Estimates for $\bar\partial$ and Optimal Constants

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Dedicated to Professor Yum-Tong Siu on the occasion of his 70th birthday

1 Introduction

The fundamental extension result of Ohsawa-Takegoshi [12] says that if Ω is a pseudoconvex domain and H is an affine complex subspace of $\mathbb{C}^n$ then for any plurisubharmonic φ in Ω ($\varphi \equiv 0$ is an especially interesting case) and any holomorphic f in $\Omega' := \Omega \cap H$ there exists a holomorphic extension F to Ω satisfying the estimate

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \le C\pi \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda', \tag{1}$$

where C is a constant depending only on n and the diameter of Ω.

The original proof of this result used $\bar\partial$-theory on complete Kähler manifolds and complicated commutator identities. This approach was simplified by Siu [13] who used only Hörmander's formalism in $\mathbb{C}^n$ and proved in addition that the constant C depends only on the distance of Ω from H: he showed that if $\Omega \subset \{|z_n| < 1\}$ and $H = \{z_n = 0\}$ then one can take $C = 64/9\sqrt{1 + 1/4e} = 6.80506\ldots$ in (1). This was improved to $C = 4$ in [1] and $C = 1.95388\ldots$ in [10]. The optimal constant here, $C = 1$, was recently obtained in [6]. A slightly more general result was shown: if $\Omega \subset \mathbb{C}^{n-1} \times D$ and $0 \in D$ then (1) holds with $C = c_D(0)^{-2}$, where $c_D(0)$ is the logarithmic capacity of $\mathbb{C} \setminus D$ with respect to 0. This gave in particular

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a one-dimensional estimate

$$c_D(z)^2 \le \pi K_D(z,z),$$

where K_D is the Bergman kernel, and settled a conjecture of Suita [14].

The main tool in proving the optimal version of (1) was a new L^2–estimate for $\bar\partial$. On one hand, this new result, using some ideas of Berndtsson [2] and B.-Y. Chen [8], easily follows from the classical Hörmander estimate [11]. On the other hand, it also implies some other $\bar\partial$-estimates due to Donnelly-Fefferman and Berndtsson, even with optimal constants as will turn out. Important contribution here is due to B.-Y. Chen [8] who showed that the Ohsawa-Takegoshi theorem, unlike in [12, 13] or [1], can be deduced directly from Hörmander's estimate.

2 Estimates for $\bar\partial$

Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. For

$$\alpha = \sum_j \alpha_j d\bar z_j \in L^2_{loc,(0,1)}(\Omega)$$

we look for $u \in L^2_{loc}(\Omega)$ solving the equation

$$\bar\partial u = \alpha. \tag{2}$$

Such u always exists and we are interested in weighted L^2-estimates for solutions of (2).

The classical one is due to Hörmander [11]: for smooth, strongly plurisubharmonic φ in Ω one can find a solution of (2) satisfying

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda, \tag{3}$$

where

$$|\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k} \varphi^{jk} \bar\alpha_j \alpha_k$$

is the length of α with respect to the Kähler metric with potential φ. (Here (φ^{jk}) is the inverse transposed of the complex Hessian $(\partial^2\varphi/\partial z_j \partial\bar z_k)$.) It was observed in [4] that the Hörmander estimate (3) also holds for arbitrary plurisubharmonic φ but one should replace $|\alpha|^2_{i\partial\bar\partial\varphi}$ with any nonnegative $H \in L^\infty_{loc}(\Omega)$ satisfying

$$i\bar\alpha \wedge \alpha \le H\, i\partial\bar\partial\varphi.$$

Another very useful estimate (see e.g. [7]) for (2) is due to Donnelly-Feffermann [9]: if ψ is another plurisubharmonic function in Ω such that

$$i\partial\psi \wedge \bar\partial\psi \le i\partial\bar\partial\psi$$

(that is $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le 1$) then there exists a solution of (2) with

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le C \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{-\varphi} d\lambda, \tag{4}$$

where C is an absolute constant. We will show that $C = 4$ is optimal here.

