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INTERIOR REGULARITY OF THE COMPLEX MONGE-AMPÈRE EQUATION IN CONVEX DOMAINS

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0. Introduction. For C^2-smooth plurisubharmonic (psh) functions, we consider the complex Monge-Ampère equation

$$\det\left(u_{i\bar{j}}\right) = \psi, \tag{0.1}$$

where $u_{i\bar{j}} = \partial^2 u/\partial z_i \partial \bar{z}_j$, $i, j = 1, \dots, n$. The main result of this paper is the following theorem.

Theorem A. *Let Ω be a bounded, convex domain in $\mathbb{C}^n$. Assume that ψ is a C^∞ function in Ω such that $\psi > 0$ and $|D\psi^{1/n}|$ is bounded. Then there exists a C^∞-psh solution u of (0.1) in Ω with $\lim_{z\to\partial\Omega} u(z) = 0$.*

The theory of fully nonlinear elliptic operators of second order can be applied to the operator $(\det(u_{i\bar{j}}))^{1/n}$. It follows in particular that if u is strictly psh and $C^{2,\alpha}$ for some $\alpha \in (0,1)$, then $\det(u_{i\bar{j}}) \in C^{k,\beta}$ implies $u \in C^{k+2,\beta}$, where $k = 1, 2, \dots,$ and $\beta \in (0,1)$ (see, e.g., [9, Lemma 17.16]). Therefore, to prove Theorem A, it is enough to show existence of a solution that is $C^{2,\alpha}$ in every $\Omega' \Subset \Omega$, where $\alpha \in (0,1)$ depends on Ω'. We obtain this assuming only that $\psi^{1/n}$ is positive and Lipschitz in Ω (see Theorem 4.1).

In a special case of a polydisc, we also allow nonzero boundary values.

Theorem B. *Let P be a polydisc in $\mathbb{C}^n$. Assume that ψ is a C^∞ function in P such that $\psi > 0$ and $|D^2\psi^{1/n}|$ is bounded. Let f be a $C^{1,1}$ function on the boundary ∂P such that f is subharmonic on every analytic disc embedded in ∂P. Then (0.1) has a C^∞-psh solution in P such that $\lim_{\zeta\to z} u(\zeta) = f(z)$ for $z \in \partial P$.*

In Section 5, we explain what we precisely mean by saying that a function is $C^{1,1}$ on a (nonsmooth) set ∂P. In particular, all functions that are extendable to a $C^{1,1}$ function in an open neighborhood of ∂P are allowed.

Usually, the Dirichlet problem for the complex Monge-Ampère operator is considered on smooth, strictly pseudoconvex domains in $\mathbb{C}^n$. For these, the existence of (weak) continuous solutions was proved in [1], whereas smooth solutions were obtained, for example, in [5], [10], and [11]. Here, however, we do not assume any

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regularity of the boundary. In case of the real Monge-Ampère operator, a result corresponding to Theorem A is due to Pogorelov, and a proof without gaps can be found in [6, Theorem 7] (see also [7]).

To prove Theorem A, we need interior C^1, C^2, and $C^{2,\alpha}$ a priori estimates for the solutions of (0.1). One of the main problems in the complex case was to derive a C^1-estimate, whereas in the real case it is trivial (because for any convex function on Ω, vanishing on $\partial\Omega$, we have $|Du(x)| \le -u(x)/\operatorname{dist}(x, \partial\Omega)$). We do it in Section 2 (Theorem 2.1), and this is the only point when we need the assumption that Ω is convex. We suspect that Theorem A should hold in a broader class of hyperconvex domains.

An interior C^2-estimate for the complex Monge-Ampère equation is proved in [14]. However, it gives an L^∞-bound only for Δu and not for $|D^2u|$; therefore, we cannot use the $C^{2,\alpha}$-estimate from [15]. In Section 3, we adapt the methods of [16] for the real Monge-Ampère equation and get an interior $C^{2,\alpha}$-estimate of solutions of (0.1) using only the upper bounds of Δu and $|D\psi^{1/n}|$. To show Theorem A, we could have used a result from [13] instead of Theorem 3.1, but this would not give Theorem 4.1 in its full generality.

In the proofs of the above theorems, we use a notion of a generalized solution of (0.1) introduced in [1]. The solutions obtained in Theorems A and B are unique, even among continuous psh functions.

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1. Preliminaries. If u is a continuous psh function, then we can uniquely define a nonnegative Borel measure Mu in such a way that

(i) if $u_j \to u$ locally uniformly, then $Mu_j \to Mu$ weakly;

(ii) $Mu = \det(u_{i\bar{j}})\, d\lambda$ if u is C^2 (see, e.g., [1]).

Bedford and Taylor [1] solved the Dirichlet problem for the operator M in strictly pseudoconvex domains. This result was generalized in [2] (see also [3]) for the class of hyperconvex domains.

THEOREM 1.1. *Let Ω be a bounded, hyperconvex domain in $\mathbb{C}^n$. Assume that ψ is nonnegative, continuous, and bounded in Ω. Let f be continuous on $\partial\Omega$ and such that it can be continuously extended to a psh function on Ω. Then there exists a solution of the following Dirichlet problem:*

$$\begin{cases} u \text{ psh on } \Omega, \text{ continuous on } \overline{\Omega}, \\ Mu = \psi \text{ on } \Omega, \\ u = f \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

We recall that a domain is called hyperconvex if it admits a bounded psh exhaustion function. In particular, all bounded convex domains are hyperconvex.

In [1] Bedford and Taylor also proved the following comparison principle, which implies in particular the uniqueness of (1.1) in an arbitrary bounded domain in $\mathbb{C}^n$.

