1. Basic definitions and properties

**Bergman kernel.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ (we will assume it throughout, unless otherwise stated). By $H^2(\Omega)$ we will denote the space $L^2$-integrable holomorphic functions in $\Omega$. For such an $f$ the function $|f|^2$ is in particular sub-harmonic and thus for $B(z, r) \subset \Omega$

$$|f(z)|^2 \leq \frac{1}{\lambda(B(z, r))} \int_{B(z, r)} |f|^2 d\lambda.$$ 

Therefore

(1.1) $$|f(z)| \leq \frac{c_n}{(\text{dist}(z, \partial \Omega))^n} ||f||$$

and

$$\sup_K |f| \leq C(K, \Omega) ||f||, \quad K \in \Omega,$$

where by $||f||$ we denote the $L^2$-norm of $f$. It follows that the $L^2$-convergence in $H^2(\Omega)$ implies locally uniform convergence, and thus $H^2(\Omega)$ is a closed subspace of $L^2(\Omega)$.

Hence, $H^2(\Omega)$ is a separable Hilbert space with the scalar product

$$\langle f, g \rangle = \int_{\Omega} f \bar{g} \, d\lambda.$$ 

By (1.1), for a fixed $w \in \Omega$, the functional

$$H^2(\Omega) \ni f \mapsto f(w) \in \mathbb{C}$$
is continuous. Therefore there is a unique element in $H^2(\Omega)$, which we denote by $K_{\Omega}(\cdot, w)$, such that

$$f(w) = \langle f, K_{\Omega}(\cdot, w) \rangle,$$

or equivalently

$$f(w) = \int_{\Omega} f(z)\overline{K(z, w)} \, d\lambda(z),$$

for every $f \in H^2(\Omega)$. The function

$$K_{\Omega} : \Omega \times \Omega \rightarrow \mathbb{C}$$

is called the Bergman kernel for the domain $\Omega$.

In particular, for $f = K_{\Omega}(\cdot, z)$ we get

$$K_{\Omega}(w, z) = \langle K_{\Omega}(\cdot, z), K_{\Omega}(\cdot, w) \rangle = \overline{K(z, w)}.$$

It follows that $K_{\Omega}(z, w)$ is holomorphic in $z$ and antiholomorphic in $w$. By the Hartogs theorem on separate analyticity the function $K_{\Omega}(\cdot, \cdot)$ is holomorphic (where it is defined) and therefore in particular $K_{\Omega} \in C^\infty(\Omega \times \Omega)$.

If $F : \Omega \rightarrow D$ is a biholomorphism then the mapping

$$H^2(D) \ni f \mapsto f \circ F \circ \text{Jac} F \in H^2(\Omega)$$

is an isomorphism of the Hilbert spaces and

$$f(F(w)) = \int_D f K_D(\cdot, F(w)) \, d\lambda = \int_{\Omega} f \circ F K_D(\cdot, F(w)) \circ F |\text{Jac} F|^2 \, d\lambda.$$

Therefore

$$(1.2) \quad K_{\Omega}(z, w) = K_D(F(z), F(w)) \text{Jac} F(z) \overline{\text{Jac} F(w)}.$$

**Example.** In the unit disc $\Delta$ we have

$$f(0) = \frac{1}{\pi r^2} \int_{\Delta(0, r)} f \, d\lambda, \quad f \in H^2(\Delta), \ r < 1.$$

Therefore

$$f(0) = \frac{1}{\pi} \int_{\Delta} f \, d\lambda,$$

that is

$$K_\Delta(\cdot, 0) = \frac{1}{\pi}.$$

For arbitrary $w \in \Delta$ we use automorphisms of $\Delta$

$$T_w(z) = \frac{z - w}{1 - wz},$$

so that $T_w^{-1} = T_{-w}$ and

$$T_w'(z) = \frac{1 - |w|^2}{(1 - zw)^2}.$$
Then by (1.2)

\[ K_\Delta(z, w) = K_\Delta(T_w(z), 0) = K_\Delta(T_{w'}(z), T_{w'}(w)) = \frac{1}{\pi(1 - z\bar{w})^2}. \]

More generally, for the unit ball \( B \) in \( \mathbb{C}^n \), we similarly have

\[ K_B(z, 0) = \frac{1}{\lambda_n}, \]

where \( \lambda_n = \lambda(B) = \pi^n/n! \). For \( w \in B \) we can use the automorphism of \( B \)

\[ T_w(z) = \frac{\langle z, w \rangle - 1}{1 - \langle z, w \rangle} \]

where \( s_w = \sqrt{1 - |w|^2} \) (see e.g. [Ru]). Then \( T_w^{-1} = T_{-w} \) and

\[ \text{Jac} T_w(z) = (1 - |w|^2)^{(n+1)/2} \]

\[ (1 - \langle z, w \rangle)^{n+1}. \]

Therefore

\[ K_B(z, w) = \frac{1}{\lambda_n} \text{Jac} T_w(z) \text{Jac} T_w(w) = \frac{n!}{\pi^n(1 - \langle z, w \rangle)^{n+1}}. \]

If \( \{\phi_k\} \) is an orthonormal system in \( H^2(\Omega) \) then

\[ f = \sum_k \langle f, \phi_k \rangle \phi_k, \quad f \in H^2(\Omega), \]

and the convergence is also locally uniform. Therefore

\[ K_\Omega(z, w) = \sum_k \langle K_\Omega(\cdot, w), \phi_k(\cdot) \phi_k(z) = \sum_k \phi_k(z)\overline{\phi_k(w)} \]

and

\[ K_\Omega(z, z) = \sum_k |\phi_k(z)|^2. \]

**Exercise 1.** Find an orthonormal system for \( H^2(\mathbb{B}) \) and use it to compute in another way the Bergman kernel for \( \mathbb{B} \).

**Example.** For the annulus \( P = \{ r < |\zeta| < 1 \} \) we have for \( j, k \in \mathbb{Z} \)

\[ \langle \zeta^j, \zeta^k \rangle = \int_0^{2\pi} e^{i(j-k)t} dt \int_r^1 \rho^{j+k+1} d\rho = \begin{cases} 0, & j \neq k \\ \frac{\pi}{j+k}(1 - r^{2j+2}), & j = k \neq -1 \\ -2\pi \log r, & j = k = -1. \end{cases} \]

Therefore \( \{\zeta^j\}_{j \in \mathbb{Z}} \) is an orthogonal system and we will get

\[ (1.3) \quad K_\mathbb{P}(z, w) = \frac{1}{\pi zw} \left( \frac{1}{2 \log(1/r)} + \sum_{j \in \mathbb{Z}} \frac{j(z\bar{w})^j}{1 - r^{2j}} \right). \]
More examples can be obtained from the product formula:

\[ K_{\Omega_1 \times \Omega_2}(z^1, z^2, (w^1, w^2)) = K_{\Omega_1}(z^1, w^1) K_{\Omega_2}(z^2, w^2) \]

which easily follows directly from the definition (here \( \Omega_1 \subset C^n \) and \( \Omega_2 \subset C^m \)).

On the diagonal we have

\[ K_{\Omega}(z, z) = ||K_{\Omega}(\cdot, z)||^2 = \text{sup}\{|f(z)|^2 : f \in H^2(\Omega), ||f|| \leq 1\}. \]

It follows that \( \log K_{\Omega}(z, z) \) is a smooth plurisubharmonic function in \( \Omega \). We will show below that in fact it is strongly plurisubharmonic.

**Bergman metric.** By \( B_{\Omega}^2 \) we will denote the Levi form of \( \log K_{\Omega}(z, z) \), that is

\[ B_{\Omega}^2(z; X) := \lim_{\zeta \to 0} \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} \log K_{\Omega}(z + \zeta X, z + \zeta X) \]

\[ = \sum_{j, k=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K_{\Omega}(z, z) \]

\[ X_j \bar{X}_k, \quad z \in \Omega, \quad X \in C^n. \]

**Theorem 1.1.** We have

\[ B_{\Omega}(z; X) = \frac{1}{\sqrt{K_{\Omega}(z, z)}} \sup\{|f_X(z)| : f \in H^2(\Omega), ||f|| \leq 1, f(z) = 0\}, \]

where

\[ f_X = \sum_{j=1}^n \frac{\partial f}{\partial z_j} X_j. \]

**Proof.** Fix \( z_0 \in \Omega, \quad X \in C^n \) and set \( H := H^2(\Omega), \)

\[ H' := \{ f \in H : f(z_0) = 0 \} \]

\[ H'' := \{ f \in H' : f_X(z_0) = 0 \}. \]

Then \( H'' \subset H' \subset H \) and in both cases the codimension is 1 (note in particular that \( \langle \cdot, -z_0, X \rangle \in H'' \setminus H' \)). Let \( \phi_0, \phi_1, \ldots \) be an orthonormal system in \( H \) such that \( \phi_1 \in H' \) and \( \phi_k \in H'' \) for \( k \geq 2 \). Since \( k_{\Omega} = \sum_{p \geq 0} |\phi_k|^2 \), we have

\[ B^2_{\Omega}(z, X) = (\sum_{p} |\phi_p|^2)^{-1} \sum_p |\phi_{p, X}|^2 - (\sum_{p} |\phi_p|^2)^{-2} \sum_p \phi_{p, X} \bar{\phi}_p \phi_p \]

Therefore

\[ K_{\Omega}(z_0, z_0) = |\phi_0(z_0)|^2, \quad B^2_{\Omega}(z_0, X) = |\phi_{1, X}(z_0)|^2 / |\phi_0(z_0)|^2. \]

This gives \( \leq \). For the reverse inequality take \( f \in H' \) with \( ||f|| \leq 1 \). Then \( \langle f, \phi_0 \rangle = 0 \) and

\[ f = \sum_{p \geq 1} \langle f, \phi_p \rangle \phi_p. \]
Therefore
\[ |f_X(z_0)| = |\langle f, \phi_1 \rangle \phi_1(X(z_0))| \leq |\phi_1(X(z_0))| \]
and the result follows. \( □ \)

It follows that \( B_\Omega(z; X) > 0 \) and hence \( \log k_\Omega \) is strongly plurisubharmonic. It is thus a potential of a Kähler metric which we call the Bergman metric. Length of a curve \( \gamma \in C^1([0, 1], \Omega) \) in this metric is given by
\[
l(\gamma) = \int_0^1 B_\Omega(\gamma(t), \gamma'(t)) \, dt
\]
and the Bergman distance by
\[
\text{dist}_{B_\Omega}(z, w) = \inf \{ l(\gamma) : \gamma \in C^1([0, 1], \Omega), \gamma(0) = z, \gamma(1) = w \}.
\]

If \( F : \Omega \to D \) is a biholomorphism then
\[
B_\Omega(z; X) = B_D(F(z); F'(z) X)
\]
and
\[
\text{dist}_{B_\Omega}(z, w) = \text{dist}_{B_D}(F(z), F(w)),
\]
that is the Bergman metric is biholomorphically invariant.

**Kobayashi’s construction.** Define a mapping
\[
\iota : \Omega \ni w \mapsto [K_\Omega(\cdot, w)] \in \mathbb{P}(L^2(\Omega)).
\]
It is well defined since \( K_\Omega(\cdot, w) \neq 0 \). One can easily show that \( \iota \) is one-to-one.

For any Hilbert space \( H \) one can define the Fubini-Study metric on \( \mathbb{P}(H) \) as follows:
\[
FS_{\mathbb{P}(H)} := \pi^* \mathbb{P}, \quad \pi : H_* \ni f \mapsto [f] \in \mathbb{P}(H),
\]
where
\[
H_* = H \setminus \{0\}
\]
and
\[
P^2(f; F) := \left( \frac{\partial^2}{\partial \zeta \partial \zeta} \log \|f + \zeta F\|^2 \right)_{\zeta=0} = \frac{\|F\|^2}{\|f\|^2} - \frac{\langle F, f \rangle^2}{\|f\|^4}, \quad f \in H_*, \quad F \in H.
\]
One can show that \( FS_{\mathbb{P}(H)} \) is well defined.