The Donnelly-Feffermann estimate (4) was generalized by Berndtsson [1]: if $0 \le \delta < 1$ then we can find appropriate u with

$$\int_\Omega |u|^2 e^{\delta\psi-\varphi} d\lambda \le \frac{4}{(1-\delta)^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\delta\psi-\varphi} d\lambda. \tag{5}$$

This particular constant was obtained in [3] (originally in [1] it was $\frac{4}{\delta(1-\delta)^2}$) and we will prove in Sect. 3 that it is the best possible.

Berndtsson's estimate (5) is closely related to the Ohsawa-Takegoshi extension theorem [12] but the latter cannot be deduced from it directly (it could be if (5) were true for $\delta = 1$). The following version from [5] makes up for this disadvantage: if in addition $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le \delta < 1$ on $\operatorname{supp}\alpha$ then we can find a solution of (2) with

$$\int_\Omega |u|^2 (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\psi}) e^{\psi-\varphi} d\lambda \le \frac{1}{(1-\sqrt{\delta})^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\psi-\varphi} d\lambda. \tag{6}$$

The best constant in the Ohsawa-Takegoshi theorem that one can get from (6) is $1.95388\ldots$ (see [5]), originally obtained in [10].

To get the optimal constant 1 in the Ohsawa-Takegoshi theorem the following estimate for $\bar\partial$ was obtained in [6]:

Theorem 1 *Assume that $\alpha \in L^2_{loc,(0,1)}(\Omega)$ is $\bar\partial$-closed form in a pseudoconvex domain Ω in $\mathbb{C}^n$. Let φ be plurisubharmonic in Ω and $\psi \in W^{1,2}_{loc}(\Omega)$, locally bounded from above, satisfy $|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \le 1$ in Ω and $|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \le \delta$ on $\operatorname{supp}\alpha$. Then there exists $u \in L^2_{loc}(\Omega)$ solving (2) and such that*

$$\int_\Omega |u|^2 (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}) e^{2\psi-\varphi} d\lambda \le \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}} \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda. \tag{7}$$

Theorem 1 can be quite easily deduced from the Hörmander estimate (3) using some ideas of Berndtsson [2] and Chen [8], see [6] for details. On the other hand, note that we can recover (3) from Theorem 1 if we take $\psi \equiv 0$. We can also easily get (5): take $\tilde\varphi = \varphi + \psi$ and $\tilde\psi = \frac{1+\delta}{2}\psi$. Then $2\tilde\psi - \tilde\varphi = \delta\psi - \varphi$ and

$$|\bar\partial\tilde\psi|^2_{i\partial\bar\partial\tilde\varphi} \le \frac{(1+\delta)^2}{4} =: \tilde\delta$$

(since $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le 1$). From (7) we obtain (5) with the constant

$$\frac{1+\sqrt{\tilde\delta}}{(1-\sqrt{\tilde\delta})(1-\tilde\delta)} = \frac{4}{(1-\delta)^2}.$$

3 Optimal Constants

We will show that the constant in (5) is optimal for every δ. For $\delta = 0$ this gives $C = 4$ in the Donnelly-Fefferman estimate (4). We consider $\Omega = \Delta$, the unit disc, $\varphi \equiv 0$ and $\psi(z) = -\log(-\log|z|)$, so that

$$\psi_{z\bar z} = |\psi_z|^2 = \frac{1}{4|z|^2\log^2|z|}.$$

We also take functions of the form

$$v(z) = \frac{\eta(-\log|z|)}{z} \tag{8}$$

for $\eta \in C_0^1([0,\infty))$, and set

$$\alpha := \bar\partial v = -\frac{\eta'(-\log|z|)}{2|z|^2}\,d\bar z. \tag{9}$$

The crucial observation is that v is the minimal solution to $\bar\partial u = \alpha$ in $L^2(\Delta, e^{\delta\psi})$. Indeed, using polar coordinates we can easily show that $\{z^n\}_{n\ge 0}$ is an orthogonal system in $L^2(\Delta, e^{\delta\psi}) \cap \ker\bar\partial$ and that

$$\langle v, z^n\rangle_{L^2(\Delta, e^{\delta\psi})} = 0, \quad n = 0, 1, \dots$$