PROPOSITION 1.2. *Let Ω be a bounded domain in $\mathbb{C}^n$. If u, v are psh in Ω, continuous on $\overline{\Omega}$, and such that $u \le v$ on $\partial\Omega$ and $Mu \ge Mv$ in Ω, then $u \le v$ in Ω.*

The following regularity result can be also found in [1].

THEOREM 1.3. *Let $\Omega = B$ be a Euclidean ball in $\mathbb{C}^n$. Assume that f is $C^{1,1}$ on ∂B and $\psi^{1/n}$ is $C^{1,1}$ on $\overline{B}$ (i.e., it is $C^{1,1}$ inside B and the second derivative is bounded there). Then a solution of (1.1) is $C^{1,1}$ in B. Moreover, for any $B' \Subset B$, we have*

$$\|D^2 u\|_{B'} \le C,$$

where C depends only on n, $\|D^2 f\|_{\partial B}$, $\|D^2 \psi^{1/n}\|_B$, $\operatorname{dist}(B', \partial B)$, and the radius of B.

In Section 5, we prove a similar result for a polydisc in $\mathbb{C}^n$.

The following theorem was proved in [5].

THEOREM 1.4. *Assume that Ω is strictly pseudoconvex with C^∞ boundary, ψ is C^∞ on $\overline{\Omega}$, $\psi > 0$, and f is C^∞ on $\partial\Omega$. Then u, the solution of (1.1), is C^∞ on $\overline{\Omega}$.*

It is well known that

$$\big(M(u_1+u_2)\big)^{1/n} \ge (Mu_1)^{1/n} + (Mu_2)^{1/n}, \quad u_1, u_2 \text{ psh and } C^2. \tag{1.2}$$

The above inequality does not make sense if u_1 and u_2 are just continuous, since then Mu_1 and Mu_2 are only measures. However, we can generalize it as follows (see [3, Theorem 3.11]).

PROPOSITION 1.5. *Let u_1 and u_2 be psh and continuous with $Mu_1 \ge \psi_1$, $Mu_2 \ge \psi_2$, where ψ_1 and ψ_2 are continuous and nonnegative. Then*

$$M(u_1+u_2) \ge \left(\psi_1^{1/n} + \psi_2^{1/n}\right)^n.$$

The following C^2-estimate was proved by F. Schulz [14].

THEOREM 1.6. *Let Ω be a bounded, hyperconvex domain in $\mathbb{C}^n$, and let u be a C^3-psh function in Ω with $\lim_{z\to\partial\Omega} u(z) = 0$. Assume, moreover, that for some positive constants K_0, K_1, b, B_0, and B_1, we have*

$$|u| \le K_0, \qquad |Du| \le K_1$$

and

$$b \le \psi \le B_0, \qquad |D\psi| \le B_1$$

in Ω*, where* $\psi = \det(u_{i\bar{j}})$*. Then for any* $\varepsilon > 0$*, there exists a constant* C*, depending only on* n*,* ε*,* b*,* B_0*,* B_1*,* K_0*,* K_1 *and on the upper bound for the volume of* Ω *such that*

$$\Delta u(-u)^{2+\varepsilon} \leq C$$

in Ω*.*

In the proof of Theorem B, instead of applying Theorems 1.4 and 1.6, we use the following proposition.

PROPOSITION 1.7. *Let* Ω *be a bounded domain in* $\mathbb{C}^n$*. Assume that* u *is a psh function in a neighborhood of* $\overline{\Omega}$ *and such that, for a positive constant* K *and* h *sufficiently small, it satisfies the estimate*

$$u(z+h)+u(z-h)-2u(z) \leq K|h|^2, \quad z \in \Omega.$$

Then u *is* $C^{1,1}$ *in* Ω *and* $|D^2u| \leq K$*.*

This result was essentially proved in [1, pp. 34–35]. The arguments from [1] were simplified in [8], and we present Demailly's proof for the convenience of the reader.

Proof of Proposition 1.7. Let $u_\varepsilon = u * \rho_\varepsilon$ denote the standard regularizations of u. Then for $z \in \Omega_\varepsilon := \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \varepsilon\}$ and h sufficiently small, we have

$$u_\varepsilon(z+h)+u_\varepsilon(z-h)-2u_\varepsilon(z) \leq K|h|^2.$$

This implies that

$$D^2u_\varepsilon.h^2 \leq K|h|^2. \tag{1.3}$$

Since u_ε is psh, we have

$$D^2u_\varepsilon.h^2 + D^2u_\varepsilon.(ih)^2 = 4\sum_{j,k=1}^{\infty} \frac{\partial^2 u_\varepsilon}{\partial z_j \partial \bar{z}_k} h_j \bar{h}_k \geq 0.$$

Therefore, by (1.3),

$$D^2u_\varepsilon.h^2 \geq -D^2u_\varepsilon.(ih)^2 \geq -K|h|^2.$$

This implies that $|D^2u_\varepsilon| \leq K$ on Ω_ε, and the proposition follows. □

2. A C^1-estimate in convex domains. In this section we prove the following interior a priori gradient estimate for the complex Monge-Ampère operator in convex domains.

THEOREM 2.1. *Let u be psh and continuous in a bounded, convex domain Ω in $\mathbb{C}^n$ with $\lim_{z\to\partial\Omega} u(z) = 0$. Assume, moreover, that $Mu = \psi$ is continuous and $\psi^{1/n}$ is Lipschitz in Ω with a constant K_1. Then for any $\Omega' \Subset \Omega$, u is Lipschitz in Ω' with the constant*

$$\widetilde{K} = D^2\left(\frac{2K_0}{d} + K_1\left(1+\frac{D}{d}\right)\right),$$

where $D = \operatorname{diam}\Omega$, $d = \operatorname{dist}(\Omega', \partial\Omega)$, and $K_0 = \sup_\Omega \psi^{1/n}$.