We have the following result of Kobayashi [K]:

**Theorem 1.2.** \( B_\Omega = \iota^* FS_{\mathbb{P}(L^2(\Omega))} \).

**Proof.** We have to show that \( B_\Omega = A^* P \), where
\[
A : \Omega \ni w \mapsto K_\Omega(\cdot, w) \in L^2(\Omega),
\]
that is that \( B_\Omega(w; X) = P(f; F) \), where \( f = K_\Omega(\cdot, w) \) and \( F = D_X K_\Omega(\cdot, w) \) with \( D_X \) being the derivative in direction \( X \in C^n \) w.r.t. \( w \). Let \( \phi_0, \phi_1, \ldots \) be an orthonormal system chosen as in the proof of Theorem 1.1. Then
\[
f = \phi_0(w)\phi_0, \quad F = \phi_{0, X}(w)\phi_0 + \phi_{1, X}(w)\phi_1
\]
and one can easily show that
\[ P^2(f; F) = \frac{|\phi_{1,X}(z_0)|^2}{|\phi_{0}(z_0)|^2} = B_{1\Omega}(w; X) \]
by the proof of Theorem 1.1. \( \square \)

The mapping \( \iota \) embeds \( \Omega \) equipped with the Bergman metric into infinitely dimensional manifold \( \mathbb{P}(H^2(\Omega)) \) equipped with the Fubini-Study metric. In particular, it must be distance decreasing. Since the distance in \( \mathbb{P}(H) \) is given by
\[ d([f], [g]) = \arccos \frac{|(f, g)|}{||f|| \cdot ||g||}, \]
we have thus obtained the following:

**Theorem 1.3.** \( \text{dist}_{\Omega}^B(z, w) \geq \arccos \frac{|K_\Omega(z, w)|}{\sqrt{K_\Omega(z, z)K_\Omega(w, w)}}. \) \( \quad \square \)

**Corollary 1.4.** If \( K_\Omega(z, w) = 0 \) then \( \text{dist}_{\Omega}^B(z, w) \geq \pi/2. \)

The constant \( \pi/2 \) in Corollary 1.4 turns out to be optimal, it was shown for the annulus in \([\text{Di}2]\).

**Curvature.** The sectional curvature of the Bergman metric is given by
\[ R_\Omega(z; X) := -\frac{(\log B)_{\zeta\bar{\zeta}}}{B}|_{\zeta=0}, \quad z \in \Omega, \ X \in \mathbb{C}^n, \]
where \( B(\zeta) = B_{1\Omega}^2(z + \zeta X; X). \)

**Theorem 1.5.** We have
\[ R_\Omega(z; X) = 2 - \sup \{|f_{XX}(z)|^2 : f \in H^2(\Omega), \ ||f|| \leq 1, \ f(z) = 0, \ f_X(z) = 0\}. \]

**Proof.** Fix \( z_0 \in \Omega, \ X \in \mathbb{C}^n \) and let \( \phi_0, \phi_1, \ldots \) be as in the proof of Theorem 1.1, satisfying in addition that \( \phi_k \in H^m \) for \( k \geq 3. \) Denoting \( K(\zeta) := K_\Omega(z + \zeta X) \) we will get
\[ -\frac{(\log(\log K))_{\zeta\bar{\zeta}}}{(\log K)_{\zeta\bar{\zeta}}} = 2 - \frac{(\log(KK_{\zeta\bar{\zeta}} - |K_{\zeta}\bar{\zeta}|^2))_{\zeta\bar{\zeta}}}{(\log K)_{\zeta\bar{\zeta}}}
= 2 - \frac{KK_{\zeta\zeta\bar{\zeta}} - |K_{\zeta\zeta}\bar{\zeta}|^2}{K^2((\log K)_{\zeta\zeta})^2} + \frac{|KK_{\zeta\zeta\bar{\zeta}} - K_{\zeta}\bar{\zeta}K_{\zeta\zeta}|^2}{K^4((\log K)_{\zeta\zeta})^3}. \]

Denoting \( \varphi_p(\zeta) = \phi_p(z + \zeta X) \) we have \( K = \sum_{p \geq 0} |\varphi_p|^2 \) and, for \( \zeta = 0, \)
\[ K = |\varphi_0|^2, \quad K_\zeta = \varphi_0'\bar{\varphi}_0, \quad K_{\zeta\zeta} = |\varphi_0'|^2 + |\varphi_1'|^2, \quad K_{\zeta\zeta\zeta} = \varphi_0''\bar{\varphi}_0, \]
\[ K_{\zeta\bar{\zeta}} = \varphi_0''\bar{\varphi}_0 + \varphi_1''\bar{\varphi}_1, \quad K_{\zeta\zeta\bar{\zeta}} = |\varphi_0'|^2 + |\varphi_1'|^2 + |\varphi_2'|^2. \]

We will get, for \( \zeta = 0, \)
\[ K_\Omega(z_0, z_0) = |\varphi_0|^2, \quad B_{1\Omega}^2(z_0; X) = \frac{|\varphi_1'|^2}{|\varphi_0|^2}, \quad R_\Omega(z_0; X) = 2 - \frac{|\varphi_0|^2|\varphi_2'|^2}{|\varphi_1'|^4}. \]

We thus obtain \( \leq \) and the reverse inequality can be obtained the same way as in the proof of Theorem 1.1. \( \square \)

We conclude in particular that always \( R_\Omega(z; X) < 2. \) This estimate is in fact optimal, as can be shown for the annulus \( \{r < |\zeta| < 1\} \) with \( r \to 0, \) see \([\text{Di}1]\) (and a simplification in \([\text{Z2}2]\)).

The following result will be useful:
Theorem 1.6. Assume that $\Omega_j$ is a sequence of domains increasing to $\Omega$ (that is $\Omega_j \subset \Omega_{j+1}$ and $\sum_j \Omega_j = \Omega$). Then we have locally uniform convergences $K_{\Omega_j} \to K_\Omega$ (in $\Omega \times \Omega$), $B_{\Omega_j}(\cdot, X) \to B_\Omega(\cdot, X)$, $R_{\Omega_j}(\cdot, X) \to R_\Omega(\cdot, X)$ (in $\Omega$), for every $X \in \mathbb{C}^n$.

Proof. It is enough to prove the first convergence as the other will then be a consequence of it using the following elementary result: if $h_j$ is a sequence of harmonic functions converging locally uniformly to $h$ then $D^\alpha h_j \to D^\alpha h$ locally uniformly for any multi-index $\alpha$.

For $\Omega' \Subset \Omega$ by the Schwarz inequality for $j$ sufficiently big we have

$$ |K_{\Omega_j}(z, w)|^2 \leq K_{\Omega_j}(z, z)K_{\Omega_j}(w, w) \leq K_{\Omega'}(z, z)K_{\Omega'}(w, w), \quad z, w \in \Omega', $$

and thus the sequence $K_{\Omega_j}$ is locally uniformly bounded in $\Omega \times \Omega$. By the Montel theorem (applied to holomorphic functions $K_{\Omega_j}(\cdot, \cdot)$) there is a subsequence of $K_{\Omega_j}$ converging locally uniformly. Therefore, to conclude the proof it is enough to show that if $K_{\Omega_j} \to K$ locally uniformly then $K = K_{\Omega}$.

Fix $w \in \Omega$. We have

$$ \|K(\cdot, w)\|_{L^2(\Omega')}^2 = \lim_{j \to \infty} \|K_{\Omega_j}(\cdot, w)\|_{L^2(\Omega')}^2 \leq \liminf_{j \to \infty} \|K_{\Omega_j}(\cdot, w)\|_{L^2(\Omega_j)}^2 = \liminf_{j \to \infty} K_{\Omega_j}(w, w) = K(w, w). $$

Therefore $\|K(\cdot, w)\|^2 \leq K(w, w)$, in particular $K(\cdot, w) \in H^2(\Omega)$ and it remains to show that for any $f \in H^2(\Omega)$

$$ f(w) = \int_{\Omega} f \overline{K(\cdot, w)} d\lambda. $$

For $j$ big enough we have

$$ f(w) - \int_{\Omega_j} f \overline{K_{\Omega_j}(\cdot, w)} d\lambda = \int_{\Omega_j} f \overline{K_{\Omega_j}(\cdot, w)} d\lambda - \int_{\Omega} f \overline{K(\cdot, w)} d\lambda = \int_{\Omega'} f (K_{\Omega_j}(\cdot, w) - K(\cdot, w)) d\lambda + \int_{\Omega_j \setminus \Omega'} f \overline{K_{\Omega_j}(\cdot, w)} d\lambda - \int_{\Omega_j \setminus \Omega'} f \overline{K(\cdot, w)} d\lambda. $$

The first integral converges to 0, whereas the other two are arbitrarily small if $\Omega'$ is chosen to be sufficiently close to $\Omega$. □

2. The one dimensional case

We assume that $\Omega$ is a bounded domain in $\mathbb{C}$. We first show that in this case the Bergman kernel can be obtained as a solution of the Dirichlet problem:
Theorem 2.1. Assume that $\Omega$ is regular. Then for $w \in \Omega$ we have
\[ K_\Omega(\cdot, w) = \frac{\partial v}{\partial z}, \]
where $v$ is a complex-valued harmonic function in $\Omega$, continuous on $\bar{\Omega}$, such that
\[ v(z) = \frac{1}{\pi(z - w)}, \quad z \in \partial \Omega. \]

Proof. We have to show that for $f \in H^2(\Omega)
\[ f(w) = \int_\Omega f \bar{v} d\lambda. \]
By Theorem 1.6 we may assume that $\partial \Omega$ is smooth and $f$ is defined in a neighborhood of $\Omega$. Then we have
\[ \int_\Omega f \bar{v} d\lambda = -\frac{i}{2} \int_\Omega d(f \bar{v} dz) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z)}{z - w} dz = f(w). \]

The Green function of $\Omega$ with pole at $w \in \Omega$ can be defined as
\[ G_\Omega(\cdot, w) := \sup\{ v \in SH^- (\Omega) : \lim_{\zeta \to w} (v(\zeta) - \log |\zeta - w|) < \infty \}. \]
Then $G_\Omega(\cdot, w)$ is a negative subharmonic function in $\Omega$ such that $G_\Omega(z, w) - \log |z - w|$ is harmonic in $z$. The Green function $G_\Omega$ is symmetric. If $\Omega$ is regular then $G_\Omega(\cdot, w)$ is continuous on $\bar{\Omega} \setminus \{w\}$ and vanishes on $\partial \Omega$.

We have the following relation due to Schiffer:

Theorem 2.2. Away from the diagonal of $\Omega \times \Omega$ we have
\[ K_\Omega = 2 \frac{\partial^2 G_\Omega}{\pi \partial z \partial \bar{w}}. \]

Proof. We may assume that $\partial \Omega$ is smooth. The function
\[ \psi(z, w) := G_\Omega(z, w) - \log |z - w| \]
is then smooth in $\Omega \times \Omega$. For a fixed $w_0 \in \Omega$ set
\[ u := \frac{\partial \psi}{\partial \bar{w}}(\cdot, w_0). \]
Then $u$ is harmonic in $\Omega$, continuous on $\bar{\Omega}$ and
\[ u(z) = \frac{1}{2(z - w)}, \quad z \in \partial \Omega. \]
Therefore by Theorem 2.1
\[ K_\Omega(\cdot, w_0) = \frac{2}{\pi} \frac{\partial u}{\partial z} = \frac{2}{\pi} \frac{\partial^2 G_\Omega(\cdot, w_0)}{\partial z \partial \bar{w}}. \]

On the diagonal we have the following formula due to Suita [Su]:

**Theorem 2.3.** We have
\[ K_\Omega(z, z) = \frac{1}{\pi} \frac{\partial^2 \rho_\Omega}{\partial z \partial \bar{z}}, \]
where
\[ \rho_\Omega(w) = \lim_{z \to w} (G_\Omega(z, w) - \log |z - w|) \]
is the Robin function for \( \Omega \).

**Proof.** This in fact follows easily from the previous result: we have
\[ \rho_\Omega(\zeta) = \psi(\zeta, \zeta), \]
where \( \psi \) is as in the proof of Theorem 2.2. We will get
\[ \frac{\partial^2 \rho_\Omega}{\partial \zeta \partial \bar{\zeta}} = \psi_{zz} + 2 \psi_{zw} + \psi_{w\bar{w}}. \]
The result now follows from Theorem 2.2, since \( \psi \) is harmonic in both \( z \) and \( w \). \( \square \)

**Suita metric.** Assume for a moment that \( M \) is a Riemann surface such that the Green function \( G_M \) exists. (This is equivalent to the existence of a nonconstant bounded subharmonic function on \( M \).) Then for \( w \in M \) the Robin function
\[ \rho_M(w) = \lim_{z \to w} (G_M(z, w) - \log |z - w|) \]
is ambiguously defined: it depends on the choice of local coordinates. In fact, if change local coordinates by \( z = f(\zeta) \), where \( f \) is a local biholomorphism with \( f(w) = w \), then it is easy to check that
\[ \rho_M(w) = \tilde{\rho}_M(w) + \log |f'(w)|, \]
where \( \tilde{\rho}_M(w) \) is the Robin constant w.r.t. the new coordinates. It follows that the metric
\[ e^{\rho_M} |dz| \]
is invariantly defined on \( M \), we call it the **Suita metric**.

