1}{b} + b\big(1 + 2\,\mathrm{Re}\,\alpha_2 + |\alpha_2|^2\big) - 1 - |\alpha_2|^2 \\ &= \frac{1}{b} + b - 1 + 2bx - (1 - b)\left(x^2 + y^2\right), \end{aligned}$$

and we write $\alpha_2 = x + iy$. This means that

$$\left|\alpha_2 - \frac{b}{1 - b}\right| < \frac{1 - 2b}{\sqrt{b}(1 - b)}, \tag{4–11}$$

and the set U will be the intersection of this disk with Δ. By (4–2) and (4–10), we have

$$\varphi'(0) = 2b\left(b(1 + \bar{\alpha}_2) - \frac{T}{2}, b - (1 - b)\bar{\alpha}_2\right),$$

and therefore ,

$$f = 2b^2(1+x) - bT - 2b^2yi,$$
$$g = 2b^2 - 2b(1-b)x + 2b(1-b)yi.$$

We can compute that

$$\begin{aligned} H &= 4(1-b)b^2\big[b^2(1+2x) - (1-b)\big(1+b(x^2+y^2)\big)\big] \\ &\quad \times \big[-1+2b+b^3-2b^2(1-b)x + b(1-b)^2(x^2+y^2)\big] \\ &= 4(1-b)b^3(b+b^2-(1-b)T)(b^2+2b-2+bT). \end{aligned}$$

One can check that $H > 0$ everywhere on U.

If $b \le 1/4$, then $U = \Delta$, and using polar coordinates in Δ and Lemma 3.1, we obtain

$$\lambda(I_1) = \frac{2\pi^2}{3}(1-b)b^2\left(3-9b+2b^2+6b^3-6b^4+10b^5\right). \tag{4–12}$$

For $b > 1/4$, it is more convenient to use polar coordinates in the disk (4–11) instead:

$$x = \frac{b}{1-b} + r\cos t, \quad y = r\sin t.$$

Then

$$H = 4b^2(1-2b)^2 - 4b^4(1-b)^4r^4.$$

For r with

$$\frac{1-2b}{1-b} < r < \frac{1-2b}{\sqrt{b}(1-b)},$$

the circles $\{|\alpha_2 - b/(1-b)| = r\}$ and $\{|\alpha_2| = 1\}$ intersect when $t = \pm t(r)$, where

$$t(r) = \arccos\frac{1-2b-(1-b)^2r^2}{2br(1-b)}. \tag{4–13}$$

Therefore,

$$\begin{aligned} \lambda(I_1) &= \pi^2 \int_0^{(1-2b)/(1-b)} rH\,dr \\ &\quad + \pi\int_{(1-2b)/(1-b)}^{(1-2b)/(\sqrt{b}(1-b))} r(\pi - t(r))H\,dr. \end{aligned}$$

We can compute the second integral using the following indefinite integrals:

$$\begin{aligned} \int v\arccos\left(\frac{a}{v} - v\right)dv &= \frac{1}{4}\sqrt{-a^2+2av^2-v^4+v^2} \\ &+ \frac{4a+1}{8}\arctan\frac{2a-2v^2+1}{2\sqrt{-a^2+2av^2-v^4+v^2}} \\ &+ \frac{v^2}{2}\arccos\left(\frac{a}{v}-v\right) + \text{const} \end{aligned} \tag{4–14}$$

and

$$\begin{aligned} &\int v^5\arccos\left(\frac{a}{v}-v\right)dv \\ &= \frac{1}{288}\left(15+78a+80a^2+(10+32a)v^2+8v^4\right) \\ &\quad \times\sqrt{-a^2+2av^2-v^4+v^2} \\ &\quad + \frac{5+36a+72a^2+32a^3}{192} \\ &\quad \times \arctan\frac{2a-2v^2+1}{2\sqrt{-a^2+2av^2-v^4+v^2}} \\ &\quad + \frac{v^6}{6}\arccos\left(\frac{a}{v}-v\right) + \text{const}. \end{aligned} \tag{4–15}$$

We obtain

$$\begin{aligned} \lambda(I_1) &= -\frac{\pi^2 b}{3(1-b)^2}\big(10b^9 - 36b^8 + 54b^7 + 84b^6 - 375b^5 \\ &\qquad + 414b^4 - 166b^3 - 6b^2 + 21b - 4\big) \\ &\quad + \frac{\pi b(1-2b)}{9(1-b)}\big(30b^6 - 58b^5 + 43b^4 - 19b^3 - 26b^2 \\ &\qquad + 32b - 8\big)\sqrt{4b-1} \\ &\quad + \frac{4\pi(1-2b)^4b\left(2b^2-2b-1\right)}{3(1-b)^2}\arccos\frac{3b-1}{2b^{3/2}} \\ &\quad + \frac{2}{3}\pi(1-b)b^2\left(10b^5 - 6b^4 + 6b^3 + 2b^2 - 9b + 3\right) \\ &\qquad \times \arctan\frac{2b^2-4b+1}{(1-2b)\sqrt{4b-1}} \end{aligned} \tag{4–16}$$

for $b > 1/4$.