In the proof of Theorem 2.1, we use the following elementary lemma.

LEMMA 2.2. *Assume that Ω is a bounded convex domain in $\mathbb{R}^n$ containing the origin. Then, if $0 < \alpha < 1$, we have*

$$\operatorname{dist}\big(\alpha\Omega, \partial\Omega\big) = (1-\alpha)\operatorname{dist}(0, \partial\Omega).$$

Proof. The inequality "$\le$" is clear. To prove the reverse, we take $x, y \in \partial\Omega$. We have to show that $|x-\alpha y| \ge (1-\alpha)\,d$, where $d := \operatorname{dist}(0, \partial\Omega)$. Let l be a line passing through x and y. If 0, x, and y form an acute-angled triangle, then

$$|x-\alpha y| \ge |x-\alpha x| \ge (1-\alpha)\,d.$$

Otherwise, from the convexity of Ω, it follows that $d \le \operatorname{dist}(0, l)$ and, consequently,

$$|x-\alpha y| \ge (1-\alpha)\operatorname{dist}(0, l) \ge (1-\alpha)\,d. \qquad \square$$

Proof of Theorem 2.1. We may assume that Ω' is convex. Fix $a, b \in \Omega'$ with $|a-b| < d$. It is enough to show that

$$u(b) - u(a) \le \widetilde{K}|a-b|. \tag{2.1}$$

For $z \in \Omega$, put

$$T(z) := \left(1 - \frac{|a-b|}{d}\right)(z-a) + b.$$

Then $T(a) = b$ and, by Lemma 2.2,

$$\operatorname{dist}\left(\left(1-\frac{|a-b|}{d}\right)(\Omega - a), \Omega - a\right) = \frac{|a-b|}{d}\operatorname{dist}(a, \partial\Omega) \ge |a-b|,$$

and it follows that $T(\Omega) \subset \Omega$. Moreover, simple calculation shows that

$$\big|T(z) - z\big| \le \left(1+\frac{D}{d}\right)|a-b|, \quad z \in \Omega,$$

and, since $\psi^{1/n}$ is Lipschitz,

$$\psi^{1/n}\big(T(z)\big) \geq \psi^{1/n}(z) - K_1\left(1 + \frac{D}{d}\right)|a-b|. \tag{2.2}$$

For $z \in \Omega$, put

$$v(z) := u\big(T(z)\big) + \frac{\widetilde{K}^2}{D^2}\big(|z-a|^2 - D^2\big)|a-b|.$$

(It is well defined because $T(\Omega) \subset \Omega$.) The function v is psh, continuous, and negative on Ω. From Proposition 1.5 and (2.2), we infer that

$$\begin{aligned} Mv &\geq \left(\left(1 - \frac{|a-b|}{d}\right)^2 \psi^{1/n}\big(T(z)\big) + \frac{\widetilde{K}^2}{D^2}|a-b|\right)^n \\ &\geq \left(\left(1 - \frac{2|a-b|}{d}\right)\psi^{1/n}\big(T(z)\big) + \frac{\widetilde{K}^2}{D^2}|a-b|\right)^n \\ &\geq \left(\psi^{1/n}\big(T(z)\big) + \left(\frac{\widetilde{K}^2}{D^2} - \frac{2K_0}{d}\right)|a-b|\right)^n \\ &\geq \left(\psi^{1/n}(z) + \left(\frac{\widetilde{K}^2}{D^2} - \frac{2K_0}{d} - K_1\left(1 + \frac{D}{d}\right)\right)|a-b|\right)^n \\ &= \psi(z). \end{aligned}$$

The comparison principle now implies that $v \leq u$; thus

$$u(a) \geq v(a) = u(b) - \widetilde{K}|a-b|,$$

and we get (2.2). □

3. A $C^{2,\alpha}$-estimate and local regularity. The aim of this section is to show the following result.

THEOREM 3.1. *Let u be a C^4-psh function in an open $\Omega \subset \mathbb{C}^n$. Assume that for some positive K_0, K_1, K_2, b, B_0, and B_1, we have*

$$|u| \leq K_0, \qquad |Du| \leq K_1, \qquad \Delta u \leq K_2$$

and

$$b \leq \psi \leq B_0, \qquad \left|D\psi^{1/n}\right| \leq B_1$$

in Ω, where $\psi = \det(u_{i\bar{j}})$. Let $\Omega' \Subset \Omega$. Then there exist $\alpha \in (0,1)$ depending only on n, K_0, K_1, K_2, b, B_0, B_1 and a positive constant C depending, besides those *quantities, on* $\operatorname{dist}(\Omega', \partial\Omega)$ *such that*

$$\left\| D^2 u \right\|_{C^\alpha(\Omega')} \le C.$$

We use similar methods, as in other papers on nonlinear elliptic operators, especially the methods in [16]. Note that if we knew that $|D^2 u| \le K_2$, then Theorem 3.1 would be a consequence of [15]. On the other hand, if we additionally assumed that $|D^2 \psi^{1/n}| \le B_2$, then from [13, Theorem 1] we would get the estimate

$$\left\| D(\Delta u) \right\|_{\Omega'} \le C,$$

and Theorem 3.1 would follow from the Schauder estimates.

It is interesting to generalize Theorem 3.1 to arbitrary, continuous psh functions u (since $\Delta u \in L^\infty$, u would have to be at least in $W^{2,p}$ for every $p < \infty$).

