# The Complex Monge–Ampère Equation in Kähler Geometry

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*Course given at CIME Summer School in Pluripotential Theory*
*Cetraro, Italy, July 11-16, 2011*

**Abstract** We will discuss two main cases where the complex Monge–Ampère equation (CMA) is used in Käehler geometry: the Calabi–Yau theorem which boils down to solving nondegenerate CMA on a compact manifold without boundary and Donaldson's problem of existence of geodesics in Mabuchi's space of Käehler metrics which is equivalent to solving homogeneous CMA on a manifold with boundary. At first, we will introduce basic notions of Käehler geometry, then derive the equations corresponding to geometric problems, discuss the continuity method which reduces solving such an equation to a priori estimates, and present some of those estimates. We shall also briefly discuss such geometric problems as Käehler–Einstein metrics and more general metrics of constant scalar curvature.

## 1 Introduction

We present two situations where the complex Monge–Ampère equation (CMA) appears in Kähler geometry: the Calabi conjecture and geodesics in the space of Kähler metrics. In the first case the problem is to construct, in a given Kähler class, a metric with prescribed Ricci curvature. It turns out that this is equivalent to finding a metric with prescribed volume form, and thus to solving nondegenerate CMA on a manifold with no boundary. This was eventually done by Yau [47], building up on earlier work by Calabi, Nirenberg and Aubin. On the other hand, to find a geodesic in a Kähler class (the problem was posed by Donaldson [20]) one has to solve a homogeneous CMA on a manifold with boundary (this was observed independently by Semmes [39] and Donaldson [20]). Existence of weak geodesics was proved

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G. Patrizio et al., *Pluripotential Theory*, Lecture Notes in Mathematics 2075, 95
DOI 10.1007/978-3-642-36421-1_2, 

by Chen [18] but Lempert and Vivas [33] showed recently that these geodesics do not have to be smooth. Their partial regularity is nevertheless of interest from the geometric point of view.

In Sects. 2–6 we discuss mostly geometric aspects, whereas Sects. 7–13 concentrate on the PDE part, mostly a priori estimates. We start with a very elementary introduction to Kähler geometry in Sect. 2, assuming the reader is familiar with Riemannian geometry. The Calabi conjecture and its equivalence to CMA are presented in Sect. 3, where the problem of extremal metrics is also briefly discussed. Basic properties of the Riemannian structure of the space of Kähler metrics (introduced independently by Mabuchi [35] and Donaldson [20]) are presented in Sect. 4. The Aubin–Yau functional and the Mabuchi K-energy as well as relation to constant scalar curvature metrics are discussed there as well. The Lempert–Vivas example is described in Sect. 5. Assuming Sects. 7–13, where appropriate results on CMA are shown, in Sect. 6 we present a theorem due to Chen [18] that a Kähler class with the distance defined by this Riemannian structure is a metric space.

The fundamental results on CMA are formulated in Sect. 7, where also basic uniqueness results as well as the comparison principle are showed. The continuity method, used to prove existence of solutions, is described in Sect. 8. It reduces the problem to a priori estimates. Yau's proof of the $L^\infty$-estimate using Moser's iteration is presented in Sect. 9, whereas Sects. 10–12 deal with the first and second order estimates (Sects. 11–12 are not needed in the empty boundary case, that is in the proof of the Calabi conjecture). Higher order estimates then follow from the general, completely *real* Evans–Krylov theory, this is explained in Sect. 13. A slight novelty of this approach in the proof of Yau's theorem is the use of Theorem 25 below which enables us to use directly this real Evans–Krylov theory, instead of proving its complex version (compare with [10, 40] or [13]).

The author would like to thank the organizers of the CIME school in Pluripotential Theory, Filippo Bracci and John Erik Fornæss, for the invitation and a very good time he had in Cetraro in July 2011.

## 2 Basic Notions of Kähler Geometry

Let $M$ be a complex manifold of dimension $n$ and by $J : TM \to TM$ denote its complex structure. We start with a Hermitian metric $h$ on $M$ and set

$$\langle X, Y\rangle := \operatorname{Re} h(X,Y), \quad \omega(X,Y) := -\operatorname{Im} h(X,Y), \quad X, Y \in TM.$$

Then $\langle\cdot,\cdot\rangle$ is a Riemannian metric on $M$, $\omega$ a real 2-form on $M$ and

$$\langle JX, Y\rangle = \omega(X,Y), \quad \langle JX, JY\rangle = \langle X, Y\rangle. \tag{1}$$

The Riemannian metric $\langle\cdot,\cdot\rangle$ determines unique Levi–Civita connection $\nabla$.

By $T_{\mathbb{C}}M$ denote the complexification of $TM$ (treated as a real space) and extend $J$, $\langle\cdot,\cdot\rangle$, $\omega$, and $\nabla$ to $T_{\mathbb{C}}M$ in a $\mathbb{C}$-linear way. In local coordinates $z^j = x^j + iy^j$ the vector fields $\partial/\partial x^j$, $\partial/\partial y^j$ span TM over $\mathbb{R}$. We also have

$$J(\partial/\partial x_j) = \partial/\partial y_j, \quad J(\partial/\partial y_j) = -\partial/\partial x_j.$$

The vector fields

$$\partial_j := \frac{\partial}{\partial z^j}, \quad \partial_{\bar{j}} := \frac{\partial}{\partial \bar{z}^j},$$

span $T_{\mathbb{C}}M$ over $\mathbb{C}$ and

$$J(\partial_j) = i\partial_j, \quad J(\partial_{\bar{j}}) = -i\partial_{\bar{j}}.$$

Set

$$g_{j\bar{k}} := \langle\partial_j, \partial_{\bar{k}}\rangle \; \big(= \langle\partial_{\bar{k}}, \partial_j\rangle\big).$$

Then $\overline{g_{j\bar{k}}} = g_{k\bar{j}}$ and by (1)

$$\langle\partial_j, \partial_k\rangle = \langle\partial_{\bar{j}}, \partial_{\bar{k}}\rangle = 0.$$

If $X = X^j\partial_j + \bar{X}^j\partial_{\bar{j}}$ then $X \in TM$ and it follows that

$$|X|^2 = 2g_{j\bar{k}}X^j\bar{X}^k,$$

thus $(g_{j\bar{k}}) > 0$. By (1)

$$\omega = ig_{j\bar{k}}dz^j \wedge d\bar{z}^k \tag{2}$$

(we see in particular that $\omega$ is a form of type $(1,1)$).

**Proposition 1.** *For a Hermitian metric $h$ the following are equivalent*

*(i)* $\nabla J = 0$*;*
*(ii)* $d\omega = 0$*;*
*(iii)* $\omega = i\partial\bar{\partial}g$ *locally for some smooth real-valued function $g$.*

*Proof.* (i)⇒(ii) By (1)

$$\begin{aligned} 3\,d\omega(X,Y,Z) &= X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y) \\ &\qquad - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y) \\ &= \langle(\nabla_X J)Y, Z\rangle + \langle(\nabla_Y J)Z, X\rangle + \langle(\nabla_Z J)X, Y\rangle. \end{aligned}$$

(ii)⇒(i) Similarly one can show that

$$3\,d\omega(X,Y,Z) - 3\,d\omega(X,JY,JZ) = 2\,\langle(\nabla_X J)Y, Z\rangle + \langle X, N(Y,JZ)\rangle,$$

where

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

is the Nijenhuis tensor (in our case it vanishes, because $J$ is integrable).

