THE COMPLEX MONGE-AMPÈRE EQUATION IN KÄHLER GEOMETRY

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Abstract. We discuss two cases when the complex Monge-Ampère equation appears in Kähler geometry: the Calabi conjecture (with its solution by Yau) and the equation for geodesics in the Mabuchi space of Kähler metrics, introduced independently by Semmes and Donaldson.

Introduction

This is a slightly expanded version of the talk given on the 2nd Forum of Polish Mathematicians in Częstochowa on 1st July 2008.

1. Kähler manifolds

Let M be a complex manifold of dimension n. A complex structure induces an endomorphism of the tangent bundle $J: TM \to TM$, in local coordinates given by

$$J(\partial/\partial x^j) = \partial/\partial y^j, \qquad J(\partial/\partial y^j) = -\partial/\partial x^j$$

It can be extended in a C-linear way to the complexified tangent bundle $J: T_C M \to T_C M$, so that

$$J(\partial_j) = i\partial_j, \qquad J(\partial_{\bar{j}}) = -i\partial_{\bar{j}}$$

where $\partial_j := \partial/\partial z^j$, $\partial_{\bar{j}} := \partial/\partial \overline{z^j}$.

Remark. The celebrated Newlander-Nirenberg theorem [17] (see also [14]) says that if M is a real manifold and $J: TM \to TM$ satisfies $J^2 = -id$, then J comes from some complex structure on M if and only if

$$[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0 \quad \text{for} \quad X, Y \in TM$$

Coming back to the case when M is a complex manifold, let $h: TM \times TM \to C$ be a hermitian form on M. Locally we may write

$$h = g_{j\bar{k}} dz^j d\overline{z^k}$$

where $(g_{j\bar{k}})$ is a hermitian matrix of smooth functions (i.e. $\overline{g_{j\bar{k}}} = g_{k\bar{j}}$, $(g_{j\bar{k}}) > 0$). Every such a hermitian form can be associated with the (1,1)-form

$$\omega := g_{j\bar{k}} i dz^j \wedge d\overline{z^k}$$

(one can easily check that h and ω behave the same way under holomorphic change of coordinates).

The hermitian metric h gives a Riemannian metric $\Re h$, which in turn generates uniquely defined Levi-Civita connection ∇ on M. One can then show that

$$\nabla J = 0 \Leftrightarrow \nabla \omega = 0 \Leftrightarrow d\omega = 0 \Leftrightarrow g_{j\bar{k}} = \frac{\partial^2 g}{\partial z^j \partial \overline{z^k}}$$

locally for some smooth function g. We then say that the (1,1)-form ω is *Kähler*. (In other words, a smooth (1,1)-form ω is Kähler if $\overline{\omega} = \omega$, $\omega = 0$ and $d\omega = 0$). Kähler metrics are thus those hermitian metrics for which the Riemannian structure is compatible with the complex structure.

For compact complex manifolds existence of a Kähler metric imposes topological constraints: the Betti numbers $b_{2k} = 0$ for $1 \le k \le n$. Namely, $\omega^k = \omega \wedge \cdots \wedge \omega$ is a closed real *2k*-form which is not exact (if $\omega^k = d\alpha$ for some α then by the Stokes theorem we would have

$$\int_M \omega^n = \int_M d(\alpha \wedge \omega^{n-k}) = 0$$

which is a contradiction), and thus $0 \ne \{\omega^k\} \in H^{2k}(M, R)$.

Example (Hopf surface). $M := (C^2 \setminus \{0\}) / \{2^n : n \in Z\}$. Then M is homemorphic to $S^1 \times S^3$ and thus $b_2(M) = 0$. Therefore, M does not admit any Kähler metric.