In the proof of Theorem 3.1, we need the following fact from the matrix theory.

LEMMA 3.2. *Let* λ *and* Λ *be such that* $0 < \lambda < \Lambda < +\infty$. *By* $S[\lambda, \Lambda]$ *we denote the set of positive Hermitian matrices in* $\mathbb{C}^{n \times n}$ *with eigenvalues in* $[\lambda, \Lambda]$. *Then we can find unit vectors* $\gamma_1, \dots, \gamma_N$ *in* $\mathbb{C}^n$ *and* λ^*, Λ^* *depending only on* n, λ, *and* Λ *such that* $0 < \lambda^* < \Lambda^* < +\infty$. *For every* $A = (a_{ij}) \in S[\lambda, \Lambda]$, *we can write*

$$A = \sum_{k=1}^{N} \beta_k \gamma_k \otimes \overline{\gamma}_k, \qquad \text{that is, } a_{ij} = \sum_{k=1}^{N} \beta_k \gamma_{ki} \overline{\gamma}_{kj},$$

where $\beta_1, \dots, \beta_N \in [\lambda^*, \Lambda^*]$. *The set* $\{\gamma_1, \dots, \gamma_N\}$ *can be chosen so that it contains a given finite subset of the unit sphere in* $\mathbb{C}^n$, *for example, the set of the coordinate unit vectors.*

The proof of Lemma 3.2 for real symmetric matrices can be found, for example, in [9, Lemma 17.13], and it readily extends to the case of Hermitian matrices.

Proof of Theorem 3.1. If we consider constants depending only on the quantities used in the assumption, we say that those constants are under control, and we usually denote them by C_1, C_2, etc. Let $a^{i\bar{j}}$ denote the i, j-cominor of the matrix $(u_{i\bar{j}})$, so that $a^{k\bar{l}} = \partial \det(u_{i\bar{j}}) / \partial u_{k\bar{l}}$. If we set $u^{i\bar{j}} := a^{i\bar{j}}/\psi$, then we have $(u^{i\bar{j}})^T = (u_{i\bar{j}})^{-1}$. If we differentiate both sides of the equation

$$u^{i\bar{j}} u_{i\bar{k}} = \delta_{jk}$$

with respect to z_p and solve a suitable system of linear equations, we obtain

$$\left(u^{i\bar{j}}\right)_p = -u^{i\bar{l}} u^{k\bar{j}} u_{k\bar{l}p}.$$

Since $\psi_p = a^{k\bar{l}} u_{k\bar{l}p}$, we get

$$\left(a^{i\bar{j}}\right)_p = \psi \left(u^{i\bar{j}} u^{k\bar{l}} - u^{i\bar{l}} u^{k\bar{j}}\right) u_{k\bar{l}p}.$$

Therefore,

$$\left(a^{i\bar{j}_0}\right)_i = \left(a^{i_0\bar{j}}\right)_{\bar{j}} = 0 \tag{3.1}$$

for every $i_0, j_0 = 1, \dots, n$. Take $\gamma \in \mathbb{C}^n$, $|\gamma| = 1$, and for arbitrary function v denote $v_\gamma = \sum_p v_p \gamma_p$. The operator $F(A) := (\det A)^{1/n}$ is concave on the set of nonnegative Hermitian matrices. If we differentiate the equation $F((u_{i\bar{j}})) = \psi^{1/n}$ with respect to γ and $\bar{\gamma}$, we obtain

$$F_{u_{i\bar{j}}, u_{k\bar{l}}} u_{i\bar{j}\gamma} u_{k\bar{l}\bar{\gamma}} + F_{u_{i\bar{j}}} u_{i\bar{j}\gamma\bar{\gamma}} = \left(\psi^{1/n}\right)_{\gamma\bar{\gamma}}.$$

Since $F_{u_{i\bar{j}}} = (1/n)\psi^{-1+1/n} a^{i\bar{j}}$ and since F is concave, by (3.1) we have

$$a^{i\bar{j}} u_{\gamma\bar{\gamma}i\bar{j}} = \left(a^{i\bar{j}} u_{\gamma\bar{\gamma}i}\right)_{\bar{j}} \geq n\psi^{1-1/n}\left(\psi^{1/n}\right)_{\gamma\bar{\gamma}} = \psi_{\gamma\bar{\gamma}} - \left(1 - \frac{1}{n}\right)\psi^{-1}\left|\psi_\gamma\right|^2,$$

and we arrive at the estimate

$$\left(a^{i\bar{j}} u_{\gamma\bar{\gamma}i}\right)_{\bar{j}} \geq -C_1 + \sum_{s=1}^{2n} \frac{\partial f^s}{\partial x_s}, \tag{3.2}$$

where $\|f^s\|_{L^\infty(\Omega)} \leq C_2$.

From the assumptions of the theorem, it follows that the eigenvalues of the matrix $(u_{i\bar{j}})$ are in $[\lambda, \Lambda]$, where $\lambda, \Lambda > 0$ are under control. By Lemma 3.2, there are unit vectors $\gamma_1, \dots, \gamma_N$ such for $z, w \in \Omega$ we write

$$a^{i\bar{j}}(w)\left(u_{i\bar{j}}(w) - u_{i\bar{j}}(z)\right) = \sum_{k=1}^{N} \beta_k(w)\left(u_{\gamma_k\bar{\gamma}_k}(w) - u_{\gamma_k\bar{\gamma}_k}(z)\right),$$

where $\beta_k(w) \in [\lambda^*, \Lambda^*]$ and $\lambda^*, \Lambda^* > 0$ are under control. It is a consequence of the inequality between geometric and arithmetic means that for any nonnegative Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$ we have

$$\frac{1}{n}\operatorname{trace}\left(AB^T\right) \geq (\det A)^{1/n}(\det B)^{1/n}.$$

Therefore,

$$a^{i\bar{j}}(w) u_{i\bar{j}}(z) \geq n\left(\psi(w)\right)^{1-1/n}\left(\psi(z)\right)^{1/n}.$$

We conclude that

$$\sum_{k=1}^{N} \beta_k(w)\left(u_{\gamma_k\bar{\gamma}_k}(w) - u_{\gamma_k\bar{\gamma}_k}(z)\right) \leq C_3|z - w| \tag{3.3}$$

since $|D\psi^{1/n}| \leq K_1$.