(ii)⇒(iii) Locally we can find a real 1-form $\gamma$ such that $\omega = d\gamma$. We may write $\gamma = \bar{\beta} + \beta$, where $\beta$ is a $(0, 1)$-form. Then, since $d = \partial + \bar{\partial}$,

$$\omega = \partial\bar{\beta} + \partial\beta + \bar{\partial}\bar{\beta} + \bar{\partial}\beta.$$

It follows that $\bar{\partial}\beta = 0$, because $\omega$ is a $(1, 1)$-form. Therefore we can find (locally) a complex-valued, smooth function $f$ with $\beta = \bar{\partial} f$ and

$$\omega = \partial\beta + \bar{\partial}\bar{\beta} = 2i\,\partial\bar{\partial}(\mathrm{Im}\, f).$$

We can thus take $g = 2\mathrm{Im}\, f$.

(iii)⇒(ii) is obvious. □

The metric satisfying equivalent conditions in Proposition 1 is called *Kähler*. It is thus a Hermitian metric on a complex manifold for which the Riemannian structure is compatible with the complex structure. The corresponding form $\omega$ is also called *Kähler*, it is characterized by the following properties: $\omega$ is a smooth, real, positive, closed $(1, 1)$-form.

From now on we will use the lower indices to denote partial differentiation w.r.t. $z^j$ and $\bar{z}^k$, so that for example $\partial^2 g/\partial z^j \partial\bar{z}^k = g_{j\bar{k}}$ and (2) is compatible with $\omega = i\,\partial\bar{\partial} g$.

**Volume form.** Since $\langle \partial_j, \partial_{\bar{k}} \rangle = g_{j\bar{k}}$ and $\langle \partial_j, \partial_k \rangle = 0$, we can easily deduce that

$$\Big\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \Big\rangle = \Big\langle \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k} \Big\rangle = 2\mathrm{Re}\, g_{j\bar{k}}, \quad \Big\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k} \Big\rangle = -\Big\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^j} \Big\rangle = 2\mathrm{Im}\, g_{j\bar{k}}.$$

From this, using the notation $x^{j+n} = y^n$,

$$\sqrt{\det\Big(\Big\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \Big\rangle\Big)_{1 \le j,k \le 2n}} = 2^n \det(g_{j\bar{k}}).$$

It follows that the volume form on $M$ is given by

$$2^n \det(g_{j\bar{k}})\, d\lambda = \frac{\omega^n}{n!},$$

where $d\lambda$ is the Euclidean volume form and $\omega^n = \omega \wedge \cdots \wedge \omega$. In the Kähler case it will be however convenient to get rid of the constant and define the volume as

$$dV := \omega^n.$$

**Christoffel symbols.** From now on we assume that $\omega$ is a Kähler form on $M$ and $\langle\cdot,\cdot\rangle$ is the associated metric. Write

$$\nabla_{\partial_j}\partial_k = \Gamma^l_{jk}\partial_l + \Gamma^{\bar l}_{jk}\partial_{\bar l}, \quad \nabla_{\partial_{\bar j}}\partial_k = \Gamma^l_{\bar j k}\partial_l + \Gamma^{\bar l}_{\bar j k}\partial_{\bar l}.$$

Since $\nabla J = 0$, we have for example $i\nabla_{\partial_j}\partial_k = \nabla_{\partial_j}(J\partial_k) = J\nabla_{\partial_j}\partial_k$ and it follows that $\Gamma^{\bar l}_{jk} = 0$. Similarly we show that $\Gamma^l_{\bar j k} = \Gamma^{\bar l}_{\bar j k} = 0$, so the only non-vanishing Christoffel symbols are $\Gamma^l_{jk} = \overline{\Gamma^{\bar l}_{\bar j \bar k}}$. Denoting further $g_j = \partial g/\partial z^j$, $g_{\bar k} = \partial g/\partial \bar z^k$ (which by Proposition 1(iii) is consistent with the previous notation) we have

$$g_{j\bar k l} = \partial_l\langle\partial_l,\partial_{\bar k}\rangle = \Gamma^p_{lj}g_{p\bar q},$$

which means that

$$\Gamma^l_{jk} = \overline{\Gamma^{\bar l}_{\bar j \bar k}} = g^{l\bar q}g_{j\bar q k}, \tag{3}$$

where $g^{p\bar q}$ is determined by

$$g^{j\bar q}g_{k\bar q} = \delta_{jk}. \tag{4}$$

**Riemannian curvature.** Recall that it is defined by

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

(we extend it to $T_{\mathbb C}M$) and

$$R(X,Y,Z,W) = \langle R(X,Y)Z,W\rangle. \tag{5}$$

The classical properties are

$$\begin{gathered} R(Y,X) = -R(X,Y), \\ R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z) = R(Z,W,X,Y), \\ R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \end{gathered} \tag{6}$$

(the latter is the first Bianchi identity). From $\nabla J = 0$ it follows that

$$R(X,Y)J = JR(X,Y)$$

and from (6) we infer

$$R(X,Y,Z,W) = R(X,Y,JZ,JW) = R(JX,JY,Z,W).$$

It follows that

$$R(JX,JY) = R(X,Y),$$

thus

$$R(\partial_j, \partial_k) = R(\partial_{\bar{j}}, \partial_{\bar{k}}) = 0.$$

We have

$$R(\partial_j, \partial_{\bar{k}})\partial_p = -\nabla_{\partial_{\bar{k}}}\nabla_{\partial j}\partial_p = -\partial_{\bar{k}}(\Gamma^l_{jp})\partial_l$$

and

$$R(\partial_j, \partial_{\bar{k}})\partial_{\bar{q}} = \nabla_{\partial_j}\nabla_{\partial_{\bar{k}}}\partial_{\bar{q}} = \partial_j(\overline{\Gamma^l_{kq}})\partial_{\bar{l}}.$$

Therefore, if we write

$$R(\partial_j, \partial_{\bar{k}})\partial_p = R^l_{j\bar{k}p}\partial_l, \quad R(\partial_j, \partial_{\bar{k}})\partial_{\bar{q}} = R^{\bar{l}}_{j\bar{k}\bar{q}}\partial_{\bar{l}},$$

then

$$R^l_{j\bar{k}p} = -\overline{R^{\bar{l}}_{k\bar{j}\bar{p}}} = -(g^{l\bar{t}}g_{j\bar{t}p})_{\bar{k}}.$$

The relevant coefficients for (5) are

$$R_{j\bar{k}p\bar{q}} := R(\partial_j, \partial_{\bar{k}}, \partial_p, \partial_{\bar{q}}) = g_{l\bar{q}}R^l_{j\bar{k}p}$$

by (3). Applying a first-order differential operator (with constant coefficients) $D$ to both sides of (4) we get

$$Dg^{p\bar{q}} = -g^{p\bar{t}}g^{s\bar{q}}Dg_{s\bar{t}} \tag{7}$$

and thus

$$R_{j\bar{k}p\bar{q}} = -g_{j\bar{k}p\bar{q}} + g^{s\bar{t}}g_{j\bar{t}p}g_{s\bar{k}\bar{q}}.$$

**Ricci curvature.** Recall that the Ricci curvature is defined by

$$Ric(X, Y) := tr(Z \mapsto R(Z, X)Y).$$

We extend it to $T_{\mathbb{C}}M$. If we write $Z = Z^p\partial_p + \tilde{Z}^q\partial_{\bar{q}}$ then

$$R(Z, \partial_j)\partial_k = -\tilde{Z}^q R^l_{j\bar{q}k}\partial_l, \quad R(Z, \partial_{\bar{j}})\partial_{\bar{k}} = Z^p R^{\bar{l}}_{p\bar{j}\bar{k}}\partial_{\bar{l}},$$

$$R(Z, \partial_j)\partial_{\bar{k}} = -\tilde{Z}^q R^{\bar{l}}_{j\bar{q}k}\partial_{\bar{l}}, \quad R(Z, \partial_{\bar{k}})\partial_j = Z^p R^l_{p\bar{k}j}\partial_l.$$