Complex (p,q) - forms may be locally written as

$$\sum_{|J|=p, |K|=q} f_{JK} dz^J \wedge d\overline{z^K}$$

where $J=(j_1,\dots,j_p)$, $1\le j_1<\dots<j_p\le n$, $dz^J=dz^{j_1}\wedge\dots\wedge dz^{j_p}$ (and similarly for K). We define the operators

$$\partial : C^\infty_{(p,q)} \to C^\infty_{(p+1,q)}, \qquad \bar\partial : C^\infty_{(p,q)} \to C^\infty_{(p,q+1)}$$

as follows

$$\partial f := \sum_j \frac{\partial f}{\partial z^j} dz^j, \qquad \partial(f dz^J \wedge d\overline{z^K}) := \partial f \wedge dz^J \wedge d\overline{z^K}$$

$$\bar\partial f := \sum_j \frac{\partial f}{\partial \overline{z^j}} d\overline{z^j}, \qquad \bar\partial(f dz^J \wedge d\overline{z^K}) := \bar\partial f \wedge dz^J \wedge d\overline{z^K}$$

We then have $d=\partial+\bar\partial$, and since $d^2=0$, we get $\partial^2=\bar\partial^2=\partial\bar\partial+\bar\partial\partial=0$

It is convenient to introduce the operator

$$d^c := \frac{i}{2}(\bar\partial - \partial)$$

It is a real operator on M (in the sense that it maps real forms into real forms) depending however on the complex structure on M. One can easily check that $dd^c = i\partial\bar\partial.$

dd^c*-lemma.* On a compact Kähler manifold a (p,q) - form is d-exact if and only if it is dd^c-exact.

This result can be proved using the Hodge theory and some (local) commutator formulas on Kähler manifolds (see eg [10]).

2. Calabi conjecture

Assume that $\omega = g_{j\bar k} i dz^j \wedge d\overline{z^k}$ is a Kähler form. One can then show that the Ricci curvature is given by

$$Ric_\omega = -dd^c \log\det(g_{j\bar k})$$

If $\tilde\omega = \tilde g_{j\bar k} i dz^j \wedge d\overline{z^k}$ is another Kähler form on M then

$$Ric_\omega - Ric_{\tilde\omega} = dd^c \eta$$

where

$$\eta = \log \frac{\det(\widetilde{g}_{j\bar{k}})}{\det(g_{j\bar{k}})}$$

is a globally defined function on M. We see therefore that the cohomology class of $(1,1)$-forms $\{Ric_\omega\}$ is independent of ω. It is in fact the first Chern class of M, we denote it $c_1(M)$ (more precisely it is equal to $c_1(M)_R/2\pi$).

From now on we will assume that (M,ω) is a compact Kähler manifold. Calabi [8] conjectured that for any (1,1)-form R which is cohomologous to Ric_ω (that is $R \in c_1(M)$) there exists a Kähler form $\widetilde{\omega}$ cohomologous to ω such that $Ric_{\widetilde{\omega}} = R$. We thus have $R = Ric_\omega + dd^c\eta$ for some $\eta \in C^\infty(M)$, and we look for $\varphi \in C^\infty(M)$ such that $\omega + dd^c\varphi > 0$ and

$$dd^c\left(\log \frac{\det(g_{j\bar{k}} + \varphi_{j\bar{k}})}{\det(g_{j\bar{k}})} + \eta\right) = 0$$

where $\varphi_{j\bar{k}} = \partial^2\varphi/\partial z^j \partial\overline{z^k}$. This means that for some constant c

$$\det(g_{j\bar{k}} + \varphi_{j\bar{k}}) = e^{-\eta + c}\det(g_{j\bar{k}})$$

or equivalently

$$(\omega + dd^c\varphi)^n = e^{-\eta + c}\omega^n$$

The constant c is uniquely determined, since by the Stokes theorem

$$\int_M (\omega + dd^c\varphi)^n = \int_M \omega^n$$

Solving the Calabi conjecture is thus equivalent to proving the following result:

Theorem (Yau [20]). Assume that $f \in C^\infty(M)$, $f > 0$, is such that

$$\int_M f\omega^n = \int_M \omega^n$$

Then there exists a unique (up to an additive constant) $\varphi \in C^\infty(M)$ satisfying $\omega + dd^c\varphi > 0$ and solving the Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = f\omega^n \tag{1}$$

If we consider the Kähler class of Kähler metrics cohomologous to ω:

$$H := \{\omega + dd^c\varphi : \varphi \in C^\infty(M), \omega + dd^c\varphi > 0\} \tag{2}$$

the Calabi conjecture can be formulated as follows: the mapping

$$H \ni \widetilde{\omega} \mapsto Ric_{\widetilde{\omega}} \in c_1(M)$$

is bijective.