Fix $z_0 \in \Omega$ and denote $B_R = B(z_0, R)$ for $R < 1$ such that $0 < 4R < \operatorname{dist}(z_0, \partial\Omega)$. Set $M_{k,R} = \sup_{B_R} u_{\gamma_k\overline{\gamma}_k}$ and $m_{k,R} = \inf_{B_R} u_{\gamma_k\overline{\gamma}_k}$. By (3.2) and the weak Harnack inequality (see [9, Theorem 8.18]), it follows that

$$R^{-2n}\int_{B_R}\big(M_{k,4R} - u_{\gamma_k\overline{\gamma}_k}\big)\,d\lambda \le C_4\big(M_{k,4R} - M_{k,R} + R\big). \tag{3.4}$$

Summing (3.4) over $k \neq k_0$, where k_0 is fixed, we obtain

$$R^{-2n}\int_{B_R}\sum_{k\neq k_0}\big(M_{k,4R} - u_{\gamma_k\overline{\gamma}_k}\big)\,d\lambda \le C_4\big(\omega(4R) - \omega(R) + R\big), \tag{3.5}$$

where $\omega(R) = \sum_{k=1}^{N}(M_{k,R} - m_{k,R})$. By (3.3) for $z \in B_{4R}$, $w \in B_R$, we have

$$\begin{aligned}\beta_{k_0}(w)\big(u_{\gamma_{k_0}\overline{\gamma}_{k_0}}(w) - u_{\gamma_{k_0}\overline{\gamma}_{k_0}}(z)\big) &\le C_3|z-w| + \sum_{k\neq k_0}\beta_k(w)\big(u_{\gamma_k\overline{\gamma}_k}(z) - u_{\gamma_k\overline{\gamma}_k}(w)\big)\\ &\le C_5 R + \Lambda^*\sum_{k\neq k_0}\big(M_{k,4R} - u_{\gamma_k\overline{\gamma}_k}(w)\big).\end{aligned}$$

Thus,

$$u_{\gamma_{k_0}\overline{\gamma}_{k_0}}(w) - m_{k_0,4R} \le \frac{1}{\lambda^*}\left(C_5 R + \Lambda^*\sum_{k\neq k_0}\big(M_{k,4R} - u_{\gamma_k\overline{\gamma}_k}(w)\big)\right),$$

and (3.5) gives

$$R^{-2n}\int_{B_R}\big(u_{\gamma_{k_0}\overline{\gamma}_{k_0}} - m_{k_0,4R}\big)\,d\lambda \le C_6\big(\omega(4R) - \omega(R) + R\big).$$

This, coupled with (3.4), easily implies that

$$\omega(R) \le C_7\big(\omega(4R) - \omega(R) + R\big);$$

hence

$$\omega(R) \le \delta\omega(4R) + R,$$

where $\delta \in (0,1)$ is under control. In an elementary way (see [9, Lemma 8.23]), we deduce that for any $\mu \in (0,1)$,

$$\omega(R) \le \frac{1}{\delta}\left(\frac{R}{R_0}\right)^{(1-\mu)(-\log\delta)/\log 4}\omega(R_0) + \frac{1}{1-\delta}R^{\mu}R_0^{1-\mu},$$

where $0 < R < R_0 < \min\{1, \operatorname{dist}(z_0, \partial\Omega)\}$. Therefore, if we choose μ so that $(1-\mu)(-\log\delta)/\log 4 \le \mu$, we obtain $\omega(R) \le CR^{\alpha}$, where $\alpha \in (0,1)$ is under control and C depends additionally on $\operatorname{dist}(z_0, \partial\Omega)$.

Since $\gamma_1, \dots, \gamma_N$ can be chosen so that they contain the coordinate vectors, we deduce that $\|\Delta u\|_{C^\alpha(\Omega')} \le C$ for some $\alpha \in (0,1)$ under control. The conclusion of the theorem follows from the Schauder estimates. □

We now prove the following local regularity of the Monge-Ampère operator.

THEOREM 3.3. *Assume that u is a $C^{1,1}$-psh function such that Mu is C^∞ and $Mu > 0$. Then u is C^∞.*

Proof. We may assume that u is defined in a neighborhood of a Euclidean ball B. There is a sequence $f_j \in C^\infty(\partial B)$ decreasing to u on ∂B and such that $\|D^2 f_j\|_{\partial B} \le C_1$. Theorem 1.4 gives $u_j \in C^\infty(\overline{B})$, u_j psh in B such that $Mu_j = Mu$, and $u_j = f_j$ on ∂B. By the comparison principle, u_j is decreasing to u in B. From Theorem 1.3 it follows that for every $B' \Subset B$ there is C_2 such that $\|D^2 u_j\|_{B'} \le C_2$. Thus, by Theorem 3.1, for every $B'' \Subset B'$ we can find $\alpha \in (0,1)$ and C_3 such that $\|D^2 u_j\|_{C^\alpha(B'')} \le C_3$. It follows that $u \in C^{2,\alpha}(B'')$, which finishes the proof. □