It follows that

$$Ric(\partial_j, \partial_k) = Ric(\partial_{\bar{j}}, \partial_{\bar{k}}) = 0$$

and

$$Ric_{j\bar{k}} := Ric(\partial_j, \partial_{\bar{k}}) = \overline{R^p_{p\bar{j}k}} = -(g^{p\bar{q}}g_{p\bar{q}\bar{k}})_j.$$

Since

$$D\det(g_{p\bar{q}}) = M^{p\bar{q}}Dg_{p\bar{q}},$$

where $(M^{p\bar{q}}) = \det(g_{s\bar{t}})(g^{p\bar{q}})$ is the adjoint matrix to $(g_{p\bar{q}})$, we have

$$D\big(\log\det(g_{p\bar q})\big) = g^{p\bar q} D g_{p\bar q}. \tag{8}$$

Therefore

$$Ric_{j\bar k} = -(\log\det(g_{p\bar q}))_{j\bar k}. \tag{9}$$

From the proceeding calculations we infer in particular

$$Ric(JX, JY) = Ric(X, Y).$$

The associated Ricci 2-form is defined by

$$Ric_\omega(X, Y) := Ric(JX, Y)$$

(since $Ric$ is symmetric, $Ric_\omega$ is antisymmetric). We then have

$$Ric_\omega = i Ric_{j\bar k} dz^j \wedge d\bar z^k = -i\,\partial\bar\partial(\log\det(g_{p\bar q})).$$

An important consequence of this formula is the following: if $\tilde\omega$ is another Kähler form on $M$ then

$$Ric_\omega - Ric_{\tilde\omega} = i\,\partial\bar\partial\log\frac{\tilde\omega^n}{\omega^n}. \tag{10}$$

In particular, $Ric_\omega$ and $Ric_{\tilde\omega}$ are $\partial\bar\partial$-cohomologous.

**Scalar curvature.** It is the trace of the mapping $Ric : T_{\mathbb{C}}M \to T_{\mathbb{C}}M$ defined by the relation

$$\langle Ric\, X, Y\rangle = Ric(X, Y).$$

Since

$$Ric\,\partial_j = g^{p\bar q} Ric_{j\bar q}\partial_p,$$

we will obtain

$$S = 2g^{p\bar q} Ric_{p\bar q} = 2n\frac{Ric_\omega \wedge \omega^{n-1}}{\omega^n}.$$

**Bisectional curvature.** It is defined by

$$\sigma(X, Y) = R(X, JX, Y, JY) = R(X, Y, X, Y) + R(X, JY, X, JY),$$

where the last equality is a consequence of the first Bianchi identity. If we write $X = X^j\partial_j + \tilde X^k\partial_{\bar k}$, $Y = Y^p\partial_p + \tilde Y^q\partial_{\bar q}$, then

$$\sigma(X, Y) = -2iX^j\tilde X^k R(\partial_j, \partial_{\bar k}, Y, JY) = 4R_{j\bar k p\bar q}X^j\tilde X^k Y^p\tilde Y^q.$$

An upper bound for the bisectional curvature is a positive constant $C > 0$ satisfying

$$\sigma(X, Y) \le C|X|^2|Y|^2, \quad X, Y \in TM. \tag{11}$$

Since $|X^j\partial_j + \bar{X}^k\partial_{\bar{k}}|^2 = 2g_{j\bar{k}}X^j\bar{X}^k$, it follows that (11) is equivalent to

$$R_{j\bar{k}p\bar{q}}a^j\bar{a}^k b^p\bar{b}^q \le C g_{j\bar{k}}a^j\bar{a}^k g_{p\bar{q}}b^p\bar{b}^q, \quad a, b \in \mathbb{C}^n. \tag{12}$$

Similarly we can define a lower bound.

**Gradient.** For a real-valued function $\varphi$ on $M$ its gradient $\nabla\varphi$ is defined by the relation

$$\langle\nabla\varphi, X\rangle = X\varphi.$$

Therefore

$$\nabla\varphi = g^{j\bar{k}}(\varphi_{\bar{k}}\partial_j + \varphi_j\partial_{\bar{k}})$$

and

$$|\nabla\varphi|^2 = 2g^{j\bar{k}}\varphi_j\varphi_{\bar{k}}.$$

**Laplacian.** It is given by

$$\Delta\varphi := tr(X \mapsto \nabla_X\nabla\varphi).$$

For $X = X^j\partial_j + \tilde{X}^k\partial_{\bar{k}}$ we have

$$\begin{aligned}\nabla_X\nabla\varphi = {} & X^j\big[(g^{p\bar{q}}\varphi_{\bar{q}})_j\partial_p + g^{p\bar{q}}\varphi_{\bar{q}}\Gamma^k_{jp}\partial_k + (g^{p\bar{q}}\varphi_p)_j\partial_{\bar{q}}\big] \\ & + \tilde{X}^k\big[(g^{p\bar{q}}\varphi_{\bar{q}})_{\bar{k}}\partial_p + (g^{p\bar{q}}\varphi_p)_{\bar{k}}\partial_{\bar{q}} + g^{p\bar{q}}\varphi_p\overline{\Gamma^j_{kq}}\partial_{\bar{j}}\big].\end{aligned}$$

From (3) and (7) we will get

$$\Delta\varphi = 2g^{j\bar{k}}\varphi_{j\bar{k}}.$$

**Lichnerowicz operator.** For a real-valued function $\varphi$ we can write

$$\nabla\varphi = \nabla'\varphi + \overline{\nabla'\varphi},$$

where

$$\nabla'\varphi = g^{j\bar{k}}\varphi_{\bar{k}}\partial_j \in T^{1,0}M.$$

The Lichnerowicz operator is defined by

$$\mathcal{L}\varphi := \bar{\partial}\nabla'\varphi = \big(g^{j\bar{k}}\varphi_{\bar{k}}\big)_{\bar{q}}\partial_j \otimes d\bar{z}^q,$$

so that $\nabla\varphi$ is a holomorphic vector field iff $\mathcal{L}\varphi = 0$.

**Proposition 2.** $\mathcal{L}^*\mathcal{L}\varphi = \Delta^2\varphi + \langle Ric_\omega, i\partial\bar{\partial}\varphi\rangle + \langle\nabla S, \nabla\varphi\rangle$.

*Proof.* Since

$$|\mathcal{L}\varphi|^2 = 4g^{p\bar{q}}g_{j\bar{t}}\big(g^{j\bar{k}}\varphi_{\bar{k}}\big)_{\bar{q}}\big(g^{s\bar{t}}\varphi_s\big)_p$$

and

$$\left(g^{p\bar{q}}\det(g_{j\bar{k}})\right)_p = 0$$

for every $q$, it follows that

$$\mathcal{L}^*\mathcal{L}\varphi = 4\mathrm{Re}\left[g^{s\bar{t}}\left(g^{p\bar{q}}\left(g_{j\bar{t}}\big(g^{j\bar{k}}\varphi_{\bar{k}}\big)_{\bar{q}}\right)_p\right)_s\right].$$

We can compute that

$$g^{p\bar{q}}\left(g_{j\bar{t}}\big(g^{j\bar{k}}\varphi_{\bar{k}}\big)_{\bar{q}}\right)_p = (g^{p\bar{q}}\varphi_{p\bar{q}})_{\bar{t}} - g^{p\bar{q}}(g^{a\bar{k}}g_{a\bar{t}\bar{q}})_p\varphi_{\bar{k}}$$

and thus

$$\frac{1}{4}\mathcal{L}^*\mathcal{L}\varphi = g^{s\bar{t}}(g^{p\bar{q}}\varphi_{p\bar{q}})_{s\bar{t}} - g^{s\bar{t}}g^{p\bar{q}}(g^{a\bar{k}}g_{a\bar{t}\bar{q}})_p\varphi_{s\bar{k}} - \mathrm{Re}\left[g^{s\bar{t}}\big(g^{p\bar{q}}(g^{a\bar{k}}g_{a\bar{t}\bar{q}})_p\big)_s\varphi_{\bar{k}}\right].$$