We will analyze the curvature of the Suita metric:
\[ S_M := K_{e^{\rho_M} |dz|} = -\frac{2}{e^{2\rho_M}}(\rho_M)_{zz}, \]
which is of course also invariantly defined. Coming back to the case when \( \Omega \) is a bounded domain in \( \mathbb{C} \), by Theorem 2.3 we have
\[ S_\Omega(z) = -2\pi \frac{K_\Omega(z, z)}{e^{2\rho_\Omega(z)}}. \]
Exercise 3. i) Show that if $F : \Omega \to D$ is a biholomorphism then
\[ \rho_\Omega = \rho_D \circ F + \log |F'|. \]

ii) Prove that if $\Omega$ is simply connected then $S_\Omega \equiv -2$.

iii) Set $D := \Delta \cap \Delta(1, r)$. For $w \in D$ let $F_w : D \to \Delta$ be biholomorphic and such that $F_w(w) = w$. Show that
\[ \lim_{w \to 1 \atop w \not\in D} |F'_w(w)| = 1. \]

iv) Prove that if $\Omega$ has a $C^2$ boundary then
\[ \lim_{z \to \partial \Omega} S_\Omega(z) = -2. \]

The case of annulus is less trivial and we have the following result of Suita [Su]:

Theorem 2.4. For the annulus $P = \{ r < |\zeta| < 1 \}$ we have $S_P < -2$ in $P$.

To prove this we will use the theory of elliptic functions.

3. Weierstrass elliptic functions

For $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$, let $\Lambda := \{ 2j\omega_1 + 2k\omega_2 : (j, k) \in \mathbb{Z}^2 \}$ be the lattice in $\mathbb{C}$. We define the Weierstrass elliptic function $\mathcal{P}$ by
\[ \mathcal{P}(z) = \mathcal{P}(z; \omega_1, \omega_2) := \frac{1}{z^2} + \sum_{\omega \in \Lambda} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right), \]

Since
\[ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} = \frac{-z^2 + 2\omega z}{\omega^2(z - \omega)^2} = O(|\omega|^{-3}), \]

it follows that $\mathcal{P}$ is holomorphic in $\mathbb{C} \setminus \Lambda$. From
\[ \frac{1}{(z - \omega)^2} + \frac{1}{(z + \omega)^2} = 2 \frac{z^2 + \omega^2}{(z^2 - \omega^2)^2}, \]

it follows that
\[ \mathcal{P}(-z) = \mathcal{P}(z). \]

We further have
\[ \mathcal{P}'(-z) = -\mathcal{P}'(z), \]
\[ \mathcal{P}'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z - \omega)^3}, \]

so that
\[ \mathcal{P}'(z + 2\omega_1) = \mathcal{P}'(z) = \mathcal{P}'(z + 2\omega_2). \]

It follows that $\mathcal{P}(z + 2\omega_1) = \mathcal{P}(z) + A$ for some constant $A$, but since $\mathcal{P}(-\omega_1) = \mathcal{P}(\omega_1)$, we have in fact $A = 0$, that is
\[ \mathcal{P}(z + 2\omega_1) = \mathcal{P}(z) = \mathcal{P}(z + 2\omega_2). \]
The differential equation for $\mathcal{P}$. Write
$$\mathcal{P} = z^{-2} + az^2 + bz^4 + O(|z|^6)$$
and
$$\mathcal{P}' = -2z^{-3} + 2az + 4bz^3 + O(|z|^5).$$
Then
$$\mathcal{P}^3 = \left(z^{-2} + az^2 + bz^4\right)^3 + O(|z|^2) = z^{-6} + 3az^{-2} + 3b + O(|z|^2)$$
and
$$(\mathcal{P}')^2 = \left(-2z^{-3} + 2az + 4bz^3\right)^2 + O(|z|^2) = 4z^{-6} - 8az^{-2} - 16b + O(|z|^2).$$
Therefore
$$(\mathcal{P}')^2 - 4\mathcal{P}^3 + 20a\mathcal{P} + 28b = O(|z|^2).$$
The left-hand side is an entire holomorphic function with periods $2\omega_1$ and $2\omega_2$. It is thus bounded and hence, by the Liouville theorem, constant. We thus obtained the following result:

**Theorem 3.1.** We have
$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3,$$
where
$$g_2 = 60\sum_{\omega \in \Lambda_*} \frac{1}{\omega^4}, \quad g_3 = 140\sum_{\omega \in \Lambda_*} \frac{1}{\omega^6}. \quad \square$$

**Remark.** The function $\mathcal{P}$ can be also defined using the constants $g_2, g_3$ instead of the half-periods $\omega_1, \omega_2$ by the relation
$$z = \int_{\mathcal{P}(z)}^{\infty} \frac{1}{\sqrt{4t^3 - g_2t - g_3}} \, dt.$$

The Weierstrass function $\zeta$ is determined by
$$\zeta' = -\mathcal{P}, \quad \zeta(z) = \frac{1}{z} + O(|z|).$$
One can easily compute that
$$\zeta(z) = \frac{1}{z} - \sum_{\omega \in \Lambda_*} \left( \frac{1}{z - \omega} + \frac{z}{\omega^2} + \frac{1}{\omega} \right).$$
Again, adding any pair from $\Lambda_*$ with opposite signs we easily get
$$\zeta(-z) = -\zeta(z).$$
Since $\zeta'(z + 2\omega_1) = \zeta'(z) = \zeta'(z + 2\omega_2)$, we have
$$(3.1) \quad \zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1, \quad \zeta(z + 2\omega_2) = \zeta(z) + 2\eta_2$$
where $\eta_1 = \zeta(\omega_1), \, \eta_2 = \zeta(\omega_2)$. 

Exercise 4. Show that
\begin{equation}
\eta_1 \omega_2 - \eta_2 \omega_1 = \frac{\pi i}{2}.
\end{equation}

We can also define the Weierstrass elliptic function \( \sigma \) by
\[ \frac{\sigma'}{\sigma} = \zeta, \quad \sigma(z) = z + O(|z|^2). \]

One can easily show that
\[ \sigma(z) = z \prod_{\omega \in \Lambda} \left( 1 - \frac{z}{\omega} \right) \exp \left( \frac{z}{\omega} + \frac{z^2}{2\omega^2} \right). \]

It follows that
\[ \sigma(-z) = -\sigma(z). \]

From the definition of \( \sigma \) and from (3.1) we infer \( \sigma(z + 2\omega_1) = Be^{2n\pi z} \sigma(z) \) for some constant \( B \). Substituting \( z = -\omega_1 \) we will get \( B = -e^{2\eta_1 \omega_1} \), so that
\[ \sigma(z + 2\omega_1) = -e^{2\eta_1 (z + \omega_1)} \sigma(z), \]
and, similarly,
\[ \sigma(z + 2\omega_2) = -e^{2\eta_2 (z + \omega_1)} \sigma(z). \]

The following formula will allow to express \( \rho_P \), where \( P \) is an annulus, in terms of \( \sigma \).

\textbf{Theorem 3.2.} Assume that \( \text{Im} (\omega_2/\omega_1) > 0 \). Then
\begin{equation}
\sigma(z) = \frac{2\omega_1}{\pi} \exp \frac{\eta_1 z^2}{2\omega_1} \sin \frac{\pi z}{2\omega_1} \prod_{n=1}^{\infty} \frac{\cos(2n\pi \omega_2/\omega_1) - \cos(\pi z/\omega_1)}{\cos(2n\pi \omega_2/\omega_1) - 1}
\end{equation}
and
\begin{equation}
\rho(z) = -\frac{\eta_1}{\omega_1} + \frac{\pi^2}{4\omega_1^2} \sum_{j \in \mathbb{Z}} \sin^{-2} \frac{\pi (z + 2j\omega_2)}{2\omega_1}.
\end{equation}

\textbf{Proof.} On one hand we have
\begin{equation}
\frac{\cos(2n\pi \omega_2/\omega_1) - \cos(\pi z/\omega_1)}{\cos(2n\pi \omega_2/\omega_1) - 1} = 1 - 2q^{2n} \cos \frac{\pi z}{\omega_1} + q^{4n},
\end{equation}
where \( q := \exp(\pi i \omega_2/\omega_1) \). Since \( |q| < 1 \), it follows that the infinite product is convergent. On the other hand,
\begin{equation}
1 - 2q^{2n} \cos \frac{\pi z}{\omega_1} + q^{4n} = 4q^{2n} \sin \frac{\pi (z + 2n\omega_2)}{2\omega_1} \sin \frac{\pi (z - 2n\omega_2)}{2\omega_1}.
\end{equation}
It is clear that its inverses defined in a neighborhood of the interval \((r, \rho)\), to finish the proof of (3.3) it is therefore enough to show that

\[
\tilde{\sigma}(z + 2 \omega_2) = -e^{2 \eta_2(z + \omega_2)} \tilde{\sigma}(z)
\]

and use the Liouville theorem for the function \(\sigma / \tilde{\sigma}\). We have, denoting \(A = \exp(\pi i z/2 \omega_1)\) and using (3.2)

\[
\frac{\tilde{\sigma}(z + 2 \omega_2)}{\tilde{\sigma}(z)} = \exp \frac{2 \eta_1 \omega_2(z + \omega_2)}{\omega_1} \lim_{N \to \infty} \sin \frac{\pi(z + 2(N+1)\omega_2)}{2 \omega_1} \sin \frac{\pi(z - 2N\omega_2)}{2 \omega_1}
\]

and thus (3.3) follows.

To prove (3.4) it is enough to combine (3.3) with (3.5) and (3.6) plus the fact that \(P = -(\log \sigma)''\).

**Proof of Theorem 2.4.** We first want to express \(\rho_P\) in terms of \(\sigma\). By Myrberg’s theorem we have

\[
G_{\Omega}(z, w) = \sum_j \log \left| \frac{\varphi_j(w) - \varphi_j(z)}{1 - \varphi_j(w) \varphi_j(z)} \right|,
\]

where \(\varphi_j = (p|_{V_j})^{-1}\), \(p : \Delta \to \Omega\) is a covering, \(p^{-1}(U) = \bigcup_j V_j\), \(U\) is a small neighborhood of \(w\), \(V_j\) are disjoint and \(\varphi_0(w) \in V_0\). Then

\[
\rho_{\Omega} = \log \frac{|\varphi_j'|}{1 - |\varphi_0|^2} + \sum_{j \neq 0} \log \left| \frac{\varphi_j - \varphi_0}{1 - \varphi_0 \varphi_j} \right|.
\]

For \(\Omega = P\) we can take a covering \(\Delta \to P\) given by

\[
p(\zeta) = \exp \left( \frac{\log r}{\pi i} \log \left( \frac{1 + \zeta}{1 - \zeta} \right) \right).
\]

Its inverses defined in a neighborhood of the interval \((r, 1)\) are given by

\[
\varphi_j(z) = \frac{e^{\pi i (\log r + 2j \pi i)}/ \log r - i}{e^{\pi i (\log r + 2j \pi i)}/ \log r + i}, \quad j \in \mathbb{Z},
\]

It is clear that \(\rho_P(z)\) depends only on \(|z|\). We will get

\[
e^{-\rho_P(z)} = \frac{2 |z| \log(1/r)}{\pi} \sin \frac{\pi \log |z|}{\log r} \prod_{n=1}^{\infty} \cosh \frac{2 \pi n}{\log r} - \cos \frac{2 \pi \log |z|}{\log r},
\]

Now choose \(\omega_1 = -\log r\) and \(\omega_2 = \pi i\). By Theorem 3.2 we will obtain

\[
\rho_P(z) = \frac{t}{2} - \log \sigma(t) + \frac{c}{2} t^2 =: \gamma(t),
\]
where $t = -2 \log |z| \in (0, 2\omega_1)$ and $c = \eta_1 / \omega_1$. By Theorem 2.3

\begin{equation}
K_P(z, z) = \frac{\gamma''}{\pi |z|^2} = \frac{1}{\pi} (\mathcal{P} + c)e^t.
\end{equation}

Combining this with (1.3)

\begin{equation}
\mathcal{P}(t) = \frac{1}{2\omega_1} - c + \sum_{j=\infty}^{\infty} \frac{je^{-jt}}{1 - r^{2j}}.
\end{equation}

One can easily check that $\mathcal{P}(0) = \infty$ and $\mathcal{P}$ decreases in $(0, \omega_1)$. We also have $\mathcal{P}(2\omega_1 - t) = \mathcal{P}(t)$ and $\mathcal{P}'(\omega_1) = 0$. Set

\[ F := \log \frac{\pi K_P}{e^{2\mathcal{P}'}} = \log(\mathcal{P} + c) + 2 \log \sigma - ct^2. \]

Then $F(2\omega_1 - t) = F(t)$ and

\[ F' = \frac{\mathcal{P}'}{\mathcal{P} + c} + 2\zeta - 2ct. \]

Since $\mathcal{P} = t^{-2} + O(t^2)$, $\zeta = t^{-1} + O(t)$, we get $F'(0) = 0$. We also have $F'(\omega_1) = 0$. Theorem 3.1 gives $(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2 \mathcal{P} - g_3$, and thus $\mathcal{P}'' = 6\mathcal{P}^2 - g_2/2$. Therefore

\begin{equation}
(3.9)
F'' = \frac{(g_2 - 12c^2)\mathcal{P} - cg_2 + 2g_3 - 4c^3}{2(\mathcal{P} + c)^2}.
\end{equation}

By (3.8) $\mathcal{P} + c > 0$. We also have $F(0) = 0$ and we claim that

\begin{equation}
(3.10)
F(\omega_1) > 0.
\end{equation}

This will finish the proof because from (3.9) and $F'(0) = F'(\omega_1) = 0$ we will conclude that $F''$ has precisely one zero in $(0, \omega_1)$ and thus $F' > 0$ there. It thus remains to show (3.10).