Corollary. If $c_1(M) = 0$ then there exists a Kähler metric with vanishing Ricci curvature.

This result, useful for example in algebraic geometry, is interesting because in every single case such a metric cannot be written explicitly.

The Yau theorem is proved in several steps:

1. uniqueness;
2. continuity method reducing the problem to a $C^{2,\alpha}$ a priori estimate;
3. a priori estimate for the L^∞-norm of solutions;
4. a priori estimate for the C^2-norm of solutions;
5. a priori estimate for the $C^{2,\alpha}$-norm of solutions.

Uniqueness is a simple consequence of integration by parts and was proved by Calabi already in the 50's (see [3] for a more general result). The continuity method is often used in the theory of fully nonlinear elliptic equations of second order, it relies on the implicit function theorem in infinitely dimensional Banach spaces.

The uniform a priori estimate was proved in [20] using Moser's iteration technique. It should be stressed that in many problems of this kind (eg the Yamabe problem or the problem of existence of a Kähler-Einstein metric) this estimate is crucial. The best result in this direction was proved using pluripotential theory:

Theorem (Kołodziej [15]). Assume that $\varphi \in C^2(M)$, $\omega + dd^c\varphi \geq 0$, solves (1). Then for $p > 1$ there exists a positive constant C, depending only on (M,ω), p and $\| f \|_{L^p(M)}$, such that

$$osc\varphi := \sup_M \varphi - \inf_M \varphi \leq C$$

See also [4] for a bit different proof of this result.

It is rather unusual that the C^2-estimate can be derived directly from the L^∞-estimate. This was done independently by Aubin [1] and Yau [20]. This estimate was later improved in [2] and [6] to the following form:

Theorem ([6]). Assume that $\varphi \in C^4(M)$, $\omega + dd^c\varphi > 0$, solves (1). Then there exists a positive constant *C*, depending only on *n*, on upper bounds for $osc\varphi$, $\sup f$ and the scalar curvature of *M*, and on lower bounds for the bisectional curvature of *M* and $\inf f^{1/(n-1)}\Delta(\log f)$, such that

$$\Delta\varphi \le C$$

Behaviour on the geometry of *M* in the above result is quite explicit, it is not so geometric as in the L^∞-estimate. It was shown in [18] that the exponent 1/(*n*-1) above is optimal.

In the original proof Yau used Nirenberg's estimate for the C^3-norm. In the early 80's a general theory (it is now called Evans-Krylov theory) was developed. It allows to estimate locally the $C^{2,\alpha}$-norm in terms of the C^2-norm of solutions of general equations of the form

$$F(D^2u, Du, u, x) = 0$$

provided that $(\partial F / \partial u_{jk}) > 0$ and that *F* is concave with respect to D^2u. Concerning the last condition, the crucial fact for the complex Monge-Ampère operator is that the mapping $A \mapsto (\det A)^{1/n}$ is concave on the set of positive hermitian matrices. Using the Evans-Krylov theory one can get the following estimate (see e.g. [5] - this paper contains the whole proof of the Yau theorem):