One can check that

$$\begin{aligned} -g^{s\bar{t}}g^{p\bar{q}}(g^{a\bar{k}}g_{a\bar{t}\bar{q}})_p &= g^{s\bar{q}}g^{p\bar{k}}Ric_{p\bar{q}} \\ -g^{s\bar{t}}\big(g^{p\bar{q}}(g^{a\bar{k}}g_{a\bar{t}\bar{q}})_p\big)_s &= \frac{1}{2}g^{j\bar{k}}S_j \end{aligned}$$

and the result follows. □

**Poisson bracket.** It is defined by the relation

$$\{\varphi,\psi\}\omega^n = n\, d\varphi\wedge d\psi\wedge\omega^{n-1}$$

or, in local coordinates,

$$\{\varphi,\psi\} = i g^{j\bar{k}}(\varphi_{\bar{k}}\psi_j - \varphi_j\psi_{\bar{k}}).$$

If one of $\varphi, \psi, \eta$ has a compact support then

$$\int_M\{\varphi,\psi\}\eta\,\omega^n = \int_M\varphi\{\psi,\eta\}\omega^n.$$

***$d^c$*-operator.** It is useful to introduce the operator $d^c := \frac{i}{2}(\bar{\partial}-\partial)$. It is real (in the sense that it maps real forms to real forms) and $dd^c = i\partial\bar{\partial}$. One can easily show that

$$dd^c\varphi\wedge\omega^{n-1} = \frac{1}{2n}\Delta\varphi\,\omega^n$$

and

$$d\varphi\wedge d^c\psi\wedge\omega^{n-1} = \frac{1}{2n}\langle\nabla\varphi,\nabla\psi\rangle\omega^n.$$

The operator $d^c$ clearly depends only on the complex structure. In the Kähler case we have however the formula

$$d^c\varphi = -\frac{1}{2} i_{\nabla\varphi}\omega \tag{13}$$

(where $i_X\omega(Y) = \omega(X,Y)$).

**Normal coordinates.** Near a fixed point we can holomorphically change coordinates in such a way that $g_{j\bar{p}} = \delta_{jk}$ and $g_{j\bar{k}l} = g_{j\bar{k}lm} = 0$. By a linear transformation we can obtain the first condition. Then consider the mapping

$$F^m(z) := z^m + \frac{1}{2}a^m_{jk}z^j z^k + \frac{1}{6}b^m_{jkl}z^j z^k z^l$$

(the origin being our fixed point), where $a^m_{jk}$ is symmetric in $j,k$ and $b^m_{jkl}$ symmetric in $j,k,l$. Then for $\tilde{g} = g \circ F$ we have

$$\tilde{g}_{j\bar{k}l}(0) = g_{j\bar{k}l}(0) + a^k_{jl}$$
$$\tilde{g}_{j\bar{k}lm}(0) = g_{j\bar{k}lm}(0) + 3g_{j\bar{k}p}(0)a^p_{lm} + b^k_{jlm}$$

and we can choose the coefficients of $F$ in such a way that the left-hand sides vanish.

## 3 Calabi Conjecture and Extremal Metrics

A complex manifold is called *Kähler* if it admits a Kähler metric. We will be particularly interested in compact Kähler manifolds. If $\omega$ is a Kähler form on a compact complex manifold $M$ then the $(p,p)$-form $\omega^p$ is not exact, because if $\omega^p = d\alpha$ for some $\alpha$, then

$$\int_M \omega^n = \int_M d(\alpha \wedge \omega^p) = 0$$

which is a contradiction. Since $\omega^p$ is a real closed $2p$-form, it follows that for compact Kähler manifolds $H^{2p}(M,\mathbb{R}) \neq 0$.

*Example.* Hopf surface $M := (\mathbb{C}^2 \setminus \{0\})/\{2^n : n \in \mathbb{Z}\}$ is a compact complex surface, topologically equivalent to $S^1 \times S^3$. Therefore $H^2(M,\mathbb{R}) = 0$ and thus $M$ is not Kähler. □

***dd^c*-lemma.** It follows from (10) that for two Kähler forms $\omega, \tilde{\omega}$ on $M$ the $(1,1)$-forms $Ric_\omega, Ric_{\tilde{\omega}}$ are $dd^c$-cohomologous, in particular $d$-cohomologous. The following result, called a *$dd^c$-lemma*, shows that these two notions are in fact equivalent for $(1,1)$-forms on a compact manifold:

**Theorem 3.** *If a (1,1)-form on a compact Kähler manifold is $d$-exact then it is $dd^c$-exact.*

We will follow the proof from [44]. Theorem 3 will be an easy consequence of the following:

**Lemma 4.** *Assume that $\beta$ is a (0,1)-form on a compact Kähler manifold such that $\bar\partial\beta = 0$. Then $\partial\beta = \partial\bar\partial f$ for some $f \in C^\infty(M,\mathbb{C})$.*

*Proof.* Let $\omega$ be a Kähler form on $M$. Since

$$\int_M \partial\beta \wedge \omega^{n-1} = \int_M d\beta \wedge \omega^{n-1} = \int_M d(\beta \wedge \omega^{n-1}) = 0,$$

we can find $f \in C^\infty(M,\mathbb{C})$ solving

$$\partial\bar\partial f \wedge \omega^{n-1} = \partial\beta \wedge \omega^{n-1}.$$

Set $\gamma := \beta - \bar\partial f$, we have to show that $\partial\gamma = 0$. Since $\bar\partial\gamma = 0$,

$$\int_M \partial\gamma \wedge \overline{\partial\gamma} \wedge \omega^{n-2} = \int_M d(\gamma \wedge \overline{d\gamma} \wedge \omega^{n-2}) = 0.$$

Locally we may write

$$\gamma = \gamma_{\bar k} d\bar z^k$$

and

$$\partial\gamma = \gamma_{\bar k j} dz^j \wedge d\bar z^k.$$

One can then show that

$$\partial\gamma \wedge \overline{\partial\gamma} \wedge \omega^{n-2} = \frac{1}{n(n-1)}\left(g^{j\bar k} g^{p\bar q} \gamma_{\bar k p}\overline{\gamma_{\bar j q}} - \left|g^{j\bar k}\gamma_{\bar k j}\right|^2\right)\omega^n.$$

Now $\partial\gamma \wedge \omega^{n-1} = 0$ means that $g^{j\bar k}\gamma_{\bar k j} = 0$ and it follows that $\gamma_{k\bar j} = 0$. □

*Proof of Theorem 3.* Write $\alpha = \beta_1 + \beta_2$, where $\beta_1$ is a $(1,0)$, and $\beta_2$ a $(0,1)$-form. Then

$$d\alpha = \partial\beta_1 + \partial\beta_2 + \bar\partial\beta_1 + \bar\partial\beta_2.$$

Since $d\alpha$ is of type (1,1), it follows that $\bar\partial\bar\beta_1 = \bar\partial\beta_2 = 0$. By Lemma 4 we have $\partial\bar\beta_1 = \partial\bar\partial f_1$ and $\partial\beta_2 = \partial\bar\partial f_2$ for some $f_1, f_2 \in C^\infty(M,\mathbb{C})$. Therefore

$$d\alpha = \partial\beta_2 + \bar\partial\beta_1 = \partial\bar\partial(f_2 - \bar f_1).$$ □

From now on we assume that $M$ is a compact Kähler manifold. For a Kähler form $\omega$ on $M$ by $c_1(M)$ we denote the cohomology class $\{Ric_\omega\}$. By (10) it is independent of the choice of $\omega$; in fact $c_1(M) = c_1(M)_\mathbb{R}/2\pi$, where $c_1(M)_\mathbb{R}$ is the first Chern class.