Using (3.7) we may write

\[ \gamma = \log \frac{\pi}{2\omega_1} + \frac{t}{2} - \log \sin \frac{\pi t}{2\omega_1} + \log \prod_{n=1}^{\infty} \frac{a_n - 1}{a_n - \cos(\pi t/\omega_1)}, \]

where $a_n = \cosh(2\pi^2 n/\omega_1)$. Then

\begin{equation}
(3.11)
\gamma'' = \frac{\pi^2}{4\omega_1^2 \sin^2(\pi t/2\omega_1)} + \frac{\pi^2}{\omega_1^2} \sum_{n=1}^{\infty} \frac{1 - a_n \cos(\pi t/\omega_1)}{(a_n - \cos(\pi t/\omega_1))^2}
\end{equation}

and

\[ F = \log \gamma'' + t - 2\gamma \]

\[ = \log \left( 1 + 4 \sin^2 \frac{\pi t}{2\omega_1} \sum_{n=1}^{\infty} \frac{1 - a_n \cos(\pi t/\omega_1)}{(a_n - \cos(\pi t/\omega_1))^2} \right) + 2 \sum_{n=1}^{\infty} \log \frac{a_n - \cos(\pi t/\omega_1)}{a_n - 1}.
\]
We will obtain
\[ F(\omega_1) = \log \left( 1 + 4 \sum_{n=1}^{\infty} \frac{1}{a_n + 1} \right) + 2 \sum_{n=1}^{\infty} \log \frac{a_n + 1}{a_n - 1} > 0. \]

In the proof of Theorem 2.4 we showed in particular that
\[ K_P(z, z) = \frac{1}{\pi |z|^2} \left( \mathcal{P}(2 \log |z|) + \frac{\eta_1}{\omega_1} \right), \]
where \( \mathcal{P} \) is the Weierstrass function with half-periods \( \omega_1 = -\log r \) and \( \omega_2 = \pi i \). In fact, we can show a similar formula also away from the diagonal and characterize precisely the zeros of \( K_P \) (compare with [R] and [Sk]):

**Theorem 3.4.** We have
\[ K_P(z, w) = \frac{h(z\bar{w})}{\pi z\bar{w}}, \]
where
\[ (3.12) \quad h(\lambda) = \mathcal{P}(\log \lambda) + \frac{\eta_1}{\omega_1}. \]

The function \( h \) has exactly two simple zeros in the annulus \( \{ r^2 < |\lambda| < 1 \} \), both on the interval \( (-r^2, -1) \).

**Proof.** Let \( \varphi_j \) be as in the proof of Theorem 2.4. After some calculations we will get
\[ G_P(z, w) = \sum_{j \in \mathbb{Z}} \log \left| \frac{1 - f_j(w/z)}{1 - f_j(z\bar{w})} \right|, \]
where
\[ f_j(\zeta) = \exp \left( \frac{\pi i \text{Log} \zeta + 2j\pi i}{\log r} \right). \]

By Theorem 2.2 we will get (also after some calculations)
\[ K_P(z, w) = -\frac{\pi}{\lambda \log^2 r} \sum_{j \in \mathbb{Z}} \frac{f_j(\lambda)}{(1 - f_j(\lambda))^2}, \]
where \( \lambda = z\bar{w} \). Since
\[ \frac{e^\alpha}{(1 - e^\alpha)^2} = -\frac{1}{4 \sin^2(\alpha/2)}, \]
we will get
\[ (3.13) \quad h(\lambda) = \frac{\pi^2}{4 \log^2 r} \sum_{j \in \mathbb{Z}} \sin^{-2} \frac{\pi(\text{Log} \lambda + 2j\pi i)}{2 \log r} \]
and (3.12) follows from Theorem 3.2.

By (1.3) we have
\[ h(\lambda) = \frac{1}{2\omega_1} + \sum_{j \in \mathbb{Z}} \frac{j\lambda^j}{1 - r^{2j}}. \]
It follows in particular that \( h \) is real-valued for real \( \lambda \) and that \( h(r^2/\lambda) = h(\lambda) \). We also have \( f_j(-r) = -q^{-(2j+1)} \) and \( f_j(-1) = q^{-(2j+1)} \), where \( q = e^{\pi^2/\log r} \). Therefore by (3.13)

\[
\begin{align*}
\frac{\pi^2}{\log^2 r} \sum_{j \in \mathbb{Z}} \frac{q^{2j+1}}{(1 + q^{2j+1})^2} & > 0, \\
h(-r) &= \pi^2 \log^2 r \sum_{j \in \mathbb{Z}} \frac{q^{2j+1}}{(1 + q^{2j+1})^2} < 0.
\end{align*}
\]

This implies that there are two simple zeros on the interval \((-1, -r^2)\). The following result guarantees that there are no more than two in the annulus \( \{r^2 < |\lambda| < 1\} \):

**Proposition 3.5.** In the parallelogram \( \{2t\omega_1 + 2s\omega_2 : s, t \in [0, 1)\} \) the Weierstrass function \( P \) attains every value exactly twice (counting with multiplicities).

**Proof.** For any complex number \( w \) let \( C \) be an oriented contour given by the boundary of this parallelogram moved slightly, so that it doesn’t contain neither zeros nor poles of \( P - w \). Then

\[
\frac{1}{2\pi i} \int_C \frac{P'(z)}{P(z) - w} dz = Z - P,
\]

where \( Z \) is the number of zeros an \( P \) the number of poles of \( P \) inside \( C \). We have \( P = 2 \) because \( P \) has precisely one double pole inside \( C \). On the other hand, since the function under the sign of integration is doubly periodic with periods \( 2\omega_1 \) and \( 2\omega_2 \), it follows easily that the integral must vanish. \( \square \)

4. Suita conjecture

The Suita conjecture \([Su]\) asserts that \( S_{\Omega} \leq -2 \), that is that

\[
e^{2\rho_\Omega(z)} \leq \pi K_\Omega(z, z).
\]

By approximation it is enough to prove the estimate for domains with smooth boundary. The conjecture is still open. Ohsawa \([O]\) showed, using the theory of the \( \bar{\partial} \)-equation, that

\[
e^{2\rho_\Omega(z)} \leq 750\pi K_\Omega(z, z).
\]

We want to prove the following improvement from \([Bl3]\):

**Theorem 4.1.** We have

\[
e^{2\rho_\Omega(z)} \leq 2\pi K_\Omega(z, z),
\]

that is \( S_{\Omega} \leq -1 \).

We may assume that \( \Omega \) has smooth boundary. We will use the weighted \( \bar{\partial} \)-Neumann operator and an approach of Berndtsson \([B1]\). Denote

\[
\partial_\alpha = \frac{\partial}{\partial z}, \quad \bar{\partial}_\alpha = \frac{\partial}{\partial \bar{z}}.
\]
If $\phi$ is smooth in $\bar{\Omega}$ then the formal adjoint to $\bar{\partial}$ with respect to the scalar product in $L^2(\Omega, e^{-\phi})$ is given by

$$\bar{\partial}^* \alpha = -e^\phi \partial(e^{-\phi} \alpha) = -\partial \alpha + \alpha \partial \phi.$$  

The complex Laplacian in $L^2(\Omega, e^{-\phi})$ is defined by

$$\Box \alpha = -\bar{\partial} \bar{\partial}^* \alpha = \partial \bar{\partial} \alpha - \partial \phi \bar{\partial} \alpha - \alpha \partial \bar{\partial} \phi.$$  

The following formula relating $\Box$ to the standard Laplacian can be proved by direct computation:

**Proposition 4.2.**

$$\partial \bar{\partial}(|\alpha|^2 e^{-\phi}) = (2\text{Re} (\bar{\alpha} \Box \alpha) + |\bar{\partial} \alpha|^2 + |\bar{\partial}^* \alpha|^2 + |\alpha|^2 |\partial \bar{\partial} \phi|) e^{-\phi}. \quad \Box$$

We may assume that $0 \in \Omega$. If $\phi$ is subharmonic (which we assume from now on) then by PDEs we can find $N \in C^\infty(\bar{\Omega} \setminus \{0\}) \cap L^1(\Omega)$ such that

$$\Box N = \frac{\pi}{2} e^{\phi(0)} \delta_0, \quad N = 0 \text{ on } \partial \Omega.$$  

(The constant $\pi/2$ is chosen so that $N = G$, where $G = G_\Omega(\cdot, 0)$, if $\phi \equiv 0$.)

The key in the proof of Theorem 4.1 will be the following estimate of Berndtsson [B1]:

**Theorem 4.3.** $|N|^2 \leq e^{\phi+\phi(0)} G^2$.

**Proof.** Set

$$u := |\alpha|^2 e^{-\phi} + \varepsilon.$$  

Then

$$|\partial u| = |\alpha \bar{\partial} \alpha + \bar{\alpha} \partial^* \alpha| e^{-\phi} \leq |\alpha|(|\bar{\partial} \alpha| + |\partial^* \alpha|) e^{-\phi}$$

and by Proposition 4.2

$$\partial \bar{\partial} (u^{1/2}) = \frac{1}{2} u^{-1/2} \partial \bar{\partial} \bar{\partial} u - \frac{1}{4} u^{-3/2} |\partial u|^2 \geq \frac{1}{2} u^{-3/2} |\alpha|^2 \left[ 2\text{Re} (\bar{\alpha} \Box \alpha) + |\bar{\partial} \alpha|^2 + |\partial^* \alpha|^2 - \frac{1}{2} (|\bar{\partial} \alpha| + |\partial^* \alpha|)^2 \right] e^{-2\phi} \geq -u^{-3/2} |\alpha|^3 e^{-2\phi} |\Box \alpha| \geq -|\Box \alpha| e^{-\phi}/2.$$  

Now approximating $N$ by smooth functions and letting $\varepsilon \to 0$ we will get

$$\partial \bar{\partial} \left(-|N| e^{-(\phi+\phi(0))}/2\right) \leq \frac{\pi}{2} \delta_0 = \partial \bar{\partial} G$$

and the theorem follows. \quad \Box

**Proof of Theorem 4.1.** Set

$$\phi := 2(\log |z| - G).$$
Then $\varphi$ is harmonic in $\Omega$, smooth on $\bar{\Omega}$ and

$$\varphi(0) = -2\rho_\Omega(0).$$

For harmonic weights the operators $\bar{\partial}$ and its adjoint commute

$$\square = -\bar{\partial} \bar{\partial}^* = -\bar{\partial}^* \bar{\partial}.$$ 

Therefore

$$\bar{\partial}(e^{-\varphi} \partial N) = \partial(-e^{-\varphi(0)} \bar{\partial}^* N) = \frac{\pi}{2} \delta_0.$$ 

It follows that the function

$$f := ze^{-\varphi} \partial N$$

is holomorphic in $\Omega$, smooth on $\bar{\Omega}$, and, since $\bar{\partial}(2f/z - 1/z) = 0$, $f(0) = 1/2$.

