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# The $C^{1,1}$ Regularity of the Pluricomplex Green Function

Zbigniew Błocki

If $\Omega$ is a domain in $\mathbb{C}^n$ and $\zeta \in \Omega$, then the pluricomplex Green function in $\Omega$ with pole at $\zeta$ is defined as

$$g = \sup\{u \in \mathrm{PSH}(\Omega) : u < 0,\ \limsup_{z\to\zeta}(u(z) - \log|z-\zeta|) < \infty\}$$

(see [5] for details). The main goal of this note is to prove the following result.

Theorem 1. *Let $\Omega$ be a $C^\infty$ strictly pseudoconvex domain in $\mathbb{C}^n$, and let $g$ be the pluricomplex Green function of $\Omega$ with pole at some $\zeta \in \Omega$. Then $g$ is $C^{1,1}$ in $\bar\Omega \setminus \{\zeta\}$ (that is, $g$ is $C^{1,1}$ in $\Omega \setminus \{\zeta\}$ and the second derivative of $g$ is bounded near $\partial\Omega$).*

An example given in [1] shows that $g$ need not be $C^2$ smooth up to the boundary. It remains an open problem if, in that example, $g \notin C^2(\Omega \setminus \{p\})$.

In [4], Guan claimed to prove the $C^{1,\alpha}$ regularity for every $\alpha < 1$. However, the proof was incomplete because the inequality (3.6) in [4] is false. In a correction to [4], written after I had sent him a preliminary version of this paper (with the proof of Theorem 1), Guan has given a new proof of the $C^{1,\alpha}$ regularity.

Our proof will be based on a construction from [4] of an approximating sequence for $g$ and an idea from [2] used to show $C^{1,1}$ regularity for the solutions of the complex Monge–Ampère equation in a ball (see also [3]).

Using similar methods, one can also characterize domains where the Green function is Lipschitz up to the boundary. We recall that a domain in $\mathbb{C}^n$ is called *hyperconvex* if it admits a bounded PSH exhaustion function.

Theorem 2. *Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$, and let $g$ be the Green function of $\Omega$ with a pole at $\zeta \in \Omega$. Then $g \in C^{0,1}(\bar\Omega \setminus \{\zeta\})$ if and only if there exists $\psi \in \mathrm{PSH}(\Omega)$ with*

$$-C\,\mathrm{dist}(z, \partial\Omega) \le \psi(z) < 0, \quad z \in \Omega,$$

*for some $C > 0$.*

*Proof of Theorem 1.* We may assume that $\zeta = 0$. Choose $\varepsilon > 0$ such that $B_\varepsilon \Subset \Omega$, and set $\Omega_\varepsilon = \Omega \setminus \bar B_\varepsilon$. By [4], there is a sequence of functions $u^\varepsilon \in \mathrm{PSH}(\Omega_\varepsilon) \cap C^\infty(\bar\Omega_\varepsilon)$ which increase locally uniformly to $g$ on $\bar\Omega \setminus \{0\}$ as $\varepsilon \downarrow 0$ and

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which satisfy $u^\varepsilon = 0$ on $\partial\Omega$ and $u^\varepsilon = \log|z| + \psi$ on $\partial B_\varepsilon$, where $\psi$ is smooth in $\bar{\Omega}$ and $\det(u^\varepsilon_{i\bar{j}}) = \varepsilon$. It follows that the tangential derivatives of the second order of $u^\varepsilon$ with respect to $\partial B_\varepsilon$ are bounded; that is,

$$\|\nabla^2(u^\varepsilon|_{\partial B_\varepsilon})\| \le C_1. \tag{1}$$

In addition, it was shown in [4] that the $u^\varepsilon$ satisfy

$$\|\nabla u^\varepsilon\|_{\partial\Omega}, \|\nabla^2 u^\varepsilon\|_{\partial\Omega} \le C_2. \tag{2}$$

Here $C_1$ and $C_2$ are constants depending only on $\Omega$.