**Calabi conjecture ([17]).** Let $\tilde{R}$ be a (1,1)-form on $M$ cohomologous to $Ric_\omega$ (we write $R \sim Ric_\omega$). Then we ask whether there exists another, unique Kähler form $\tilde{\omega} \sim \omega$ on $M$ such that $\tilde{R} = Ric_{\tilde{\omega}}$. In other words, the question is whether the mapping

$$\{\omega\} \ni \tilde{\omega} \longmapsto Ric_{\tilde{\omega}} \in c_1(M)$$

is bijective.

**Derivation of the Monge–Ampère equation.** By $dd^c$-lemma we have $Ric_\omega = \tilde{R} + dd^c\eta$ for some $\eta \in C^\infty(M)$. We are thus looking for $\varphi \in C^\infty(M)$ such that $\omega_\varphi := \omega + dd^c\varphi > 0$ and

$$dd^c\big(\log\frac{\omega_\varphi^n}{\omega^n} - \eta\big) = 0,$$

that is

$$\log\frac{\omega_\varphi^n}{\omega^n} - \eta = c,$$

a constant. This means that

$$\omega_\varphi^n = e^{c+\eta}\omega^n.$$

Since $\omega_\varphi^n - \omega^n$ is exact, from the Stokes theorem we infer

$$\int_M \omega_\varphi^n = \int_M \omega^n =: V.$$

Therefore the constant $c$ is uniquely determined. It follows that to solve the Calabi conjecture is equivalent to solve the following Dirichlet problem for the complex Monge–Ampère operator on $M$: for $f \in C^\infty(M)$, $f > 0$, satisfying $\int_M f\omega^n = V$, there exists, unique up to an additive constant, $\varphi \in C^\infty(M)$ such that $\omega + dd^c\varphi > 0$ and

$$(\omega + dd^c\varphi)^n = f\omega^n. \tag{14}$$

This problem was solved by Yau [47], the proof will be given in Sects. 7–13. The solution of Calabi conjecture has many important consequences (see e.g. [48]). The one which is particularly interesting in algebraic geometry is that for a compact Kähler manifold $M$ with $c_1(M) = 0$ there exists a Kähler metric with vanishing Ricci curvature. Except for the torus $\mathbb{C}^n/\Lambda$ such a metric can never be written down explicitly.

**Kähler–Einstein metrics.** A Kähler form $\omega$ is called *Kähler–Einstein* if $Ric_\omega = \lambda\omega$ for some $\lambda \in \mathbb{R}$. A necessary condition for $M$ is thus that $c_1(M)$ is definite which means that it contains a definite representative. There are three possibilities: $c_1(M) = 0$, $c_1(M) < 0$ and $c_1(M) > 0$. Assume that it is the case, we can then find a Kähler metric $\omega$ with $\lambda\omega \in c_1(M)$, that is $Ric_\omega = \lambda\omega + dd^c\eta$ for some $\eta \in C^\infty(M)$. We are looking for $\varphi \in C^\infty(M)$ such that $Ric_{\omega_\varphi} = \lambda\omega_\varphi$ which, similarly as before, is equivalent to

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi+\eta+c}\omega^n. \tag{15}$$

If $c_1(M) = 0$ then $\lambda = 0$ and (15) is covered by (14). If $c_1(M) < 0$ one can solve (15) in a similar way as (14). It was done by Aubin [1] and Yau [47], in fact, the $L^\infty$-estimate in this case is very simple.

The case $c_1(M) > 0$ (such manifolds are called Fano) is the most difficult. The first obstruction to the existence of Kähler–Einstein metrics is a result of Matsushima [36] which says that in this case the Lie algebra of holomorphic vector fields must be reductive (that is it must be a complexification of a compact real subalgebra). By the result of Tian [41] this is the only obstruction in dimension 2 but in [43] he constructed a 3-dimensional Fano manifold with no holomorphic vector fields and no Kähler–Einstein metric. In fact, the Fano surfaces can be classified: they are exactly $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up at $k$ points in general position, where $1 \le k \le 8$. Among those only $\mathbb{P}^2$ blown up at one or two points have non-reductive algebras of holomorphic vector fields, and thus all the other surfaces admit Kähler–Einstein metrics—see [43] or a recent exposition of Tosatti [45].

Uniqueness of Kähler–Einstein metrics in a given Kähler class $\{\omega\}$ (satisfying the necessary condition $\lambda\{\omega\} \subset c_1(M)$) for $c_1(M) = 0$ and $c_1(M) < 0$ follows quite easily from the equation (15). In the Fano case $c_1(M) > 0$ it holds up to a biholomorphism—it was proved by Bando and Mabuchi [3] (see also [6, 7]).

**Constant scalar curvature metrics.** Given a compact Kähler manifold $(M, \omega)$ we are interested in a metric in $\{\omega\}$ with constant scalar curvature (csc). With the notation $S_\varphi = S_{\omega_\varphi}$ we are thus looking for $\varphi$ satisfying $S_\varphi = \bar{S}$, where $\bar{S}$ is a constant. First of all we note that $\bar{S}$ is uniquely determined by the Kähler class:

$$\bar{S}\int_M \omega^n = \int_M S_\varphi \omega_\varphi^n = 2n\int_M Ric_\varphi \wedge \omega_\varphi^{n-1} = 2n\int_M Ric_\omega \wedge \omega^{n-1}. \tag{16}$$

Secondly, the csc problem is more general than the Kähler–Einstein problem. For if $\lambda\{\omega\} \subset c_1(M)$, that is $Ric_\omega = \lambda\omega + dd^c\eta$ for some $\eta \in C^\infty(M)$, and $\omega_\varphi$ is a csc metric then $\bar{S} = 2n\lambda$ and $Ric_\varphi \wedge \omega_\varphi^{n-1} = \lambda\omega_\varphi^n$. But since

$$Ric_\varphi - \lambda\omega_\varphi = dd^c[\eta - \log\frac{\omega_\varphi^n}{\omega^n} - \lambda\varphi],$$

it follows that $Ric_\varphi = \lambda\omega_\varphi$.

The equation $S_\varphi = \bar{S}$ is of order 4 and therefore very difficult to handle directly. The question of uniqueness of csc metrics was treated in [19]. A general conjecture links existence of csc metrics with stability in the sense of geometric invariant theory. So far it has been fully answered only in the case of toric surfaces (Donaldson [21]). See [37] for an extensive survey on csc metrics.

## 4 The Space of Kähler Metrics

We consider the class of Kähler potentials w.r.t. a Kähler form $\omega$:

$$\mathcal{H} := \{\varphi \in C^\infty(M) : \omega_\varphi > 0\}.$$

It is an open subset of $C^\infty(M)$ and thus has a structure of an infinite dimensional differential manifold (its differential structure is determined by the relation

$$C^\infty(U, C^\infty(M)) = C^\infty(M \times U)$$

for any region $U$ in $\mathbb{R}^m$). For $\varphi \in \mathcal{H}$ the tangent space $T_\varphi\mathcal{H}$ may be thus identified with $C^\infty(M)$. On $T_\varphi\mathcal{H}$, following Mabuchi [M], we define a scalar product:

$$\langle\langle \psi, \eta \rangle\rangle := \frac{1}{V}\int_M \psi\eta\,\omega_\varphi^n, \quad \psi, \eta \in T_\varphi\mathcal{H}.$$

Also by $\varphi = \varphi(t)$ denote a smooth curve $[a,b] \to \mathcal{H}$ (which is an element of $C^\infty(M \times [a,b])$). For a vector field $\psi$ on $\varphi$ (which we may also treat as an element of $C^\infty(M \times [a,b])$) we want to define a connection $\nabla_{\dot\varphi}\psi$ (where we denote $\dot\varphi = d\varphi/dt$), so that

$$\frac{d}{dt}\langle\langle \psi, \eta \rangle\rangle = \langle\langle \nabla_{\dot\varphi}\psi, \eta \rangle\rangle + \langle\langle \psi, \nabla_{\dot\varphi}\eta \rangle\rangle \tag{17}$$