Using the fact that both $|N|^2 e^{-\varphi}$ and its derivative vanish on $\partial \Omega$, integration by parts and Proposition 4.1 give

$$\int_\Omega |N|^2 e^{-\varphi} \partial \bar{\partial}(|z|^2 e^{-\varphi})d\lambda = \int_\Omega |z|^2 (|\bar{\partial} N|^2 + |\partial^* N|^2) e^{-2\varphi} d\lambda \geq \int_\Omega |f|^2 d\lambda.$$ 

On the other hand, we have $|z|^2 e^{-\varphi} = e^{2G}$ and by Theorem 4.3

$$\int_\Omega |N|^2 e^{-\varphi} \partial \bar{\partial}(|z|^2 e^{-\varphi})d\lambda \leq e^{\varphi(0)} \int_\Omega G^2 \partial \bar{\partial} e^{2G} d\lambda.$$ 

We need the following simple lemma.

**Lemma 4.4.** For every integrable $\gamma : (-\infty, 0) \to \mathbb{R}$ we have

$$\int_\Omega \gamma \circ G |\nabla G|^2 d\lambda = 2\pi \int_{-\infty}^0 \gamma(t) dt.$$ 

**Proof.** Let $\chi : (-\infty, 0) \to \mathbb{R}$ be such that $\chi' = \gamma$ and $\chi(-\infty) = 0$. Then

$$\int_\Omega \gamma \circ G |\nabla G|^2 d\lambda = \int_\Omega \langle \nabla (\chi \circ G), \nabla G \rangle d\lambda = \int_{\partial \Omega} \chi(0) \frac{\partial G}{\partial n} d\sigma = 2\pi \chi(0). \qquad \square$$

**End of proof of Theorem 4.1.** It follows that

$$\int_\Omega G^2 \partial \bar{\partial} e^{2G} d\lambda = \int_\Omega G^2 e^{2G} |\nabla G|^2 d\lambda = \frac{\pi}{2}$$

and thus

$$\int_\Omega |f|^2 d\lambda \leq \frac{\pi}{2} e^{\varphi(0)},$$

from which the required estimate immediately follows. \quad \square
5. Hörmander’s $L^2$-estimate for the $\bar{\partial}$-equation

We will first sketch the classical theory of the $\bar{\partial}$-equation from [Hö] in the special case $p = q = 0$, namely we consider the equation

$$\bar{\partial}u = \alpha,$$

where

$$\alpha = \sum_{j=1}^{n} \alpha_j d\bar{z}_j$$

is a $(0,1)$-form satisfying the necessary condition

$$\bar{\partial} \alpha = 0.$$

We will first show how to slightly modify the proof of Lemma 4.4.1 in [Hö] to obtain the following slight improvement:

**Theorem 5.1.** Assume that $\Omega$ is a pseudoconvex domain in $\mathbb{C}^n$ (not necessarily bounded). Let $\varphi$ be a $C^2$ strongly plurisubharmonic function in $\Omega$ and $\alpha \in L^2_{loc,(0,1)}(\Omega)$ with $\bar{\partial} \alpha = 0$. Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar{\partial} u = \alpha$ and such that

$$\int_{\Omega} |u|^2 e^{-\varphi} d\lambda \leq \int_{\Omega} |\alpha|^2_{i\partial \bar{\partial} \varphi} e^{-\varphi} d\lambda,$$

where

$$|\alpha|^2_{i\partial \bar{\partial} \varphi} = \sum_{j,k=1}^{n} \varphi_j \bar{\alpha}_k \alpha_{jk}$$

is the length of the form $\alpha$ w.r.t. the Kähler metric $i\partial \bar{\partial} \varphi$ (here $(\varphi^{jk})$ is the inverse transposed of $(\partial^2 \varphi/\partial z_j \partial \bar{z}_k)$).

**Sketch of proof.** If the right hand-side of (5.1) is not finite it is enough to apply Theorem 4.2.2. in [Hö], we may thus assume that it is finite and even equal to 1. We follow the proof of Lemma 4.4.1 in [Hö] and its notation: the function $s$ is smooth, strongly plurisubharmonic in $\Omega$ and such that $\Omega_{a} := \{ s < a \} \Subset \Omega$ for every $a \in \mathbb{R}$. We fix $a > 0$ and choose $\eta_{\nu} \in C^\infty(\Omega)$, $\nu = 1, 2, \ldots$, such that $0 \leq \eta_{\nu} \leq 1$ and $\Omega_{a+1} \subset \{ \eta_{\nu} = 1 \} \uparrow \Omega$ as $\nu \uparrow \infty$. Let $\psi \in C^\infty(\Omega)$ vanish in $\Omega_{a}$ and satisfy $|\partial \eta_{\nu}|^2 \leq e^\psi$, $\nu = 1, 2, \ldots$, and let $\chi \in C^\infty(\mathbb{R})$ be convex and such that $\chi = 0$ on $(-\infty, a)$, $\chi \circ s \geq 2 \psi$ and $\chi \circ s |i\partial \bar{\partial} s| \geq (1 + a)|\partial \psi|^2 |i\partial \bar{\partial}| z|^2$. This implies that with $\varphi' := \varphi + \chi \circ s$ we have in particular

$$i\partial \bar{\partial} \varphi' \geq i\partial \bar{\partial} \varphi + (1 + a)|\partial \psi|^2 |i\partial \bar{\partial}| z|^2.$$

The $\bar{\partial}$-operator gives the densely defined operators $T$ and $S$ between Hilbert spaces:

$$L^2(\Omega, \varphi_1) \xrightarrow{T} L^2_{(0,1)}(\Omega, \varphi_2) \xrightarrow{S} L^2_{(0,2)}(\Omega, \varphi_3),$$

where $\varphi_j := \varphi' + (j-3)\psi$, $j = 1, 2, 3$. (Recall that, if

$$F = \sum_{|J|=p} \sum_{|K|=q} F_{JK} dz_J \wedge d\bar{z}_K \in L^2_{loc,(p,q)}(\Omega),$$

then...
then
\[ |F|^2 = \sum_{J,K}' |F_{JK}|^2, \]

\[ L^2_{(p,q)}(\Omega,\varphi) = \{ F \in L^2_{\text{loc}}(\Omega) : ||F||_\varphi^2 := \int_{\Omega} |F|^2 e^{-\varphi} \, d\lambda < \infty \}, \]

\[ \langle F, G \rangle_\varphi := \int_{\Omega} \sum_{J,K}' F_{JK} G_{JK} e^{-\varphi} \, d\lambda, \quad F, G \in L^2_{(p,q)}(\Omega,\varphi). \]

For \( f = \sum_j f_j d z_j \in C^\infty_{0,(0,1)}(\Omega) \) one can then compute

\[ |Sf|^2 = \sum_{j<k} \left| \frac{\partial f_j}{\partial z_k} - \frac{\partial f_k}{\partial z_j} \right|^2 = 2 \sum_{j,k} \frac{\partial f_j}{\partial z_k} \frac{\partial \overline{f}_k}{\partial z_j}, \]

and

\[ e^\psi T^* f = -\sum_j \delta_j f_j - \sum_j f_j \frac{\partial \psi}{\partial z_j}, \]

where
\[ \delta_j w := e^{\psi'} \frac{\partial}{\partial z_j} (we^{-\varphi'}) = \frac{\partial w}{\partial z_j} - w \frac{\partial \varphi'}{\partial z_j}. \]

Therefore

\[ |\sum_j \delta_j f_j|^2 \leq (1 + a^{-1}) e^{2\psi} |T^* f|^2 + (1 + a) |f|^2 |\partial \psi|^2. \]

Integrating by parts we get

\[ \int_{\Omega} \sum_j \delta_j f_j |e^{-\varphi'}|^2 \, d\lambda = \int_{\Omega} \sum_{j,k} \left( \frac{\partial^2 \varphi'}{\partial z_j \partial z_k} f_j \overline{f}_k + \frac{\partial f_j}{\partial z_k} \frac{\partial \overline{f}_k}{\partial z_j} \right) e^{-\varphi'} \, d\lambda. \]

Combining this with (5.2)-(5.4) we arrive at

\[ \int_{\Omega} \sum_{j,k} \frac{\partial^2 \varphi'}{\partial z_j \partial z_k} f_j \overline{f}_k e^{-\varphi'} \, d\lambda \leq (1 + a^{-1}) ||T^* f||_{\varphi_1}^2 + ||Sf||_{\varphi_3}^2. \]

We have

\[ |\sum_j \alpha_j f_j|^2 \leq |\alpha|^2 \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial z_k} f_j \overline{f}_k. \]

Hence, from the Schwarz inequality, (5.5) and from the fact that \( \varphi - 2\varphi_2 \leq -\varphi' \) we obtain

\[ |\langle \alpha, f \rangle_{\varphi_2}|^2 \leq (1 + a^{-1}) ||T^* f||_{\varphi_1}^2 + ||Sf||_{\varphi_3}^2 \]

for all \( f \in C^\infty_{0,(0,1)}(\Omega) \) and thus also for all \( f \in D_{T^*} \cap D_S \) (recall that we have assumed that the right hand-side of (5.1) is 1).
If \( f' \in L^2_{(0,1)}(\Omega, \varphi_2) \) is orthogonal to the kernel of \( S \) then it is also orthogonal to the range of \( T \) and thus \( T^* f' = 0 \). Moreover, since \( S\alpha = 0 \), we then also have \( \langle \alpha, f' \rangle_{\varphi_2} = 0 \). Therefore by (5.7)
\[
|\langle \alpha, f \rangle_{\varphi_2}| \leq \sqrt{1 + a^{-1}} ||T^* f||_{\varphi_1}, \quad f \in D_{T^*}.
\]
By the Hahn-Banach theorem there exists \( u_a \in L^2(\Omega, \varphi_1) \) with \( ||u_a||_{\varphi_1} \leq \sqrt{1 + a^{-1}} \) and
\[
\langle \alpha, f \rangle_{\varphi_2} = \langle u_a, T^* f \rangle_{\varphi_1}, \quad f \in D_{T^*}.
\]
This means that \( Tu_a = \alpha \) and, since \( \varphi_1 \geq \varphi \) with equality in \( \Omega_a \), we have
\[
\int_{\Omega_a} |u_a|^2 e^{-\varphi} d\lambda \leq 1 + a^{-1}.
\]
We may thus find a sequence \( a_j \uparrow \infty \) and \( u \in L^2_{\text{loc}}(\Omega) \) such that \( u_{a_j} \) converges weakly to \( u \) in \( L^2(\Omega_a, \varphi) = L^2(\Omega_a) \) for every \( a \).

It will be convenient to have a version of Theorem 5.1 for nonsmooth \( \varphi \). Note that (5.6) holds pointwise for every \( f \) precisely when
\[
i\partial \alpha \wedge \alpha \leq |\alpha|^2_i\partial \varphi \iota \partial \varphi.
\]
This observation allows to formulate the following generalization of Theorem 5.1:

**Theorem 5.1'.** Assume that \( \Omega \) is pseudoconvex and \( \varphi \) plurisubharmonic in \( \Omega \). Let \( \alpha \in L^2_{\text{loc},(0,1)}(\Omega) \) be such that \( \partial \alpha = 0 \) and
\[
i\alpha \wedge i\varphi \leq h i\partial \varphi \iota \partial \varphi
\]
for some nonnegative function \( h \in L^1_{\text{loc}}(\Omega) \) such that the right hand-side of (5.8) makes sense as a current of order 0 (that is the coefficients of \( h i\partial \varphi \iota \partial \varphi \) are complex measures; this is always the case if \( h \) is locally bounded). Then there exists \( u \in L^2_{\text{loc}}(\Omega) \) with \( \partial u = \alpha \) and
\[
\int_{\Omega} |u|^2 e^{-\varphi} d\lambda \leq \int_{\Omega} h e^{-\varphi} d\lambda.
\]