Theorem. For $u \in C^4(\Omega)$, where Ω is a domain in C^n, with $(u_{j\bar{k}}) > 0$ set $f := \det(u_{j\bar{k}})$. Then for $\Omega' \subset \Omega$ there exists $\alpha \in (0,1)$, depending only on *n*, on upper bounds for $\| u \|_{C^1(\Omega)}$, $\sup \Delta u$, $\| f \|_{C^1(\Omega)}$, and on a lower bound for $\inf f$, and positive constant *C* depending in addition on $dist(\Omega', \partial\Omega)$, such that

$$\| u \|_{C^{2,\alpha}(\Omega')} \le C$$

3. Mabuchi space of Kähler metrics

Let H be the space of Kähler metrics from one cohomology class given by (2). We can treat it as an open subset of $C^\infty(M)$ (modulo an additive constant), so for $\varphi \in H$ the tangent space $T_\varphi H$ is equal to $C^\infty(M)$. On $T_\varphi M$ we define the norm

$$\| \psi \|_\varphi^2 := \frac{1}{n!}\int_M \psi^2 (\omega + dd^c\varphi)^n, \qquad \psi \in T_\varphi H$$

Accordingly, a length of a curve $\Phi \in C^1([1,2], H) \subset C^1([1,2]\times M)$ is given by

$$l(\Phi) := \int_1^2 \| \dot{\varphi}_t \|_{\varphi_t} \, dt$$

where $\varphi_t := \Phi(t,\cdot)$ and $\dot{\varphi}_t := (\partial \Phi / \partial t)(t,\cdot)$.

The above metric, introduced by Mabuchi [16], gives a riemannian structure on an infinitely dimensional manifold H. It determines a Levi-Civita connection and the geodesic equation turns out to be

$$\ddot{\varphi} - \frac{1}{2} \| \nabla \dot{\varphi} \|_{\varphi}^2 = 0$$

It was shown independently by Semmes [19] and Donaldson [11] that it is equivalent to

$$\det \begin{pmatrix} & & & \dot{\varphi}_1 \\ & (g_{j\bar{k}} + \varphi_{j\bar{k}}) & & \vdots \\ & & & \dot{\varphi}_n \\ \dot{\varphi}_{\bar{1}} & \cdots & \dot{\varphi}_{\bar{n}} & \ddot{\varphi} \end{pmatrix} = 0$$

Therefore, to find a geodesic connecting two metrics $\omega + dd^c\varphi_1$, $\omega + dd^c\varphi_2 \in H$ is equivalent to solving the homogeneous Monge-Ampère equation

$$(\Omega + dd^c\varphi)^{n+1} = 0 \tag{3}$$

on a compact Kähler manifold (with boundary) $M \times \{1 \le |z_{n+1}| \le 2\}$, where

$$\Omega := \omega + dd^c |z_{n+1}|^2$$

with the boundary condition $\varphi = \varphi_j$ on $M \times \{|z_{n+1}| = j\}$, $j = 1,2$.

Donaldson [11] conjectured that any two metrics in H can be joined by a C^∞-geodesic, and that the function

$$d(\omega_1, \omega_2) := \inf\{l(\Phi) : \Phi \text{ is a curve in H joining } \omega_1 \text{ with } \omega_2\} \text{ for } \omega_1, \omega_2 \in H$$

is a distance on H. The latter conjecture was proved by X.X. Chen [9]. He also showed that any two metrics can be joined by a $C^{1,1}$- geodesic (although it may possibly leave H, as the intermediate metrics are only assured to be (almost) $C^{1,1}$-smooth and nonnegative). The existence of C^∞- geodesics remains an open problem.

In general, one should not expect solutions of a degenerate Monge-Ampère equation (such as (3)) to be C^∞-smooth. On one hand, in the non-degenerate case we have the following counterpart of the Yau theorem for domains in C^n:

Theorem ([7]). Let Ω be a smooth, bounded, strongly pseudoconvex domain in C^n (for example a ball). Then for $f \in C^\infty(\overline{\Omega})$, $f > 0$, and $\varphi \in C^\infty(\partial\Omega)$ there exists the unique $u \in C^\infty(\overline{\Omega})$, such that $(u_{j\bar{k}}) \geq 0$, $u = \varphi$ on $\partial\Omega$ and

$$\det(u_{j\bar{k}}) = f \quad \text{in } \Omega$$

However, when we only assume that $f \geq 0$, the best possible regularity is $C^{1,1}$, as the following example of Gamelin and Sibony shows.