(where $\eta$ is another vector field on $\varphi$). Since

$$\frac{d}{dt}\,\omega_\varphi^n = n\,dd^c\dot\varphi \wedge \omega_\varphi^{n-1} = \frac{1}{2}\Delta\dot\varphi\,\omega_\varphi^n, \tag{18}$$

where $\Delta$ denotes the Laplacian w.r.t. $\omega_\varphi$, we have

$$\begin{aligned}
\frac{d}{dt}\langle\langle \psi, \eta \rangle\rangle &= \frac{1}{V}\int_M \big(\dot\psi\eta + \psi\dot\eta + \frac{1}{2}\psi\eta\Delta\dot\varphi\big)\omega_\varphi^n \\
&= \frac{1}{V}\int_M \left(\dot\psi\eta + \psi\dot\eta - \frac{1}{2}\langle\nabla(\psi\eta), \nabla\dot\varphi\rangle\right)\omega_\varphi^n \\
&= \frac{1}{V}\int_M \left[\left(\dot\psi - \frac{1}{2}\langle\nabla\psi, \nabla\dot\varphi\rangle\right)\eta + \psi\left(\dot\eta - \frac{1}{2}\langle\nabla\eta, \nabla\dot\varphi\rangle\right)\right]\omega_\varphi^n.
\end{aligned}$$

This shows that the right way to define a connection on $\mathcal{H}$ is

$$\nabla_{\dot\varphi}\psi := \dot\psi - \frac{1}{2}\langle\nabla\psi, \nabla\dot\varphi\rangle,$$

where $\nabla$ on the right-hand side denotes the gradient w.r.t. $\omega_\varphi$. A curve $\varphi$ in $\mathcal{H}$ is therefore a *geodesic* if $\nabla_{\dot\varphi}\dot\varphi = 0$, that is

$$\ddot\varphi - \frac{1}{2}|\nabla\dot\varphi|^2 = 0. \tag{19}$$

**Curvature.**

**Theorem 5 (Mabuchi [35], Donaldson [20]).** *We have the following formula for the curvature of* $\langle\langle\cdot,\cdot\rangle\rangle$

$$R(\psi,\eta)\gamma = -\frac{1}{4}\{\{\psi,\eta\},\gamma\}, \quad \psi,\eta,\gamma\in T_\varphi\mathcal{H},\ \varphi\in\mathcal{H}.$$

*In particular, the sectional curvature is given by*

$$K(\psi,\eta) = -\frac{1}{4}||\{\psi,\eta\}||^2 \le 0.$$

*Proof.* Without loss of generality we may evaluate the curvature at $0\in\mathcal{H}$. Let $\varphi\in C^\infty([0,1]\times[0,1],\mathcal{H})$ be such that $\varphi(0,0)=0$ and at $s=t=0$ we have $\varphi_s(=d\varphi/ds)=\psi$, $\varphi_t=\eta$. Take $\gamma\in C^\infty([0,1]^2, C^\infty(M)) = C^\infty(M\times[0,1]^2)$. We have

$$\begin{aligned}
\nabla_{\varphi_s}\nabla_{\varphi_t}\gamma - \nabla_{\varphi_t}\nabla_{\varphi_s}\gamma &= \nabla_{\varphi_s}\big(\gamma_t - \frac{1}{2}\langle\nabla\varphi_t,\nabla\gamma\rangle\big) - \nabla_{\varphi_t}\big(\gamma_s - \frac{1}{2}\langle\nabla\varphi_s,\nabla\gamma\rangle\big)\\
&= -\frac{1}{2}\frac{d}{ds}\langle\nabla\varphi_t,\nabla\gamma\rangle - \frac{1}{2}\langle\nabla\varphi_s,\nabla\gamma_t\rangle + \frac{1}{4}\langle\nabla\varphi_s,\nabla\langle\nabla\varphi_t,\nabla\gamma\rangle\rangle\\
&\quad + \frac{1}{2}\frac{d}{dt}\langle\nabla\varphi_s,\nabla\gamma\rangle + \frac{1}{2}\langle\nabla\varphi_t,\nabla\gamma_s\rangle - \frac{1}{4}\langle\nabla\varphi_t,\nabla\langle\nabla\varphi_s,\nabla\gamma\rangle\rangle.
\end{aligned}$$

Denoting $u = g+\varphi$ we get

$$\begin{aligned}
\frac{d}{dt}\langle\nabla\varphi_s,\nabla\gamma\rangle &= \frac{d}{dt}\Big[u^{j\bar k}\big(\gamma_j\varphi_{s\bar k} + \gamma_{\bar k}\varphi_{sj}\big)\Big]\\
&= -\frac{1}{4}\langle i\partial\bar\partial\eta, i\partial\psi\wedge\bar\partial\gamma + i\partial\gamma\wedge\bar\partial\psi\rangle + \langle\nabla\varphi_s,\nabla\gamma_t\rangle + \langle\nabla\varphi_{st},\nabla\gamma\rangle
\end{aligned}$$

where in the last line we have evaluated at $s=t=0$. Therefore at $s=t=0$ we have

$$\begin{aligned}
\nabla_{\varphi_s}\nabla_{\varphi_t}\gamma - \nabla_{\varphi_t}\nabla_{\varphi_s}\gamma &= \frac{1}{4}\langle\nabla\psi,\nabla\langle\nabla\eta,\nabla\gamma\rangle\rangle - \frac{1}{4}\langle\nabla\eta,\nabla\langle\nabla\psi,\nabla\gamma\rangle\rangle\\
&\quad + \frac{1}{8}\langle i\partial\bar\partial\psi, i\partial\eta\wedge\bar\partial\gamma + i\partial\gamma\wedge\bar\partial\eta\rangle\\
&\quad - \frac{1}{8}\langle i\partial\bar\partial\eta, i\partial\psi\wedge\bar\partial\gamma + i\partial\gamma\wedge\bar\partial\psi\rangle.
\end{aligned}$$

We can now show, using for example normal coordinates, that the right-hand side is equal to $-\frac{1}{4}\{\{\psi, \eta\}, \gamma\}$. □

**Derivation of the homogeneous complex Monge–Ampère equation.** Writing locally $u = g + \varphi$, since $g$ is independent of $t$, we can rewrite (19) as

$$\ddot{u} - u^{p\bar{q}}\dot{u}_p\dot{u}_{\bar{q}} = 0.$$

Multiplying both sides by $\det(u_{j\bar{k}})$ (which is non-vanishing) we arrive at the equation

$$\det\begin{pmatrix} & & & u_{1t} \\ & (u_{j\bar{k}}) & & \vdots \\ & & & u_{nt} \\ u_{t\bar{1}} & \dots & u_{t\bar{n}} & u_{tt} \end{pmatrix} = 0.$$

This suggests to complexify the variable $t$, either simply by adding an imaginary variable, or introducing the new one $\zeta(= z^{n+1}) \in \mathbb{C}_*$, so that $t = \log|\zeta|$. Then for $v(\zeta) = u(\log|\zeta|)$ we have $v_\zeta = \dot{u}/2\zeta$ and $v_{\zeta\bar{\zeta}} = \ddot{u}/4|\zeta|^2$. We have thus obtained the following characterization of geodesics in $\mathcal{H}$:

**Proposition 6 ((Semmes [39], Donaldson [20])).** *For $\varphi_0, \varphi_1 \in \mathcal{H}$ existence of a geodesic in $\mathcal{H}$ joining $\varphi_0$ and $\varphi_1$ is equivalent to solving the following Dirichlet problem for the homogeneous complex Monge–Ampère equation:*

$$\begin{cases} \varphi \in C^\infty(M \times \{e^0 \le |\zeta| \le e^1\}) \\ \omega + dd^c\varphi(\cdot, \zeta) > 0, \quad e^0 \le |\zeta| \le e^1 \\ (\omega + dd^c\varphi)^{n+1} = 0 \\ \varphi(\cdot, \zeta) = \varphi_j, \quad |\zeta| = e^j, \ j = 0, 1. \end{cases}$$

□

Although $\omega$ is a degenerate form on $M \times \mathbb{C}$, it is not a problem: write

$$\omega + dd^c\varphi = \tilde{\omega} + dd^c(\varphi - |\zeta|^2), \tag{20}$$

where $\tilde{\omega} = \omega + dd^c|\zeta|^2$ is a Kähler form on $M \times \mathbb{C}$, and consider the related problem.