4. Proof of Theorem A. As mentioned in the introduction, Theorem A is an immediate consequence of the following result.

THEOREM 4.1. *Let Ω be a bounded, convex domain in $\mathbb{C}^n$. Assume that ψ is a positive function in Ω such that $\psi^{1/n}$ is (globally) Lipschitz in Ω, and let u be the (unique) solution of (1.1) with $f = 0$. Then for every $\Omega' \Subset \Omega$ there exists $\alpha \in (0,1)$ such that $u \in C^{2,\alpha}(\Omega')$.*

Proof. Let Ω'' be a convex domain such that $\Omega' \Subset \Omega'' \Subset \Omega$, and let Ω_j be a sequence of smooth strictly convex domains such that $\Omega'' \Subset \Omega_j \Subset \Omega_{j+1} \Subset \Omega$ and $\bigcup_{j=1}^\infty \Omega_j = \Omega$. Then one can find functions ψ_j, which are positive, C^∞ in a neighborhood of $\overline{\Omega}_j$ and such that $\lim_{j\to\infty} \|\psi_j - \psi\|_{\overline{\Omega}_j} = 0$, and $\|D\psi_j^{1/n}\|_{\overline{\Omega}_j} \le C_1$. (The functions ψ_j can be chosen as $\psi * \rho_\varepsilon$, the standard regularizations of ψ, where ε is sufficiently small.)

Theorem 1.4 provides C^∞ functions u_j on $\overline{\Omega}_j$, psh in Ω_j with $u_j = 0$ on $\partial\Omega_j$, and $Mu_j = \psi_j$. We claim that the sequence u_j tends locally uniformly to u in Ω. The following two inequalities can be easily deduced from superadditivity of the complex Monge-Ampère operator and from the comparison principle:

$$u(z) + \left(|z - z_0|^2 - D^2\right)\left\|\psi_j - \psi\right\|_{\overline{\Omega}_j}^{1/n} \le u_j(z), \quad z \in \Omega_j,$$

and

$$u_j(z) + \left(|z - z_0|^2 - D^2\right)\left\|\psi_j - \psi\right\|_{\overline{\Omega}_j}^{1/n} \le u(z) + \|u\|_{\partial\Omega_j}, \quad z \in \Omega_j.$$

Here, z_0 is a fixed point of Ω and $D = \operatorname{diam}\Omega$. This implies that

$$\left\|u - u_j\right\|_{\overline{\Omega}_j} \le \|u\|_{\partial\Omega_j} + D^2\left\|\psi_j - \psi\right\|_{\overline{\Omega}_j}^{1/n},$$

and the right-hand side converges to 0 as $j \to \infty$.

We claim that the sequence Δu_j is uniformly bounded in Ω''. Choose a and b so that $\max_{\Omega''} u < a < b < 0$. For j big enough, we have

$$\Omega'' \subset \left\{u_j < a\right\} \subset \{u < a\} \subset \left\{u_j < b\right\} \subset \{u < b\} \subset \Omega_j.$$

By Theorem 2.1, applied to convex domains Ω_j, there is C_2 such that for every j,

$$\left\| Du_j \right\|_{\{u<b\}} \leq C_2.$$

By Theorem 1.6, applied to domains $\{u_j < b\}$ and functions $u_j - b$ for every $\varepsilon > 0$, there exists C_3 such that

$$\Delta u_j \left(b - u_j\right)^{2+\varepsilon} \leq C_3 \quad \text{on } \left\{u_j < b\right\}.$$

Therefore,

$$\left\| \Delta u_j \right\|_{\Omega''} \leq \frac{C_3}{(b-a)^{2+\varepsilon}},$$

which proves the claim. Now, from Theorem 3.1, it follows that there exists $\alpha \in (0,1)$ such that $\|D^u_j\|_{C^\alpha(\Omega')} \leq C_4$; hence, $u \in C^{2,\alpha}(\Omega')$. □

We conjecture that Theorem 4.1 (as well as Theorem A) holds if Ω is only hyperconvex. It would be sufficient if we knew that the sequence $|Du_j|$ is locally bounded in Ω, where u_j is the sequence constructed in the proof of Theorem 4.1. This would require a counterpart of Theorem 2.1 for nonconvex domains.

Theorem A implies the following analogue of the local regularity of the real Monge-Ampère operator.

THEOREM 4.2. *Let u be a convex function defined on an open subset of $\mathbb{C}^n$ such that its graph contains no line segment. Suppose that Mu is positive and C^∞. Then u is C^∞.*

Proof. By Ω denote a domain where u is defined. Fix $z_0 \in \Omega$. Let T be an affine function such that $T \leq u$ and $T(z_0) = u(z_0)$. Since the graph of u contains no line segment, one can easily show that for some $\varepsilon > 0$ a convex domain $\{u - T + \varepsilon < 0\}$ is relatively compact in Ω. Now we apply Theorem A to this domain. By the uniqueness of the Dirichlet problem, we conclude that u must be smooth in some neighborhood of z_0. □

5. Interior regularity in a polydisc. Throughout this section, P denotes the unit polydisc in $\mathbb{C}^n$; that is, $P = \Delta^n = \{z \in \mathbb{C}^n : |z_j| < 1,\ j = 1, \ldots, n\}$.

Similarly as before, our starting point in proving Theorem B is Theorem 1.1. In order to use it, we need the following proposition.