Fix $K \Subset \Omega \setminus \{0\}$. By $C_3, C_4, \dots$ we will denote positive constants depending only on $\Omega$ and $K$. We need to show that

$$\|\nabla^2 u^\varepsilon\|_K \le C_3. \tag{3}$$

For $\zeta \in \mathbb{C}^n \setminus \{0\}$ with $|\zeta| = 1$, let $\partial_\zeta$ denote the directional derivative in the direction $\zeta$. Since $u^\varepsilon$ is plurisubharmonic, we have

$$\partial^2_\zeta u^\varepsilon + \partial^2_{i\zeta} u^\varepsilon \ge 0.$$

This easily gives

$$|\nabla^2 u^\varepsilon(a)| = \sup_{|\zeta|=1} \partial^2_\zeta u^\varepsilon(a) = \limsup_{h\to 0} \frac{u^\varepsilon(a+h) + u^\varepsilon(a-h) - 2u^\varepsilon(a)}{|h|^2} \tag{4}$$

for $a \in K$.

We will need a lemma as follows.

LEMMA. *Let* $0 < \varepsilon_0 < r_1 < r_2$ *and* $R > 0$. *Then there exist* $\delta > 0$ *and a* $C^\infty$ *smooth mapping*

$$T : [0, \varepsilon_0] \times (\bar{B}_{r_2} \setminus B_{r_1}) \times \bar{B}_\delta \times \bar{B}_R \longmapsto \mathbb{C}^n$$

($B_r$ *stands for an open ball centered at the origin with radius* $r$) *such that*

$$\begin{aligned} &T(\varepsilon, a, h, \cdot) \text{ is holomorphic in } B_R, \\ &T(\varepsilon, a, h, \cdot) \text{ maps } \partial B_\varepsilon \text{ onto } \partial B_\varepsilon, \\ &T(\varepsilon, a, h, a) = a + h, \\ &T(\varepsilon, a, 0, z) = z. \end{aligned} \tag{5}$$

*Proof*. Let $T(\varepsilon, a, h, \cdot)$ be a holomorphic automorphism of $B_\varepsilon$ (defined, in fact, on $B_R$) of the form $U \circ P$, where

$$P(z) = \varepsilon \frac{\dfrac{\langle z, b\rangle}{|b|^2} b + \sqrt{1 - |b|^2}\left(z - \dfrac{\langle z, b\rangle}{|b|^2} b\right) - \varepsilon b}{\varepsilon - \langle z, b\rangle},$$

$|b| < R/\varepsilon$ (see [6]), and $U$ is a linear orthogonal mapping with

$$P(a) = \frac{|a+h|}{|a|} a, \qquad U\left(\frac{|a+h|}{|a|} a\right) = a + h.$$

One can check that the first condition is satisfied if $b = \varepsilon\alpha a$, where

$$\alpha = \frac{|a+h| - |a|}{|a|(|a+h||a| - \varepsilon^2)}.$$

This gives

$$P(z) = \frac{\frac{\langle z, a\rangle}{|a|^2}a + \sqrt{1-\varepsilon^2\alpha^2|a|^2}\left(z - \frac{\langle z, a\rangle}{|a|^2}a\right) - \varepsilon^2\alpha a}{1 - \alpha\langle z, a\rangle}.$$

The existence of an appriopriate $U$, depending smoothly on $a$ and $h$ and in fact independent of $\varepsilon$, is clear. □

*Proof of Theorem 1 (cont.).* Let $\Omega'$ and $\Omega''$ be domains such that $K \Subset \Omega' \Subset \Omega'' \Subset \Omega$. We will use the foregoing lemma with $r_1, r_2$ and $R$ such that $K \subset \bar{B}_{r_2} \setminus B_{r_1}$ and $\Omega \subset B_R$. For $z \in \overline{\Omega''}$ and $h, \varepsilon$ small enough, set

$$v(z) := u^\varepsilon(T(\varepsilon, a, h, z)) + u^\varepsilon(T(\varepsilon, a, -h, z))$$

so that it is well-defined and $v(a) = u^\varepsilon(a+h) + u^\varepsilon(a-h)$.

A Taylor expansion about the origin of an arbitrary smooth function $f$ gives

$$f(h) + f(-h) = 2f(0) + \tfrac{1}{2}(\nabla^2 f(h') + \nabla^2 f(h'')) \cdot h^2$$

for some $h' \in [0, h]$ and $h'' \in [0, -h]$. Therefore, by (1) and (2),

$$v(z) \le 2u^\varepsilon(z) + C_4|h|^2, \quad z \in \partial B_\varepsilon. \tag{6}$$

On the other hand,

$$v(z) \le 2u^\varepsilon(z) + \tilde{C}|h|^2, \quad z \in \partial\Omega'', \tag{7}$$

where

$$\tilde{C} = \sup_{|h'| \le |h|, z \in \partial\Omega''} |\nabla_h^2(u^\varepsilon \circ T)(\varepsilon, a, h', z)|.$$