The existence of geodesic is thus equivalent to solving the homogeneous Monge–Ampère equation on a compact Kähler manifold with boundary. From the uniqueness of this equation (see e.g. the next section) it follows in particular that given two potentials in $\mathcal{H}$ there exists at most one geodesic joining them.

As shown recently by Lempert and Vivas [33], it is not always possible to join two metrics by a smooth geodesic (see Sect. 6). However, for $\varepsilon > 0$ we can introduce a notion of an *$\varepsilon$-geodesic*: instead of (19) it solves

$$(\ddot{\varphi} - \frac{1}{2}|\nabla\dot{\varphi}|^2)\omega_\varphi^n = \varepsilon\omega^n$$

which is equivalent to the following non-degenerate complex Monge–Ampère equation:

$$(\omega + dd^c\varphi)^{n+1} = \frac{\varepsilon}{4|\zeta|^2}(\omega + dd^c|\zeta|^2)^{n+1}. \tag{21}$$

As shown by Chen [18] (see also [15]), smooth $\varepsilon$-geodesics always exist (see Theorem 19 below) and they approximate weak geodesics. Existence of $\varepsilon$-geodesics will be used in Sect. 6 below to show that $\mathcal{H}$ with a distance defined by its Riemannian structure is a metric space (this result is due to Chen [18], see also [15]).

**Normalization, Aubin–Yau functional.** The Riemannian structure on $\mathcal{H}$ will induce such a structure on the Kähler class $\{\omega\} = \mathcal{H}/\mathbb{R}$, which is independent of the choice of $\omega$. For this we need a good normalization on $\mathcal{H}$. The right tool for this purpose is the Aubin–Yau functional (see e.g. [2])

$$I : \mathcal{H} \to \mathbb{R}$$

which is characterized by the following properties

$$I(0) = 0, \quad d_\varphi I.\psi = \frac{1}{V}\int_M \psi\,\omega_\varphi^n, \quad \varphi \in \mathcal{H},\ \psi \in C^\infty(M). \tag{22}$$

This means that we are looking for $I$ with $dI = \alpha$, where the 1-form $\alpha$ is given by

$$\alpha(\varphi).\psi = \frac{1}{V}\int_M \psi\,\omega_\varphi^n. \tag{23}$$

Such an $I$ exists provided that $\alpha$ is closed. But by (18)

$$d\alpha(\varphi).(\psi,\tilde{\psi}) = d_\varphi(\alpha(\varphi).\psi).\tilde{\psi} - d_\varphi(\alpha(\varphi).\tilde{\psi}).\psi = \frac{n}{V}\int_M (\psi\Delta\tilde{\psi} - \tilde{\psi}\Delta\psi)\,\omega_\varphi^n = 0$$

and it follows that there is $I$ satisfying (22).

For any curve $\tilde{\varphi}$ in $\mathcal{H}$ joining 0 with $\varphi$ we have

$$I(\varphi) = \int_0^1 \frac{1}{V}\int_M \dot{\tilde{\varphi}}\,\omega_{\tilde{\varphi}}^n\,dt.$$

Taking $\tilde{\varphi}(t) = t\varphi$, since (with some abuse of notation)

$$\frac{d}{dt}\,\frac{(\omega + t\,dd^c\varphi)^{n+1} - \omega^{n+1}}{(n+1)\,dd^c\varphi} = (\omega + t\,dd^c\varphi)^n = \omega_{t\varphi}^n,$$

we obtain the formula

$$I(\varphi) = \frac{1}{n+1} \sum_{p=0}^{n} \frac{1}{V} \int_M \varphi\, \omega_\varphi^p \wedge \omega^{n-p}.$$

We also get

$$I(\varphi + c) = I(\varphi) + c$$

for any constant $c$.

Now for any curve $\varphi$ in $\mathcal{H}$ by (22) and (17) we have

$$\left(\frac{d}{dt}\right)^2 I(\varphi) = \frac{d}{dt}\langle\langle \dot\varphi, 1\rangle\rangle = \langle\langle \nabla_{\dot\varphi}\dot\varphi, 1\rangle\rangle$$

and it follows that $I$ is affine along geodesics. Moreover, if $\varphi$ is a geodesic then so is $\varphi - I(\varphi)$. Therefore, by uniqueness of geodesics, $\mathcal{H}_0 := I^{-1}(0)$ is a totally geodesic subspace of $\mathcal{H}$. The bijective mapping

$$\mathcal{H}_0 \ni \varphi \longmapsto \omega_\varphi \in \{\omega\}$$

induces the Riemannian structure on $\{\omega\}$. By (22) we have

$$T_\varphi \mathcal{H}_0 = \{\psi \in C^\infty(M) : \int_M \psi \omega_\varphi^n = 0\}.$$

One can easily show that this Riemannian structure on $\{\omega\}$ is independent of the choice of $\omega$.

**Mabuchi K-energy [34].** It is defined by the condition

$$K(0) = 0, \quad d_\varphi K.\psi = -\frac{1}{V} \int_M \psi (S_\varphi - \bar{S}) \omega_\varphi^n, \tag{24}$$

where $\bar{S}$ is the average of scalar curvature $S_\varphi$ (it is given by (16)). We are thus looking for $K$ satisfying $dK = \beta + \bar{S}\alpha$, where $\alpha$ is given by (23) and

$$\beta(\varphi).\psi = -\frac{1}{V} \int_M \psi S_\varphi\, \omega_\varphi^n = -\frac{2n}{V} \int_M \psi Ric_\varphi \wedge \omega_\varphi^{n-1},$$

we have to show that $d\beta = 0$. We compute

$$d_\varphi(Ric_\varphi).\psi = d_\varphi\big(Ric_\omega - dd^c \log \frac{\omega_\varphi^n}{\omega^n}\big).\psi = -\frac{1}{2} dd^c \Delta\psi \tag{25}$$

and thus

$$d_\varphi(\beta(\varphi).\psi).\tilde\psi = \frac{1}{V} \int_M \psi \big(\frac{1}{2}\Delta^2 \tilde\psi\, \omega_\varphi^n - 2n(n-1) dd^c \tilde\psi \wedge Ric_\varphi \wedge \omega_\varphi^{n-2}\big).$$

It is clear that the latter expression is symmetric in $\psi$ and $\tilde{\psi}$ and therefore $d\beta = 0$.