PROPOSITION 5.1. *Let f be a continuous function on ∂P. Then the following are equivalent:*

(i) *f is subharmonic on every disc embedded in ∂P;*

(ii) *f can be continuously extended to a psh function on P.*

Proof. (ii)⇒(i) is clear. To show the converse, define

$$u := \sup\big\{v : v \text{ psh on } P,\ v^* \le f \text{ on } \partial P\big\}.$$

Here v^* denotes the upper regularization of v which is defined on $\overline{P}$; the lower regularization is denoted by v_*. By a result from [17] (see also [3, Theorem 1.5]), it is enough to show that $u^* = u_* = f$ on ∂P. By the classical potential theory, we can find a harmonic function h on P, continuous on $\overline{P}$ and such that $h = f$ on ∂P. Therefore, $u \le h$, and it remains to show that $u_* \ge f$ on ∂P.

Take any $\varepsilon > 0$ and $w \in \partial P$. We assume that $w = (1, 0, \dots, 0)$. For $z \in \overline{P}$ and A positive, we can define

$$v(z) := f\big(1, z_2, \dots, z_n\big) + A\big(\operatorname{Re} z_1 - 1\big) - \varepsilon.$$

Then v is continuous on $\overline{P}$, psh on P, and we claim that for A big enough, $v \le f$ on ∂P. We can find positive r such that $f(1, z_2, \dots, z_n) - \varepsilon \le f(z)$ if $|z_1 - 1| \le r$ and $z \in \partial P$. Therefore, it is enough to take A, which is not smaller than

$$\sup_{z\in\partial P,\, |z_1-1|\ge r} \frac{f\big(1, z_2, \dots, z_n\big) - f(z) - \varepsilon}{1 - \operatorname{Re} z_1}.$$

Eventually, $u_*(w) \ge v(w) \ge f(w) - \varepsilon$, which completes the proof. □

In case of a bidisc, Theorem 1.1 was earlier proved in [12] with probabilistic methods. In fact, similarly as in [12], if $\Omega = P$, then the assumption in Theorem 1.1 that ψ is bounded can be relaxed. One can allow nonnegative, continuous ψ with

$$\psi(z) \le \frac{C}{\big(1 - |z_1|\big)^\beta \cdots \big(1 - |z_n|\big)^\beta}, \quad z \in P,$$

for some positive C and $\beta < 2$. This arises from the subsolution

$$u(z) = -\big(1 - |z_1|^2\big)^\varepsilon \cdots \big(1 - |z_n|^2\big)^\varepsilon,$$

where $0 < \varepsilon \le 1/n$; then

$$Mu(z) = \varepsilon^n \big(1 - |z_1|^2\big)^{(n\varepsilon-2)} \cdots \big(1 - |z_n|^2\big)^{(n\varepsilon-2)} \big(1 - \varepsilon|z|^2\big).$$

Before stating the main result of this section, we explain the notation. We say that a function is $C^{1,1}$ on $\overline{P}$ if it is $C^{1,1}$ on P and its second derivative is (globally) bounded. By saying that a function is $C^{1,1}$ on ∂P, we mean that it is continuous on ∂P, $C^{1,1}$ on the $(2n-1)$-real-dimensional manifold

$$R := \bigcup_{j=1}^{n} \Delta^{j-1} \times \partial\Delta \times \Delta^{n-j},$$

and the second derivative is bounded on R.

In order to prove Theorem B, we show the following counterpart of Theorem 1.3 for a polydisc.

THEOREM 5.2. *Assume that* $\psi \geq 0$ *is such that* $\psi^{1/n} \in C^{1,1}(\overline{P})$. *Let* f *be* $C^{1,1}$ *on* ∂P *and subharmonic on every disc embedded in* ∂P. *Then a solution of (1.1) is* $C^{1,1}$ *on* P.

Note that, contrary to Theorem 3.1, we do not assume here that $\psi > 0$. We conjecture that for arbitrary bounded, hyperconvex domain Ω in $\mathbb{C}^n$, if $f = 0$ and $\psi \geq 0$, $\psi^{1/n} \in C^{1,1}(\overline{\Omega})$, then a solution of (1.1) belongs to $C^{1,1}(\Omega)$. The analogous problem can be stated for the real Monge-Ampère operator and bounded, convex domains in $\mathbb{R}^n$. By [11], the answer in both the complex and real case is positive if Ω is $C^{3,1}$ strictly pseudoconvex (resp., convex); we then get a solution in $C^{1,1}(\overline{\Omega})$. However, we cannot expect global boundedness of the second derivatives in general because if, for example, $\psi = 1$, then all eigenvalues of the complex (resp., real) Hessian of u would be bounded away from zero. This would imply in particular that there are no analytic discs (resp., line segments) in $\partial\Omega$, but this is allowed in general.

Proof of Theorem 5.2. The proof is similar to the proof of [1, Proposition 6.6]. Let D be open and relatively compact in P. Define

$$\begin{aligned} T_{a,h}(z) &= T\big(a,h,z\big) \\ &:= \left(\frac{h_1 + \big(1 - |a_1|^2 - \overline{a}_1 h_1\big) z_1}{1 - |a_1|^2 - a_1 \overline{h}_1 + \overline{h}_1 z_1}, \dots, \frac{h_n + \big(1 - |a_n|^2 - \overline{a}_n h_1\big) z_n}{1 - |a_n|^2 - a_n \overline{h}_n + \overline{h}_n z_n} \right). \end{aligned}$$

Then T is C^∞-smooth in a neighborhood of the set $\{(a,h,z) : a \in \overline{D},\ |h| \leq d/2,\ z \in \overline{P}\}$, where $d = \operatorname{dist}(D, \partial P)$. Moreover, $T_{a,h}$ is a holomorphic automorphism of P mapping a to $a+h$ and such that $T_{a,0}(z) = z$.