4.3. The Case $A = \emptyset$

We have $a_1 = a_2 = b$ and $\alpha_0 = b(\alpha_1 + \alpha_2)$. Therefore,

$$-b(1-b)(|\alpha_1|^2 + |\alpha_2|^2) + 2b^2\,\mathrm{Re}(\alpha_1\bar{\alpha}_2) + 1 - 2b = 0. \tag{4–17}$$

Again, we may assume that $\alpha_1 > 0$. We may also assume that $\mathrm{Re}\,\alpha_2 \ge 0$ and then multiply the resulting integral by 2. The equation (4–17) has a solution α_1 if

$$D := -b(1-b)^2|\alpha_2|^2 + b^3(\mathrm{Re}\,\alpha_2)^2 + (1-b)(1-2b) \ge 0.$$

It satisfies $\alpha_1 < 1$ if

$$Q := \frac{b^{3/2}\,\mathrm{Re}\,\alpha_2 + \sqrt{D}}{\sqrt{b}(1-b)} < 1.$$

This means that

$$\left|\alpha_2 - \frac{b}{1-b}\right| > \frac{1-2b}{\sqrt{b}(1-b)}. \tag{4–18}$$

By U we will denote the set of $\alpha_2 \in \Delta$ satisfying (4–18). For $b \le 1/4$, we have $U = \emptyset$ and thus $\lambda(I_0) = 0$. This together with (4–3), (4–8), and (4–12) gives (1–2).

Assume that $b > 1/4$. By (4–2), we have

$$\varphi'(0) = 2b\big((b-1)Q + b\bar\alpha_2, bQ + (b-1)\bar\alpha_2\big),$$

so that

$$f = 2b\big((b-1)Q + bx\big) - 2b^2 yi,$$
$$g = 2b\big(bQ + (b-1)x\big) + 2b(1-b)yi.$$

One can compute that

$$H = \frac{16b^3(1-2b)^3}{1-b}\left(1 + \frac{b^{3/2}x}{\sqrt{D}}\right).$$

By Lemma 3.1,

$$\lambda(I_0) = \pi \int_{-1}^{-1+(4b-1)/(2b^2)} \int_{y_2(x)}^{\sqrt{1-x^2}} H\, dy\, dx,$$

where

$$y_2(x) = 0 \quad \text{if } -1 \le x \le \frac{b^{3/2} + 2b - 1}{\sqrt{b}(1-b)},$$

while

$$y_2(x) = \sqrt{\frac{(1-2b)^2}{b(1-b)^2} - \left(x - \frac{b}{1-b}\right)^2}$$

if

$$\frac{b^{3/2} + 2b - 1}{\sqrt{b}(1-b)} \le x \le -1 + \frac{4b-1}{2b^2}.$$

It is clear from this formula that $\lambda(I_0)$ is analytic for $b \in (1/4, 1/2)$. We may therefore restrict our attention to the interval

$$\left(\frac{1}{4}, \frac{3-\sqrt{5}}{2}\right).$$

Then $0 \notin U$, and we will use polar coordinates in Δ, that is,

$$x = r\cos t, \quad y = r\sin t.$$

We get

$$\lambda(I_0) = \frac{16\pi b^3(1-2b)^3}{1-b} \times \int_{r_0}^{1} r\left(\arccos\frac{1-3b+b^2-b(1-b)r^2}{2b^2 r} - \arctan\frac{\sqrt{4b^4r^2 - (1-3b+b^2-b(1-b)r^2)^2}}{1-b-b^2-b(1-b)r^2}\right) dr,$$

where

$$r_0 = \frac{1-2b-b^{3/2}}{\sqrt{b}(1-b)}.$$

Using (4–14), one can compute that

$$\int_{r_0}^{1} r \arccos\frac{1-3b+b^2-b(1-b)r^2}{2b^2 r}\, dr = \frac{\pi\left(2b^3 - 8b^2 + 6b - 1\right)}{4(b-1)^2 b} - \frac{1}{2}\arccos\left(-1 + \frac{4b-1}{2b^2}\right) + \frac{1-2b}{4b(1-b)}\sqrt{4b-1} + \frac{(1-2b)^2}{2b(1-b)^2}\arctan\frac{1-3b}{(1-b)\sqrt{4b-1}}.$$