**Proof.** First assume that \( \varphi \) is strongly plurisubharmonic (but otherwise arbitrary, that is possibly even not locally bounded). By the Radon-Nikodym theorem there exists \( \beta = \sum_{j, k} \beta_{j,k} i dz_j \wedge d\bar{z}_k \in L^1_{\text{loc},(1,1)}(\Omega) \) such that \( 0 < \beta \leq i\partial \varphi \iota \partial \varphi \) and \( i\alpha \wedge \alpha \leq \beta \). For \( \varepsilon > 0 \) let \( a(\varepsilon) \) be such that \( \varphi_{\varepsilon} := \varphi \ast \rho_{\varepsilon} \in C^\infty(\Omega_a(\varepsilon)) \) (where \( \Omega_a \) is as in the proof of Theorem 5.1). Set \( h_{\varepsilon} := |\alpha|^2_{i\partial \varphi_{\varepsilon}} \), so that \( h_{\varepsilon} \) is the least function satisfying \( i\alpha \wedge \alpha \leq h_{\varepsilon} i\partial \varphi_{\varepsilon} \). By Theorem 5.1 we can find \( u_{\varepsilon} \in L^2_{\text{loc}}(\Omega_a(\varepsilon)) \) such that \( \partial u_{\varepsilon} = \alpha \) in \( \Omega_a(\varepsilon) \) and
\[
\int_{\Omega_a(\varepsilon)} |u_{\varepsilon}|^2 e^{-\varphi_{\varepsilon}} d\lambda \leq \int_{\Omega_a(\varepsilon)} h_{\varepsilon} e^{-\varphi_{\varepsilon}} d\lambda \leq \int_{\Omega_a(\varepsilon)} h_{\varepsilon} e^{-\varphi} d\lambda.
\]
We have $\beta := \beta * \rho \leq i\partial\bar{\partial}\varphi$ and the coefficients of $\beta$ converge pointwise almost everywhere to the respective coefficients of $\beta$. Therefore

$$\lim_{\varepsilon \to 0} h_{\varepsilon} \leq \lim_{\varepsilon \to 0} \sum_{j,k} \beta_{jk}^{\varepsilon} \alpha_j \alpha_k = \sum_{j,k} \beta_{jk} \alpha_j \alpha_k \leq h,$$

where $(\beta_{jk})$ and $(\beta_{jk}^{\varepsilon})$ denote the inverse matrices of $(\beta_{jk})$ and $(\beta_{jk} * \rho)$, respectively. By the Fatou lemma we thus have

$$\lim_{\varepsilon \to 0} \int_{\Omega_{a(\varepsilon)}} |u_{\varepsilon}|^2 e^{-\varphi} d\lambda \leq \int_{\Omega} he^{-\varphi} d\lambda.$$

Since $\varphi$ is decreasing as $\varepsilon$ decreases to 0, we see that the $L^2(\Omega_a, \varphi)$ norm of $u_{\varepsilon}$ is bounded for every $\varepsilon \leq \tilde{\varepsilon}$ and fixed $a$ and $\tilde{\varepsilon}$. Therefore, we can find a subsequence $u_{\varepsilon_l}$ converging weakly in $\Omega_a$ for every $a$ to $u \in L^2_{\text{loc}}(\Omega)$. Moreover, for every $\delta > 0$, and $l$ sufficiently big we then have

$$\int_{\Omega_a} |u|^2 e^{-\varphi_{\varepsilon_l}} d\lambda \leq \delta + \int_{\Omega} he^{-\varphi} d\lambda$$

and thus by the Lebesgue monotone convergence theorem we can conclude the proof for strongly plurisubharmonic $\varphi$.

If $\varphi$ is not necessarily strongly plurisubharmonic then we may approximate it by functions of the form $\varphi + \varepsilon |z|^2$. Note that $i\bar{\alpha} \wedge \alpha \leq h i\partial\bar{\partial}(\varphi + \varepsilon |z|^2)$ and the general case easily follows along the same lines as before. □

The next result is due to Berndtsson [B2] (see also [B3]).

**Theorem 5.2.** Let $\Omega$, $\varphi$, $\alpha$ and $h$ be as in Theorem 5.1. Fix $r \in (0,1)$ and assume in addition that $-e^{-\varphi/r} \in \text{PSH}(\Omega)$. Then for any $\psi \in \text{PSH}(\Omega)$ we can find $u \in L^2_{\text{loc}}(\Omega)$ with $\bar{\partial} u = \alpha$ and

$$\int_{\Omega} |u|^2 e^{-\varphi - \psi} d\lambda \leq \frac{1}{(1 - \sqrt{r})^2} \int_{\Omega} he^{-\varphi} d\lambda.$$

**Proof.** Approximating $-e^{-\varphi/r}$ and $\psi$ in the same way as in the proof of Theorem 5.1 we may assume that $\varphi$ and $\psi$ are smooth up to the boundary. Then we have in particular $L^2(\Omega) = L^2(\Omega, a\varphi + b\psi)$ for real $a, b$ and $-e^{-\varphi/r} \in \text{PSH}(\Omega)$ means precisely that

$$i\partial\varphi \wedge \bar{\partial}\varphi \leq r i\partial\bar{\partial}\varphi.$$

Let $u$ be the solution to $\bar{\partial} u = \alpha$ which is minimal in the $L^2(\Omega, \psi)$ norm. This means that

$$\int_{\Omega} u \bar{f} e^{-\psi} d\lambda = 0, \quad f \in H^2(\Omega).$$

Set $v := e^\varphi u$. Then

$$\int_{\Omega} v \bar{f} e^{-\varphi - \psi} d\lambda = 0, \quad f \in H^2(\Omega),$$

thus $v$ is the minimal solution in the $L^2(\Omega, \varphi + \psi)$ norm to $\bar{\partial} v = \beta$, where

$$\beta = \bar{\partial}(e^\varphi u) = e^\varphi (\alpha + u\partial\bar{\partial}\varphi).$$
For every $t > 0$ we have

$$i\beta \wedge \overline{\beta} \leq e^{2\varphi}[(1 + t^{-1})i\alpha \wedge \overline{\alpha} + (1 + t)|u|^2i\partial \overline{\partial} \varphi]$$

$$\leq e^{2\varphi}[(1 + t^{-1})h + (1 + t)r|u|^2i\partial \overline{\partial} \varphi$$

$$\leq e^{2\varphi}[(1 + t^{-1})h + (1 + t)r|u|^2i\partial \overline{\partial}(\varphi + \psi).$$

Therefore by Theorem 5.1

$$\int_{\Omega} |u|^2e^{-\varphi-\psi}d\lambda = \int_{\Omega} |v|^2e^{-\varphi-\psi}d\lambda \leq (1 + t^{-1})\int_{\Omega} he^{-\varphi-\psi}d\lambda + (1 + t)r\int_{\Omega} |u|^2e^{-\varphi-\psi}d\lambda.$$

For $t = r^{-1/2} - 1$ we obtain the required result. □

Applying Theorem 5.2 with $r = 1/4$ and $\varphi, \psi$ replaced with $\varphi/4, \psi + \varphi/4$, respectively, we obtain the following estimate essentially due to Donnelly and Fefferman [DF].

**Theorem 5.3.** Let $\Omega, \varphi, \alpha$ and $h$ satisfy the assumptions of Theorem 5.1'. Assume moreover that $-e^{-\varphi} \in PSH(\Omega)$. Then for any $\psi \in PSH(\Omega)$ we can find $u \in L^2_{loc}(\Omega)$ with $\partial u = \alpha$ and

$$\int_{\Omega} |u|^2e^{-\psi}d\lambda \leq 16 \int_{\Omega} he^{-\psi}d\lambda. \quad \square$$

One can improve the constants in Theorems 5.2 and 5.3 to $4r/(1 - r)^2$ and 4, respectively (see [BH1]).

**Exercise 5.** Let $n = 1$ and $\varphi = -\log(-\log|z|)$. Show that $u = \bar{z}$ is the minimal solution in $L^2(\Delta, \varphi)$ of the equation $\partial u = d\bar{z}$. Prove that

$$\int_{\Delta} |u|^2d\lambda = 2 \int_{\Delta} |\partial u|^2_{i\partial \bar{\partial} \varphi}d\lambda$$

and conclude that the constant in Theorem 5.3 cannot be better than 2.

### 6. Bergman completeness

Domains complete w.r.t. the Bergman metric are called **Bergman complete**.

**Proposition 6.1.** Every Bergman complete domain is pseudoconvex.

**Proof.** If $\Omega$ is not pseudoconvex then by the definition of a domain of holomorhpy there are domains $\Omega_1, \Omega_2$ such that $\emptyset \neq \Omega_1 \subset \Omega \cap \Omega_2, \Omega_2 \not\subset \Omega$ and for every $f$ holomorphic in $\Omega$ there exists $\tilde{f}$ holomorphic in $\Omega_2$ such that $f = \tilde{f}$ on $\Omega_1$. We may assume that $\Omega_1$ is a connected component of $\Omega \cap \Omega_2$ such that the set $\Omega_2 \cap \partial \Omega \cap \partial \Omega_1$ is nonempty. Since $K_\Omega(\cdot, \cdot)$ is holomorphic in $\Omega \times \Omega^*$, it follows that there exists $\tilde{K} \in C^\infty(\Omega_2 \times \Omega_2)$ such that $\tilde{K}(\cdot, \cdot)$ is holomorphic in $\Omega_2 \times \Omega_2$ and $\tilde{K} = K_\Omega$ in $\Omega_1 \times \Omega_1$. This means that every sequence $z_k \to \Omega_2 \cap \partial \Omega \cap \partial \Omega_1$ is a Cauchy sequence with respect to $\text{dist} \, \Omega$, which contradicts the completeness of $\Omega$. □
The converse is not true as the following exercise shows:

**Exercise 6.** Show that every function from $H^2(\Delta \setminus \{0\})$ extends to a function in $H^2(\Delta)$. Conclude that $\Delta \setminus \{0\}$ is not Bergman complete.

The main tool for the Bergman completeness is the following criterion of Koba- yashi [K] (from now on we again assume that $\Omega$ is a bounded domain in $\mathbb{C}^n$):

**Theorem 6.2.** Assume that

$$
\lim_{z \to \partial \Omega} \frac{|f(z)|^2}{K_\Omega(z, z)} = 0, \quad f \in H^2(\Omega).
$$

Then $\Omega$ is Bergman complete.

**Proof.** Let $z_k$ be a Cauchy sequence in $\Omega$ (with respect to the Bergman metric). Suppose that $z_k$ has no accumulation point in $\Omega$. It is easy to check that this is equivalent to the fact that $z_k \to \partial \Omega$. By Theorem 1.2 $\iota(z_k)$ is a Cauchy sequence in $P(H^2(\Omega))$ which is a complete metric space. It follows that there is $f \in H^2(\Omega) \setminus \{0\}$ such that $\iota(z_k) \to \langle f \rangle$. Therefore

$$
\frac{|f(z_k)|^2}{K_\Omega(z_k, z_k)} = \left| \langle f, \frac{K_\Omega(\cdot, z_k)}{K_\Omega(z_k, z_k)} \rangle \right|^2 \to ||f||^2
$$

as $k \to \infty$, which contradicts the assumption of the theorem. □

Zwonek [Z1] (see also [J]) showed that there exists a Bergman complete domain in $\mathbb{C}$ which does not satisfy (6.1). On the other hand, from the above proof it is clear that one can weaken (6.1) to

$$
\limsup_{z \to \partial \Omega} \frac{|f(z)|^2}{K_\Omega(z, z)} < ||f||^2, \quad f \in H^2(\Omega) \setminus \{0\}.
$$

It is not known if there exists a Bergman complete domain not satisfying (6.1').