Berndtsson's estimate (5) now gives the following version of the Hardy-Poincaré inequality

$$\int_0^\infty \eta^2 t^{-\delta}dt \le \frac{4}{(1-\delta)^2}\int_0^\infty (\eta')^2 t^{2-\delta}dt \tag{10}$$

if $0 \le \delta < 1$ and $\eta \in C_0^1([0,\infty))$.

We are thus reduced to proving that this constant is optimal:

Proposition 2 *The constant* $4/(1-\delta)^2$ *in* (10) *cannot be improved.*

Proof Set

$$\eta(t) = \begin{cases} t^{-a}, & 0 < t \le 1 \\ t^{-b}, & t \ge 1. \end{cases}$$

Then both left and right-hand sides of (10) are finite iff $a < (1-\delta)/2$ and $b > (1-\delta)/2$. Assuming this, and since $\eta(t)$ is monotone and converges to 0 as $t \to \infty$, we can find an appropriate approximating sequence in $C_0^1([0,\infty))$. Thus (10) holds also for this η. We compute

$$\int_0^\infty \eta^2 t^{-\delta} dt = \frac{1}{1-\delta-2a} + \frac{1}{\delta-1+2b}$$

and

$$\int_0^\infty (\eta')^2 t^{2-\delta} dt = \frac{a^2}{1-\delta-2a} + \frac{b^2}{\delta-1+2b}.$$

The ratio between these quantities is equal to

$$\frac{2}{(1-\delta)(a+b)-2ab}$$

and it tends to $4/(1-\delta)^2$ as both a and b tend to $(1-\delta)/2$. □

Finally, since the same argument would work for any radially symmetric weights in Δ or an annulus $\{r < |z| < 1\}$ where $0 \le r < 1$, from (5) with α given by (9) and φ, ψ of the form $\varphi = g(-\log|z|)$, $\psi = h(-\log|z|)$ we can get the following weighted Poincaré inequalities:

Theorem 3 *Let* g, h *be convex, decreasing functions on* $(0,\infty)$. *Assume in addition that* h *is* C^2 *smooth,* $h'' > 0$ *and* $(h')^2 \le h''$. *Then, if* $0 \le \delta < 1$, *for* $\eta \in C_0^1([0,\infty))$ *one has*

$$\int_0^\infty \eta^2 e^{\delta h - g} dt \le \frac{4}{(1-\delta)^2} \int_0^\infty \frac{(\eta')^2}{h''} e^{\delta h - g} dt.$$

□

Theorem 4 *Let* g, h *be convex functions on* $(0,T)$, *where* $0 < T \le \infty$. *Assume that* h *is* C^2 *smooth,* $h'' > 0$ *and* $(h')^2 \le h''$. *If* $0 \le \delta < 1$ *it follows that for any* $\eta \in W^{1,2}_{loc}((0,T))$ *with*

$$\int_0^T \eta e^{\delta h - g} dt = 0 \tag{11}$$

we have

$$\int_0^T \eta^2 e^{\delta h-g} dt \le \frac{4}{(1-\delta)^2} \int_0^T \frac{(\eta')^2}{h''} e^{\delta h-g} dt$$

provided that both integrals exist. □

The condition (11) is necessary to ensure that in the case of an annulus the solution given by (8) is minimal in the $L^2(\{r < |z| < 1\}, e^{\delta\psi-\varphi})$-norm: it is enough to check that it is perpendicular to every element of the orthogonal system $\{z^k\}_{k\in\mathbb{Z}}$ in $\ker \bar\partial$. For $k \neq -1$ it is sufficient to use the fact that the weight is radially symmetric and for $k = -1$ one has to use (11).

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