It follows that

$$\tilde{C} \le C_5(\|\nabla^2 u^\varepsilon\|_{\bar{\Omega}\setminus\Omega'} + \|\nabla u^\varepsilon\|^2_{\bar{\Omega}\setminus\Omega'}) \tag{8}$$

for $h$ small enough. Since the mapping $A \mapsto (\det A)^{1/n}$ is superadditive on the set of positive hermitian matrices, we have

$$\begin{aligned}(\det(v_{i\bar{j}}))^{1/n} &\ge \varepsilon^{1/n}\left(|JacT(\varepsilon, a, h, \cdot)|^{2/n} + |JacT(\varepsilon, a, -h, \cdot)|^{2/n}\right)\\ &\ge \varepsilon^{1/n}(2 - C_6|h|^2).\end{aligned} \tag{9}$$

Let $M > 0$ be such that $|z|^2 - M \le 0$ for $z \in \Omega$, and define

$$w(z) = v(z) - \max\{C_4, \tilde{C}\}|h|^2 + \varepsilon^{1/n}C_6|h|^2(|z|^2 - M).$$

Then $w$ is PSH in $\Omega''$, $w \le 2u^\varepsilon$ on $\partial B_\varepsilon \cup \partial\Omega''$ by (6) and (7), and $\det(w_{i\bar{j}}) \ge 2^n\varepsilon$ in $\Omega''$ by (9). The comparison principle (see e.g. [2]) now implies that $w \le 2u^\varepsilon$ in $\Omega''$. In particular, $w(a) \le 2u^\varepsilon(a)$, and this coupled with (4) and (8) gives

$$|\nabla^2 u^\varepsilon(a)| \le C_7(\|\nabla^2 u^\varepsilon\|_{\bar\Omega\setminus\Omega'} + \|\nabla u^\varepsilon\|^2_{\bar\Omega\setminus\Omega'}) + C_8.$$

Since $\Omega'$ can be chosen to be arbitrarily close to $\Omega$, (3) follows thanks to (2). □

*Proof of Theorem 2.* The "only if" part is obvious. Assume again that $\zeta = 0$ and fix $K \Subset \Omega \setminus \{0\}$. Let $r > 0$ be such that $B_r \Subset \Omega$. For $0 < \varepsilon < r$, define

$$u^\varepsilon := \sup\{v \in \mathrm{PSH}(\Omega) : v < 0,\ v|_{B_\varepsilon} \le \log(\varepsilon/r)\}.$$

Then one can easily show that $u^\varepsilon \in \mathrm{PSH}(\Omega) \cap C(\bar\Omega)$, $u^\varepsilon = 0$ on $\partial\Omega$, $u^\varepsilon = \log(\varepsilon/r)$ on $\bar B_\varepsilon$, and $u^\varepsilon \downarrow g$ as $\varepsilon \downarrow 0$ (see e.g. [5]). Since $g$ is a maximal PSH function near $\partial\Omega$, we may assume that

$$u^\varepsilon \ge g \ge \psi \ \text{ near } \partial\Omega. \tag{10}$$

For $a \in K$, $\varepsilon$ as before, and $h$ small enough, define

$$\Omega' = \{z \in \Omega : T(\varepsilon, a, h, z) \in \Omega\}.$$

By (10) and the assumption on $\psi$ we have

$$u^\varepsilon(z) \ge \psi(z) \ge -C\,\mathrm{dist}(z, \partial\Omega) \ge -C'|h|, \quad z \in \partial\Omega',$$

where $C'$ depends only on $K$ and $\Omega$. Hence, for $z \in \partial\Omega'$ we have

$$u^\varepsilon(T(\varepsilon, a, h, z)) \le 0 \le u^\varepsilon(z) + C'|h|.$$

Since $u^\varepsilon$ is maximal on $\Omega' \setminus \bar B_\varepsilon$, (1) gives

$$u^\varepsilon(T(\varepsilon, a, h, z)) \le u^\varepsilon(z) + C'|h|, \quad z \in \Omega'.$$

Thus, if $z = a$ for $a \in K$ and $|h| < \delta$, where $\delta$ depends only on $K$ and $\Omega$, we have

$$u^\varepsilon(a + h) \le u^\varepsilon(a) + C'|h|$$

and the theorem follows. □

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Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków
Poland

blocki@im.uj.edu.pl