Similarly as in the one-dimensional case one defines the pluricomplex Green function of $\Omega$ with pole at $w \in \Omega$ as

$$
G_\Omega(\cdot, w) := \sup_{\mathcal{F}_w,}
$$

where

$$
\mathcal{F}_w := \{v \in PSH^-(\Omega) : \limsup_{\zeta \to w} (v(\zeta) - \log |\zeta - w|) < \infty \}.
$$

Then $G_\Omega(\cdot, w) \in \mathcal{F}_w$ but $G_\Omega$ is not symmetric in general. We have the following estimate due to Herbort [H]:

**Theorem 6.3.** For $f \in H^2(\Omega)$ and $w \in \Omega$, where $\Omega$ is pseudoconvex, we have

$$
\frac{|f(w)|^2}{K_\Omega(w, w)} \leq c_n \int_{\{G_\Omega(\cdot, w) < -1\}} |f|^2 d\lambda.
$$

**Proof.** We will use Theorem 5.3 with $\varphi := -\log(-g)$ and $\psi := 2ng$, where $g := G_{\Omega,w}$. Since $g$ is a locally bounded plurisubharmonic function in $\Omega \setminus \{w\}$, it follows that $\partial g \in L^2_{loc,(0,1)}(\Omega \setminus \{w\})$. Set

$$
\alpha := \partial(f \cdot \gamma \circ g) = f \cdot \gamma' \circ g \partial g \in L^2_{loc,(0,1)}(\Omega),
$$

where
where $\gamma \in C^\infty(\mathbb{R})$ is such that $\gamma(t) = 0$ for $t \geq -1$, $\gamma(t) = 1$ for $t \leq -3$ and $-1 \leq \gamma' \leq 0$. We have

$$i\bar{\alpha} \wedge \alpha = |f|^2(\gamma' \circ g)^2 i\partial g \wedge \bar{\partial} g \leq |f|^2(\gamma' \circ g)^2 g^2 i\partial \bar{\partial} \psi.$$ By Theorem 5.3 we can find $u \in L^2_{loc}(\Omega)$ with $\bar{\partial} u = \alpha$ and

$$\int_{\Omega} |u|^2 e^{-2ng}d\lambda \leq 16 \int_{\Omega} |f|^2(\gamma' \circ g)^2 g^2 e^{-2ng}d\lambda.$$ Therefore

$$\|u\|_{L^2(\Omega)} \leq 12 e^{3n}\|f\|_{L^2(\{g < -1\})}.$$ The function $f \cdot \gamma \circ g - u$ is equal almost everywhere to a holomorphic $\tilde{f}$. Moreover, since $e^{-\varphi}$ is not locally integrable near $w$ it follows that $\tilde{f}(w) = f(w)$. Therefore

$$\frac{|f(w)|}{\sqrt{K_{\Omega}(w, w)}} \leq \|\tilde{f}\| \leq (1 + 12 e^{3n})\|f\|_{L^2(\{g < -1\})}. \quad \square$$

From Theorems 6.2 and 6.3 we easily deduce the following (see [C1], [BP], [H]):

**Corollary 6.4.** If pseudoconvex $\Omega$ satisfies

(6.1) $$\lim_{\lambda \to \partial H} \lambda(\{G_{\Omega}(\cdot, w) < -1\}) = 0$$

then it is Bergman complete. \quad \square

One can show that hyperconvex domains (that is domains admitting bounded plurisubharmonic exhaustion function) satisfy (6.1), and thus are Bergman complete (see [C1], [BP] and [H]).

For $f \equiv 1$ Theorem 6.3 gives

$$K_{\Omega}(w, w) \geq \frac{1}{c_n \lambda(\{G_{\Omega}(\cdot, w) < -1\})}$$

and thus in particular

(6.2) $$\lim_{\lambda \to \partial H} K_{\Omega}(w, w) = \infty$$

for hyperconvex $\Omega$ (this is originally due to Ohsawa [O]).

The following result was proved in [C2]:

**Theorem 6.5.** If $n = 1$ and $\Omega$ satisfies (6.2) then it is Bergman complete.

**Exercise 7.** Using the Hartogs triangle $\{(z, w) \in \mathbb{C}^2 : 0 < |z| < |w| < 1\}$ show that Theorem 6.5 does not hold for $n > 1$.

For the proof of Theorem 6.5 we will need the following:

**Lemma 6.6.** Assume that $f \in H^2(\Omega)$ and let $U \subset B(z_0, r)$ be such that $\Omega \cup U$ is a pseudoconvex domain contained in $B(z_0, R)$. Then there exists $F \in H^2(\Omega \cup U)$ such that

(6.3) $$\|F - f\|_{L^2(\Omega)} \leq (1 + \frac{4}{\log 2})\|f\|_{L^2(\Omega \cap B(z_0, R \sqrt{r/R})}.$$
\textbf{Proof.} Assume for simplicity that \( z_0 = 0 \). We will use Theorem 5.3 with \( \varphi = -\log(-\log(|z|/R)) \), \( \psi = 0 \) and
\[
\alpha = \bar{\partial}(f \gamma \circ \varphi) = f \gamma' \circ \varphi \bar{\partial}\varphi,
\]
where
\[
\gamma(t) = \begin{cases} 0, & t \leq -\log(-\log(r/R)) \\ \frac{t + \log(-\log(r/R))}{\log 2}, & -\log(-\log(r/R)) < t < -\log(-\log(r/R)) + \log 2 \\ 1, & t \geq -\log(-\log(r/R)) + \log 2. \end{cases}
\]
Then \( \gamma \circ \varphi = 0 \) in \( B(0, r) \) and thus \( \alpha \) is well defined in \( \Omega \cup U \). We also have
\[
i\bar{\partial}(\alpha) = |f|^2(\gamma' \circ \varphi)^2 \bar{\partial}\varphi \wedge \bar{\partial}\varphi \leq |f|^2(\gamma' \circ \varphi)^2 i\bar{\partial}\varphi.
\]
From Theorem 5.3 we obtain \( u \) with \( \bar{\partial}u = \alpha \) and
\[
\int_{\Omega \cup U} |u|^2 d\lambda \leq 16 \int_{\Omega} |f|^2(\gamma' \circ \varphi)^2 d\lambda.
\]
For \( F := f \gamma \circ \varphi - u \) the desired estimate now easily follows. \( \square \)

The point in Lemma 6.6 is that \( \Omega \cup U \) is pseudoconvex and that the r.h.s. converges to 0 as \( r \to 0 \). For \( z_0 \in \partial \Omega \) one can always find an appropriate neighborhood basis provided that \( n = 1 \).

\textbf{Proof of Theorem 6.5.} Fix \( f \in H^2(\Omega) \), \( z_0 \in \partial \Omega \) and \( \varepsilon > 0 \). By Lemma 6.6 we can find \( \tilde{f} \in H^2(\Omega) \) which is bounded near \( z_0 \) and such that \( ||\tilde{f} - f|| \leq \varepsilon \). For \( z \in \Omega \) we have
\[
\frac{|f(z)|}{\sqrt{K_\Omega(z, z)}} \leq ||\tilde{f} - f|| + \frac{|\tilde{f}(z)|}{\sqrt{K_\Omega(z, z)}}
\]
and thus by (6.2)
\[
\limsup_{z \to z_0} \frac{|f(z)|}{\sqrt{K_\Omega(z, z)}} \leq \varepsilon.
\]
It is now enough to use Theorem 6.2. \( \square \)

Our next goal is to prove the following relation between the Bergman distance and the Green function from [BI2]:

\textbf{Theorem 6.7.} Assume that \( w_1, w_2 \in \Omega \), where \( \Omega \) is pseudoconvex, are such that \( \{G_\Omega(\cdot, w_1) < -1\} \cap \{G_\Omega(\cdot, w_2) < -1\} = \emptyset \). Then \( \text{dist}_{\Omega}(w_1, w_2) \geq b_n > 0 \).

\textbf{Proof.} Set \( f := K_\Omega(\cdot, w_2)/\sqrt{K_\Omega(w_2, w_2)} \) (so that \( ||f|| = 1 \), \( \varphi := -\log(-G_\Omega(\cdot, w_1)) \) and \( \psi := 2n(G_\Omega(\cdot, w_1) + G_\Omega(\cdot, w_2)) \). Let \( \gamma \in C^\infty(\mathbb{R}) \) be such that \( \gamma = 0 \) for \( t \geq 0 \), \( \gamma = 1 \) for \( t \leq -2 \) and \( -1 \leq \gamma' \leq 0 \). Then by Theorem 5.3 we can find \( u \) with \( \bar{\partial}u = \bar{\partial}(f \gamma \circ \varphi) \) and
\[
\int_{\Omega} |u|^2 e^{-\psi} d\lambda \leq 16 \int_{\Omega} |f|^2(\gamma' \circ \varphi)^2 e^{-\psi} d\lambda \leq 16e^{2n},
\]
where the last inequality follows from the assumption, since on \( \{ \gamma' \circ \varphi \neq 0 \} \subset \{-2 \leq \varphi \leq 0\} \) we have \( \psi \geq -2n(e^2 + 1) \). Therefore \( u(w_1) = u(w_2) = 0 \) and
$F := f \gamma \circ \varphi - u$ is holomorphic with $F(w_1) = f(w_1), F(w_2) = 0$. We also have $\|F\| \leq 1 + 4e^{10n}$.

Note that $\langle F, f \rangle = F(w_2)/\sqrt{K_\Omega(w_2, w_2)} = 0$. We can therefore find an orthonormal basis $\varphi_0, \varphi_1, \ldots$ such that $\varphi_0 = f$ and $\varphi_1 = F/\|F\|$. It follows that

$$K_\Omega(z, z) \geq |f(z)|^2 + \frac{|F(z)|^2}{\|F\|^2}.$$ 

Now by Theorem 1.3

$$\text{dist}_\Omega^\beta(w_1, w_2) \geq \arccos \frac{|F(w_1)|}{\sqrt{K_\Omega(w_1, w_1)}} \geq \arccos \frac{\|F\|}{\sqrt{1 + \|F\|^2}}. \quad \Box$$

7. Ohsawa-Takegoshi extension theorem

The Ohsawa-Takegoshi extension theorem [OT] turned out to be one of the main tools in complex analysis:

**Theorem 7.1.** Let $\Omega$ be a bounded pseudoconvex domain and $H$ a complex hyperplane in $\mathbb{C}^n$. Set $\Omega' := \Omega \cap H$ and assume that $\varphi$ is a plurisubharmonic function in $\Omega$. Then for every holomorphic $f$ in $\Omega'$ there exists a holomorphic $F$ in $\Omega$ such that $F|_{\Omega'} = f$ and

$$\int_\Omega |F|^2 e^{-\varphi} \, d\lambda \leq C \int_{\Omega'} |f|^2 e^{-\varphi'} \, d\lambda',$$

where $\varphi' = \varphi|_{\Omega'}$, $d\lambda'$ is the Lebesgue measure on $\Omega'$ and $C$ depends only on $n$ and the diameter of $\Omega$.

**Sketch of proof.** We follow Berndtsson [B4] (see also [B2]). Without loss of generality we may assume that $H = \{z_1 = 0\}$ and $\Omega \subset \{|z_1| < 1\}$. By approximating $\Omega$ from inside and $\varphi$ from above we may assume that $\Omega$ is a strongly pseudoconvex domain with smooth boundary, $\varphi$ is smooth up to the boundary, and $f$ is defined in a neighborhood of $\overline{\Omega'}$ in $H$. Then it follows that $f$ extends to some holomorphic function in $\Omega$ (we may use Hörmander’s estimate with $\alpha = \overline{\partial}(\chi(z_1) f(z'))$, $\chi = 1$ near 0 but with support sufficiently close to 0, $\varphi = 2 \log |z_1|$ will ensure that $u = 0$ on $H$).

Let $F \in H^2(\Omega, e^{-\varphi}) := \mathcal{O}(\Omega) \cap L^2(\Omega, e^{-\varphi})$ be the function satisfying $F = f$ on $H$ with minimal norm in $L^2(\Omega, e^{-\varphi})$. Then $F$ is perpendicular to functions from $H^2(\Omega, e^{-\varphi})$ vanishing on $H$, and it is thus perpendicular to the space $z_1 H^2(\Omega, e^{-\varphi})$. This means that $\overline{\partial}_1 F$ is perpendicular to $H^2(\Omega, e^{-\varphi})$. Since $(H^2(\Omega, e^{-\varphi}))^\perp = (\text{ker } \overline{\partial})^\perp$ is equal to the range of $\overline{\partial}$, we have $\overline{\partial}^\perp \alpha = \overline{z}_1 F$ for some $\alpha \in L^2_{(0,1)}(\Omega, e^{-\varphi})$. Choose such $\alpha$ with the minimal norm. Then $\alpha$ is perpendicular to ker $\overline{\partial}$, and thus $\overline{\partial}_1 \alpha = 0$. We have