For $a \in D$, $|h| < d/2$, and $z \in \overline{P}$, put

$$v(z) := \frac{u\big(T_{a,h}(z)\big) + u\big(T_{a,-h}(z)\big)}{2} - K_1 |h|^2 + K_2\big(|z|^2 - n\big).$$

We claim that if K_1 and K_2 are big enough, then for all a, h, and z we have $v \leq u$. By the comparison principle, it is enough to show that $v \leq u$ on ∂P and $Mv \geq Mu$ on P.

Since $T_{a,h}$ maps R onto R, it is easy to see that if we take

$$K_1 := \frac{1}{2} \left\| \frac{\partial^2}{\partial h^2} f\big(T(a,h,z)\big) \right\|_{\{a \in \overline{D},\, |h| \leq d/2,\, z \in R\}},$$

then $v \leq u$ on R. Since both functions are continuous, the inequality holds on ∂P.

From Proposition 1.5, we infer

$$Mv \geq \left(\frac{\psi^{1/n}\big(T_{a,h}(z)\big) \big|T'_{a,h}(z)\big|^{2/n} + \psi^{1/n}\big(T_{a,-h}(z)\big) \big|T'_{a,-h}(z)\big|^{2/n}}{2} + K_2 |h|^2 \right)^n,$$

where by T' we mean the Jacobian of T. Therefore, we have $Mv \geq Mu = \psi$ if

$$K_2 = \frac{1}{2}\left\| \frac{\partial^2}{\partial h^2}\left(\psi^{1/n}\big(T_{a,h}(z)\big)\left|T'_{a,h}(z)\right|^{2/n}\right)\right\|_{\{a\in\overline{D},\,|h|\leq d/2,\,z\in P\}}.$$

Eventually, $v \leq u$ and

$$u(a) \geq v(a) \geq \frac{u(a+h)+u(a-h)}{2} - \big(K_1+nK_2\big)|h|^2, \quad a \in D,\ |h| < \frac{d}{2}.$$

The theorem follows from Proposition 1.7. □

It is clear from the proof that, similarly as in Theorem 1.3, we have an interior a priori estimate for D^2u in Theorem 5.2.

Theorem B can be deduced from Theorems 5.2 and 3.3.

The assumption that $\psi > 0$ in Theorem B is essential, as the following example shows.

Example. Let $P = \Delta^2$ be the unit bidisc. The function $f(z,w) = (\operatorname{Re} z)^2(\operatorname{Re} w)^2$ is separately subharmonic; thus, by Proposition 5.1 and Theorem 1.1, the function

$$u := \sup\left\{v : v \text{ psh in } \Delta^2,\ v^* \leq f \text{ on } \partial\big(\Delta^2\big)\right\}$$

is psh in Δ^2, continuous on $\overline{\Delta}^2$, $u = f$ on $\partial(\Delta^2)$, and $Mu = 0$ in Δ^2. By Theorem 5.2, u is $C^{1,1}$ in Δ^2.

Note that for any $z, w \in \mathbb{C}$, we have

$$4\operatorname{Re} z\operatorname{Re} w - \big(1-|z|^2\big)\big(1-|w|^2\big) = |z+w|^2 - \left|1-zw\right|^2.$$

Thus, $\{|z+w| = |1-zw|\}\cap\partial(\Delta^2) \subset \{\operatorname{Re} z\operatorname{Re} w = 0\}$. It is easy to check that the set $\{|z+w| = |1-zw|\}\cap\overline{\Delta}^2$ can be foliated by analytic discs with boundaries in $\partial(\Delta^2)$ and that $u = 0$ on $\{|z+w| \leq |1-zw|\}\cap\overline{\Delta}^2$. For $\varepsilon \in (0,1)$, set

$$\begin{aligned} v_\varepsilon(z,w) &= \frac{\varepsilon^2}{4}\left(\left|\frac{z+w}{\varepsilon+1-zw}\right|^2 - 1\right) \\ &= \frac{\varepsilon^2}{4}\,\frac{4\operatorname{Re} z\operatorname{Re} w - \big(1-|z|^2\big)\big(1-|w|^2\big) - 2\varepsilon\big(1-\operatorname{Re}(zw)\big) - \varepsilon^2}{\left|\varepsilon+1-zw\right|^2}. \end{aligned}$$

Then v_ε is psh in Δ^2, continuous on $\overline{\Delta}^2$, and $v_\varepsilon(z,w) \leq \operatorname{Re} z\operatorname{Re} w$ there. Therefore, we have $(\max\{0, v_\varepsilon\})^2 \leq u$ and $v_\varepsilon \leq \sqrt{u}$. For $t \in (\sqrt{2}-1, 1)$, an elementary calculation gives

$$\sqrt{u(t,t)} \geq \sup_{\varepsilon\in(0,1)} \frac{\varepsilon^2}{4}\left(\frac{(2t)^2}{\big(\varepsilon+1-t^2\big)^2} - 1\right) = \frac{1}{4}\Big((2t)^{2/3} - \big(1-t^2\big)^{2/3}\Big)^3,$$

since the supremum is attained for ε with $(\varepsilon+1-t^2)^3 = (2t)^2(1-t^2)$. For $t \in (0,1)$, we thus have

$$u(t,t)\begin{cases}=0 & \text{if } t\le\sqrt{2}-1,\\ \ge 2^{-4}\big((2t)^{2/3}-\big(1-t^2\big)^{2/3}\big)^6 & \text{if } t\ge\sqrt{2}-1,\end{cases}$$

and we conclude that u is not C^6. We conjecture that, in fact, u is not even C^2.

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