Example ([13]). Let $B := \{(z,w) \in C^2\colon |z|^2 + |w|^2 < 1\} \frac{n!}{r!(n-r)!}$ be the unit ball in C^2 and set

$$\varphi(z,w) := \left(|z|^2 - 1/2\right)^2 = \left(|w|^2 - 1/2\right)^2, \qquad (z,w) \in \partial B$$

Then

$$u(z,w) := (\max\{|z|^2 - 1/2,\ |w|^2 - 1/2,\ 0\})^2, \qquad (z,w) \in \overline{B}$$

is $C^{1,1}$-smooth (but not C^2 !), $(u_{j\bar{k}}) \geq 0$, $\det(u_{j\bar{k}}) = 0$, and $u = \varphi$ on ∂B

It would suggest that also in (3) the best possible regularity should be $C^{1,1}$, also among toric varieties (note that all the data in the above example depends only on $|z|$ and $|w|$). However, that problem seems to be more special, and for example for toric varieties Donaldson [12] indeed showed that a C^∞-geodesic always exists.

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References

[1] Aubin T., *Equations du type de Monge-Ampère sur les variétés Kähleriennes compactes*, C.R. Acad. Sci. Paris 283 (1976), 119-121.

[2] Błocki Z., *Regularity of the degenerate Monge-Ampère equation on compact Kähler manifolds,* Math. Z. 244 (2003), 153-161.

[3] Błocki Z., *Uniqueness and stability for the Monge-Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J. 52 (2003), 1697-1702.

[4] Błocki Z., *On uniform estimate in Calabi-Yau theorem,* Sci. China Ser. A 48 suppl. (2005), 244-247.

[5] Błocki Z., *The Monge-Ampère equation on compact Kähler manifolds*, unpublihed lecture notes based on the course given at Winter School in Complex Analysis, Toulouse, 2005, available at http://gamma.im.uj.edu.pl/~blocki/

[6] Błocki Z., *A gradient estimate in the Calabi-Yau theorem*, preprint, 2008.

[7] Caffarelli L., Kohn J.J., Nirenberg L., Spruck J., *The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations*, Comm. Pure Appl. Math. 38 (1985), 209-252.

[8] Calabi E., *The space of Kähler metrics*, Proc. Internat. Congress Math. Amsterdam 1954, vol. 2, 206-207.

[9] Chen X.X., *The space of Kähler metrics*, J. Diff. Geom. 56 (2000), 189-234.

[10] Demailly J.P., *Complex Analytic and Differential Geometry*, 1997, available at http://www-fourier.ujf-grenoble.fr/~demailly/books.html

[11] Donaldson S.K., *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, Northern California Symplectic Geometry Seminar, 13-33, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, RI, 1999.

[12] Donaldson S.K., *Scalar curvature and stability of toric varieties*, J. Diff. Geom. 62 (2002), 289-349.

[13] Gamelin T.W., Sibony N., *Subharmonicity for uniform algebras*, J. Funct. Anal. 35 (1980), 64-108.

[14] Hörmander L., *An Introduction to Complex Analysis in Several Variables*, 3rd ed., North-Holland, Amsterdam 1990.

[15] Kołodziej S., *The complex Monge-Ampère equation*, Acta Math. 180 (1998), 69-117.

[16] Mabuchi T., *Some symplectic geometry on compact Kähler manifolds. I,* Osaka J. Math. 24 (1987), 227-252.

[17] Newlander A., Nirenberg L., *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. 65 (1957), 391-404.

[18] Pliś S., *A counterexample to the regularity of the degenerate complex Monge-Ampère equation*, Ann. Polon. Math. 86 (2005), 171-175.

[19] Semmes S., *Complex Monge-Ampère and symplectic manifolds*, Amer. J. Math. 114 (1992), 495-550.

[20] Yau S.-T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, Comm. Pure Appl. Math. 31 (1978), 339-411.