$$\int_\Omega |F|^2 e^{-\varphi} \, d\lambda = \langle F/z_1, \overline{\partial}^\perp \alpha \rangle_{e^{-\varphi}} = \langle \overline{\partial}(F/z_1), \alpha \rangle_{e^{-\varphi}} = \langle F \overline{\partial}(1/z_1), \alpha \rangle_{e^{-\varphi}} = \pi \int_{\Omega'} f \overline{\alpha}_1 e^{-\varphi} \, d\lambda' \leq \pi \left( \int_{\Omega'} |f|^2 e^{-\varphi'} \, d\lambda' \right)^{1/2} \left( \int_{\Omega'} |\overline{\alpha}_1|^2 e^{-\varphi} \, d\lambda' \right)^{1/2}.$$
It is thus enough to estimate \( \int_{\Omega'} |\alpha_1|^2 e^{-\varphi} d\lambda' \). We will use the Bochner-Kodaira technique (terminology of Siu [S2], see [B2] for details). One may compute that
\[
\sum (\alpha_j \bar{\alpha}_k e^{-\varphi})_{jk} = (-2 \text{Re}(\partial \partial^* \alpha \cdot \alpha) + |\partial^* \alpha|^2 + \sum |\alpha_{j,k}|^2 - |\partial \alpha|^2 + \sum \varphi_{j,k} \alpha_j \bar{\alpha}_k) e^{-\varphi}.
\]
Integrating by parts and computing further one can show that for any (sufficiently regular) function \( w \)
\[
\int_{\Omega'} \sum w_{jk} \alpha_j \bar{\alpha}_k e^{-\varphi} d\lambda - \int_{\partial \Omega} \sum \rho_{jk} \alpha_j \bar{\alpha}_k e^{-\varphi} w \frac{d\sigma}{|\partial \rho|}
= \int_{\Omega} \left(-2 \text{Re}(\partial \partial^* \alpha \cdot \alpha) + |\partial^* \alpha|^2 + \sum |\alpha_{j,k}|^2 - |\partial \alpha|^2 + \sum \varphi_{j,k} \alpha_j \bar{\alpha}_k\right) e^{-\varphi} w d\lambda,
\]
where \( \rho \) is a defining function or \( \Omega \). In our case we have \( \partial \alpha = 0, \partial^* \alpha = \bar{z}_1 F \), and if we take negative \( w \) depending only on \( z_1 \), then
\[
\int_{\Omega} w_{11} |\alpha_1|^2 e^{-\varphi} d\lambda \leq -2 \text{Re} \int_{\Omega} F \alpha_1 e^{-\varphi} w d\lambda
\]
(since we may choose plurisubharmonic \( \rho \)). Set
\[
w := 2 \log |z_1| + |z_1|^{2\delta} - 1,
\]
where \( 0 < \delta < 1 \). Then \( w_{11} = \pi \delta_0 + \delta^2 |z_1|^{2\delta - 2} \) and for \( t > 0 \)
\[
\pi \int_{\Omega'} |\alpha_1|^2 e^{-\varphi} d\lambda' + \delta^2 \int_{\Omega} |\alpha_1|^2 |z_1|^{2\delta - 2} e^{-\varphi} d\lambda \leq t \int_{\Omega} |F|^2 e^{-\varphi} d\lambda + \frac{1}{t} \int_{\Omega} |\alpha_1|^2 w^2 e^{-\varphi} d\lambda.
\]
Choosing \( t \) with \( w^2 \leq \delta^2 t |z_1|^{2\delta - 2} \) in \( \{|z_1| \leq 1\} \) and combining this with (7.1) we arrive at
\[
\int_{\Omega} |F|^2 e^{-\varphi} d\lambda \leq t\pi \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.
\]
It is clear that iterating Theorem 7.1 we may take \( H \) to be an arbitrary complex affine subspace in \( \mathbb{C}^n \), even a point.

The original motivation behind [OT] was the following estimate:

**Theorem 7.2.** Assume that \( \Omega \) is a bounded pseudoconvex domain with \( C^2 \) boundary. Then
\[
K_\Omega \geq \frac{1}{C \text{dist} (z, \partial \Omega)^2},
\]
where \( C \) is a constant depending on \( \Omega \).

**Proof.** It follows almost immediately from Theorem 1.1. For let \( r > 0 \) be such that for any \( w \in \partial \Omega \) there exists \( w^* \in \mathbb{C}^n \setminus \Omega \) such that \( \Omega \cap \overline{B}(w^*, r) = \{w\} \). If \( z \in \Omega \),
$w \in \partial \Omega$ is such that \( \text{dist} (z, \partial \Omega) = |z - w| \), and \( w^* \) is as above then \( z, w, \) and \( w^* \) lie on the same line (normal to \( \partial \Omega \) at \( w \)). For the corresponding complex line \( H \) and \( \Omega' = \Omega \cap H \) we obtain

\[
K_{\Omega}(z) \geq \frac{1}{C_{\Omega}} K_{\Omega'}(z) \geq \frac{1}{C_{\Omega}} K_{C \setminus \{0\}}(r + |z - w|) = \frac{r^2}{\pi C_{\Omega} \text{dist} (z, \partial \Omega)^2 (2r + \text{dist} (z, \partial \Omega))^2}.
\]

□

The exponent 2 in (7.2) is optimal (for example it cannot be improved for a domain whose boundary near the origin is given by \(|z_1 - 1| = 0\)). Previously a weaker form of (7.2) was proved by Pflug [P] using Hörmander’s estimate (with arbitrary exponent lower than 2).

**Demailly approximation.** In the proof of Theorem 7.2 we used Theorem 7.1 only with \( \varphi \equiv 0 \). The fact that the weight may be an arbitrary plurisubharmonic function was used by Demailly [D] to introduce a new type of regularization of plurisubharmonic functions: by smooth plurisubharmonic functions with analytic singularities (that is functions that locally can be written in the form \( \log(|f_1|^2 + \cdots + |f_k|^2) + u \), where \( f_1, \ldots, f_k \) are holomorphic and \( u \) is \( C^\infty \) smooth) which have very similar singularities to the initial function. The Demailly approximation turned out to be an important tool in complex geometry, see e.g. [D], [DPS] or [DP]. Demailly [D] presented also a simple proof of the Siu theorem on analyticity of level sets of Lelong numbers of plurisubharmonic functions ([S1], see also [Hö]). As we will see below, the Demailly approximation shows that the Siu theorem follows rather easily from Theorem 7.1 applied when \( H \) is just a point.

Recall that the Lelong number of \( \varphi \in PSH(\Omega) \) at \( z_0 \in \Omega \) is defined by

\[
\nu_\varphi(z_0) = \lim_{z \to z_0} \frac{\varphi(z)}{\log|z - z_0|} = \lim_{r \to 0^+} \frac{\varphi^r(z_0)}{\log r},
\]

where for \( r > 0 \) we use the notation

\[
\varphi^r(z) := \max_{B(z, r)} \varphi, \quad z \in \Omega_r := \{ \delta_\Omega > r \}.
\]

One can show that \( \varphi^r \) is a plurisubharmonic continuous function in \( \Omega_r \), decreasing to \( \varphi \) as \( r \) decreases to 0. Now we are in position to prove a result from [D]:

**Theorem 7.3.** For a plurisubharmonic function \( \varphi \) in a bounded pseudoconvex domain \( \Omega \) in \( \mathbb{C}^n \) and \( m = 1, 2, \ldots \) set

\[
\varphi_m := \frac{1}{2m} \log K_{\Omega,e^{-2m\varphi}} = \frac{1}{2m} \log \sup \{|f|^2 : f \in \mathcal{O}(\Omega), \int_\Omega |f|^2 e^{-2m\varphi} \leq 1 \}.
\]

Then there exist \( C_1, C_2 > 0 \) depending only on \( \Omega \) such that

\[
\varphi - \frac{C_1}{m} \leq \varphi_m \leq \varphi^r + \frac{1}{m} \log \frac{C_2}{r^m} \quad \text{in} \quad \Omega_r.
\]

In particular, \( \varphi_m \to \varphi \) pointwise and in \( L^1_{\text{loc}}(\Omega) \). Moreover,

\[
\nu_\varphi - \frac{n}{m} \leq \nu_{\varphi_m} \leq \nu_\varphi \quad \text{in} \quad \Omega.
\]
Proof. First note that (7.4) is an easy consequence of (7.3): by the first inequality in (7.3) we get
\[ \nu_{\varphi} \leq \nu_{\varphi - C_{1}/m} = \nu_{\varphi}, \]
and by the second one
\[ \varphi^{r}_{m} \leq \varphi^{2r} + \frac{1}{m} \log \frac{C_{2}}{r^{n}}, \]
thus \( \nu_{\varphi} - n/m \leq \nu_{\varphi_{m}}. \)

By Theorem 7.1 for every \( z \in \Omega \) there exists \( f \in \mathcal{O}(\Omega) \) with \( f(z) \neq 0 \) and
\[ \int_{\Omega} |f|^2 e^{-2m \varphi} d\lambda \leq C_{\Omega} |f(z)|^2 e^{-2m \varphi(z)}. \]
We may choose \( f \) so that the right-hand side is equal to 1. Then
\[ \varphi_{m}(z) \geq \frac{1}{m} \log |f(z)| = \varphi(z) - \frac{1}{2m} \log C_{\Omega} \]
and we get the first inequality in (7.3).

To get the second one we observe that for any holomorphic \( f \) the function \( |f|^2 \) is in particular subharmonic and thus for \( z \in \Omega_{r} \)
\[ |f(z)|^2 \leq \frac{1}{\lambda(B(z,r))} \int_{B(z,r)} |f|^2 d\lambda \leq \frac{n!}{\pi^{n/2} r^{n}} e^{2m \varphi(z)} \int_{B(z,r)} |f|^2 e^{-2m \varphi} d\lambda. \]
Taking the logarithm and multiplying by \( 1/(2m) \) we will easily get the second inequality in (7.3).

By (7.4) for any real \( c \) we have
\[ \{ \nu_{\varphi} \geq c \} = \bigcap_{m} \{ \nu_{\varphi_{m}} \geq c - \frac{n}{m} \}. \]
If \( \{ \sigma_{j} \} \) is an orthonormal basis in \( H^{2}(\Omega, e^{-2m \varphi}) \) then
\[ \sum_{j} |\sigma_{j}|^2 = K_{\Omega, e^{-2m \varphi}} = \sum_{j} |\sigma_{j}|^2 \]
and one can show that
\[ \{ \nu_{\varphi_{m}} \geq c - \frac{n}{m} \} = \bigcap_{|\alpha| < m \epsilon - n} \bigcap_{j} \{ \partial^{\alpha} \sigma_{j} = 0 \}. \]
Therefore (7.5) is an analytic subset of \( \Omega \), which gives the Siu theorem [S1]:

**Theorem 7.4.** For any plurisubharmonic function \( \varphi \) and a real number \( c \) the set \( \{ \nu_{\varphi} \geq c \} \) is analytic. \( \square \)

The following sub-additivity property was proved in [DPS]. It also relies on the extension theorem, here however we will be using it for the diagonal of \( \Omega \times \Omega \).

**Theorem 7.5.** With the notation of Theorem 4.1 there exists \( C_{3} > 0 \), depending only on \( \Omega \), such that
\[ (m_{1} + m_{2}) \varphi_{m_{1} + m_{2}} \leq C_{3} + m_{1} \varphi_{m_{1}} + m_{2} \varphi_{m_{2}}. \]
Proof. Take $f \in H^2(\Omega, e^{-2(m_1+m_2)\varphi})$ with norm $\leq 1$. If we embed $\Omega$ in $\Omega \times \Omega$ as the diagonal then by Theorem 7.1 there exists $F$ holomorphic in $\Omega \times \Omega$ such that $F(z, z) = f(z)$, $z \in \Omega$, and

$$
(7.7) \quad \int_{\Omega \times \Omega} |F(z, w)|^2 e^{-2m_1\varphi(z)-2m_2\varphi(w)} d\lambda(z)d\lambda(w) \leq C = C_{\Omega \times \Omega}.
$$

If $\{\sigma_j\}$ is an orthonormal basis in $H^2(\Omega, e^{-2m_1\varphi_m})$ and $\{\sigma'_k\}$ an orthonormal basis in $H^2(\Omega, e^{-2m_1\varphi_m})$ then one can easily check that $\{\sigma_j(z)\sigma_k'(w)\}$ is an orthonormal basis in $H^2(\Omega \times \Omega, e^{-2m_1\varphi_m(z)-2m_2\varphi_m(w)})$. We may write

$$
F(z, w) = \sum_{j,k} c_{jk}\sigma_j(z)\sigma'_k(w)
$$

and by (7.7)

$$
\sum_{j,k} |c_{jk}|^2 \leq C.
$$

Therefore by the Schwarz inequality

$$
|f(z)|^2 = |F(z, z)|^2 \leq C \sum_j |\sigma_j(z)|^2 \sum_k |\sigma'_k(z)|^2 = Ce^{2m_1\varphi_m(z)}e^{2m_2\varphi_m(z)}
$$

(using (7.6)). Since $f$ was arbitrary, the theorem follows with $C_3 = (\log C)/2$. \qed

Corollary 7.6. The sequence $\varphi_{2k} + C_3/2^{k+1}$ is decreasing. \qed

It is an open problem if the whole sequence $\varphi_m$ from Theorem 7.3 (perhaps modified by constants as in Corollary 7.6) is decreasing.

References


W. Rudin, Function theory in the unit ball of $\mathbb{C}^n$, Springer-Verlag, 1980.