On the other hand, since

$$\int \frac{1}{v^2}\arctan\sqrt{-av^2+v-1}\, dv = \frac{1}{2v}\sqrt{-av^2+v-1} - \frac{1}{v}\arctan\sqrt{-av^2+v-1} - \frac{a}{2}\arctan\frac{2a\sqrt{-av^2+v-1}}{-av-2a+1} + \frac{2a-1}{4}\arctan\frac{(v-2)\sqrt{-av^2+v-1}}{2av^2-2v+2} + \text{const},$$

we obtain

$$\int_{r_0}^{1} r\arctan\frac{\sqrt{4b^4r^2-(1-3b+b^2-b(1-b)r^2)^2}}{1-b-b^2-b(1-b)r^2}\, dx = \frac{\pi(1-2b)(b+1)}{8(1-b)^2} + \frac{1-2b}{4b(1-b)}\sqrt{4b-1} - \frac{(b+2)(1-2b)}{4b(1-b)}\arctan\sqrt{4b-1} - \frac{(1+b)(1-2b)}{4(1-b)^2}\arctan\frac{1-3b}{(1-b)\sqrt{4b-1}}.$$

Therefore,

$$\lambda(I_0) = \frac{2\pi^2 b^2(1-2b)^3(-6b^2+9b-2)}{(1-b)^2} - \frac{8\pi b^3(1-2b)^3}{1-b}\arccos\left(-1+\frac{4b-1}{2b^2}\right) + \frac{4\pi b^2(1-2b)^4(b+2)}{(1-b)^2}\arctan\sqrt{4b-1} + \frac{4\pi b^2(1-2b)^4(2-b)}{(1-b)^2}\arctan\frac{1-3b}{(1-b)\sqrt{4b-1}}. \quad (4\text{–}19)$$

Using the formulas

$$\arccos\left(-1+\frac{4b-1}{2b^2}\right) = \arctan\frac{2b^2-4b+1}{(1-2b)\sqrt{4b-1}} + \frac{\pi}{2} \quad (4\text{–}20)$$

and

$$\arccos\frac{3b-1}{2b^{3/2}} = \arctan\sqrt{4b-1} - \arctan\frac{2b^2-4b+1}{(1-2b)\sqrt{4b-1}} + \frac{\pi}{2},$$

$$\begin{aligned}\lambda(I_\Omega((b,b))) &= \frac{\pi^2}{6}\left(30b^8-64b^7+80b^6-80b^5+76b^4-16b^3-8b^2+1\right)\\&+\frac{\pi(1-2b)\left(-180b^7+444b^6-554b^5+754b^4-1214b^3+922b^2-305b+37\right)}{72(1-b)}\sqrt{4b-1}\\&+\frac{4\pi b(1-2b)^4\left(7b^2+2b-2\right)}{3(1-b)^2}\arctan\sqrt{4b-1}\\&+\frac{\pi\left(30b^{10}-124b^9+238b^8-176b^7-260b^6+424b^5-76b^4-144b^3+89b^2-18b+1\right)}{6(1-b)^2}\\&\times\arctan\frac{(1-2b)\sqrt{4b-1}}{2b^2-4b+1}\\&-\frac{4\pi b^2(1-2b)^4(2-b)}{(1-b)^2}\arctan\frac{(1-b)\sqrt{4b-1}}{1-3b}.\end{aligned}$$

FIGURE 6. The formula of Figure 2 for $b \in (1/4, 1-1/\sqrt{2})$.

and combining (4–3), (4–9), (4–16), and (4–19), we get the formula of Figure 2 for $b > 1/4$.

Denoting by χ_- and χ_+ the functions defined by the right-hand sides of (1–2) and the formula of Figure 2, respectively, we can compute that at 1/4,

$$\chi_- = \chi_+ = \frac{15887}{196608}\pi^2, \quad \chi_-' = \chi_+' = -\frac{3521}{6144}\pi^2,$$
$$\chi_-'' = \chi_+'' = -\frac{215}{1536}\pi^2, \quad \chi_-^{(3)} = \chi_+^{(3)} = \frac{1785}{64}\pi^2,$$

but

$$\chi_-^{(4)} = \frac{1549}{16}\pi^2, \quad \chi_+^{(4)} = \infty.$$

This shows that our function is C^3 but not $C^{3,1}$ at 1/4. This completes the proof.

In fact, using (4–20) and

$$\arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} - \arctan x, \quad x > 0,$$

for $b \in (1/4, 1-1/\sqrt{2})$, the formula of Figure 2 can be written as the formula shown in Figure 6.

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