To get a precise formula for $K$ take as before $\tilde{\varphi} = t\varphi$. Similarly we have

$$\frac{d}{dt}\,\frac{\omega_{t\varphi}^n - \omega^n}{n\,dd^c\varphi} = \omega_{t\varphi}^{n-1}$$

and

$$\frac{d}{dt}\big[\big(\log\frac{\omega_{\tilde{\varphi}}^n}{\omega^n}\big)\omega_{\tilde{\varphi}}^n\big] = n\big(1+\log\frac{\omega_{\tilde{\varphi}}^n}{\omega^n}\big)dd^c\dot{\tilde{\varphi}}\wedge\omega_{\tilde{\varphi}}^{n-1}.$$

Using this we will easily get (see also [37, 42])

$$K(\varphi) = \frac{2}{V}\int_M \left[\big(\log\frac{\omega_\varphi^n}{\omega^n}\big)\omega_\varphi^n - \varphi\sum_{p=0}^{n-1} Ric_\omega\wedge\omega_\varphi^p\wedge\omega^{n-p-1}\right] + \bar{S}\,I(\varphi).$$

The usefulness of the K-energy in some geometric problems becomes clear in view of the following two results:

**Proposition 7 (Mabuchi [35], Donaldson [20]).** *For any smooth curve $\varphi$ in $\mathcal{H}$ we have*

$$\left(\frac{d}{dt}\right)^2 K(\varphi) = -\frac{1}{V}\int_M \nabla_{\dot{\varphi}}\dot{\varphi}(S_\varphi - \bar{S})\omega_\varphi^n + \frac{1}{2V}\int_M |\mathcal{L}\dot{\varphi}|^2\omega_\varphi^n.$$

*In particular, the K-energy is convex along geodesics.*

*Proof.* We have

$$\frac{d}{dt}K(\varphi) = -\langle\langle\dot{\varphi}, S_\varphi - \bar{S}\rangle\rangle,$$

therefore

$$\left(\frac{d}{dt}\right)^2 K(\varphi) = -\langle\langle\nabla_{\dot{\varphi}}\dot{\varphi}, S_\varphi - \bar{S}\rangle\rangle - \langle\langle\dot{\varphi}, \nabla_{\dot{\varphi}}S_\varphi\rangle\rangle.$$

Moreover

$$-\langle\dot{\varphi}, \nabla_{\dot{\varphi}}S_\varphi\rangle = \frac{1}{V}\int_M \dot{\varphi}\left(\frac{1}{2}\langle\nabla S_\varphi, \nabla\dot{\varphi}\rangle - \frac{d}{dt}S_\varphi\right)\omega_\varphi^n.$$

Write $u = g + \varphi$. Then

$$S_\varphi = -2u^{p\bar{q}}\big(\log\det(u_{j\bar{k}})\big)_{p\bar{q}}$$

and, since $\dot{g} = 0$,

$$\frac{d}{dt} S_\varphi = 2u^{p\bar{t}} u^{s\bar{q}} \big(\log\det(u_{j\bar{k}})\big)_{p\bar{q}} \dot{\varphi}_{s\bar{t}} - 2u^{p\bar{q}} \big(u^{j\bar{k}} \dot{\varphi}_{j\bar{k}}\big)_{p\bar{q}}$$
$$= -\frac{1}{2}\langle Ric_\varphi, \nabla^2\dot{\varphi}\rangle - \frac{1}{2}\Delta^2\dot{\varphi}.$$

The result now follows from Proposition 2. □

**Proposition 8 (Donaldson [20]).** *Let $\omega_{\varphi_0}$ and $\omega_{\varphi_1}$ be csc metrics. Assume moreover that $\varphi_0$ and $\varphi_1$ can be joined by a smooth geodesic. Then there exists a biholomorphism $F$ of $M$ such that $\omega_{\varphi_0} = F^*\omega_{\varphi_1}$.*

*Proof.* Let $\varphi$ be this geodesic and set $h := K(\varphi)$. Then, since $S_{\varphi_0} = S_{\varphi_1} = \bar{S}$, we have $\dot{h}(0) = \dot{h}(1) = 0$ and by Proposition 7 $h$ is convex. Therefore $\ddot{h} = 0$ and, again by Proposition 7, $\mathcal{L}\dot{\varphi} = 0$, that is $\nabla\dot{\varphi}$ is a flow of holomorphic vector fields. By $F$ denote the flow of biholomorphisms generated by $\frac{1}{2}\nabla\dot{\varphi}$ (so that $\dot{F} = \frac{1}{2}\nabla\dot{\varphi} \circ F$, $F|_{t=0} = id$).

We have to check that $\omega_{\varphi_0} = F^*\omega_\varphi$, it will be enough to show that $\frac{d}{dt}F^*\omega_\varphi = 0$. We compute

$$\frac{d}{dt}F^*\omega_\varphi = F^*\big(L_{\frac{1}{2}\nabla\dot{\varphi}}\omega_\varphi + dd^c\dot{\varphi}\big) = F^*d\big(\frac{1}{2}i_{\nabla f}\omega_\varphi + d^c\dot{\varphi}\big) = 0$$

by (13) (where $L_X = i_X \circ d + d \circ i_X$ is the Lie derivative). (This argument from symplectic geometry is called a Moser's trick.) □

In view of the Lempert–Vivas counterexample Proposition 8 is not sufficient to prove the uniqueness of csc metrics. For a more direct approach to this problem see [19].

## 5 Lempert–Vivas Example

It is well known that in general one cannot expect $C^\infty$-regularity of solutions of the homogeneous Monge–Ampère equation. The simplest example is due to Gamelin and Sibony [25]: the function

$$u(z,w) := \big(\max\{0, |z|^2 - 1/2, |w|^2 - 1/2\}\big)^2$$

satisfies $dd^c u \ge 0$, $(dd^c u)^2 = 0$ in the unit ball $\mathbb{B}$ of $\mathbb{C}^2$,

$$u(z,w) = \big(|z|^2 - 1/2\big)^2 = \big(|w|^2 - 1/2\big)^2 \in C^\infty(\partial\mathbb{B}),$$

but $u \notin C^2(\mathbb{B})$.

For some time it was however an open problem whether there exists a smooth geodesic connecting arbitrary two elements in $\mathcal{H}$. In the special case of toric Kähler manifolds it was in fact shown in [28] that it is indeed the case. This suggests that a possible counterexample would have to be more complicated, as the Gamelin–Sibony example from the flat case is *toric*.

The counterexample to the geodesic problem was found recently by Lempert and Vivas [33]. It works on Kähler manifolds with a holomorphic isometry $h : M \to M$ satisfying $h^2 = id$ and having an isolated fixed point. We will consider the simplest situation, that is the Riemann sphere $\mathbb{P}$ with the Fubini-Study metric $\omega = dd^c(\log(1+|z|^2))$ and $h(z) = -z$. The key is the following result:

**Lemma 9 (Lempert–Vivas [33]).** *Take $\varphi \in \mathcal{H}$ with*

$$\varphi(-z) = \varphi(z) \tag{26}$$

*Assume that there is a geodesic of class $C^3$ joining 0 with $\varphi$. Then either $1+\varphi_{z\bar{z}}(0) = |1-\varphi_{zz}(0)|$ or $|\varphi_{zz}(0)| \le |\varphi_{z\bar{z}}(0)|$, in particular*

$$|\varphi_{zz}(0)| \le 2 + \varphi_{z\bar{z}}(0). \tag{27}$$

*Proof.* By $\tilde{\varphi}$ denote the geodesic joining 0 with $\varphi$. We can assume that it is a $C^3$ function defined on $\mathbb{P} \times \bar{S}$, where $S = \{0 < \operatorname{Im} w < 1\}$, and such that

$$\tilde{\varphi}(z, w+\sigma) = \tilde{\varphi}(z, w), \quad \sigma \in \mathbb{R}.$$

Moreover, by uniqueness of the Dirichlet problem (see Theorem 21 below) by (26) we have

$$\tilde{\varphi}(-z, w) = \tilde{\varphi}(z, w).$$

On $\mathbb{C} \times \bar{S}$ set $u := g + \tilde{\varphi}$. Then $u \in C^3(\mathbb{C} \times \bar{S})$,

$$u_{z\bar{z}} u_{w\bar{w}} - |u_{z\bar{w}}|^2 = 0,$$

$u_{z\bar{z}} > 0$, $u$ is independent of $\sigma = \operatorname{Re} w$, $u(\cdot, 0) = g$, $u(\cdot, i) = g + \varphi$.

Since $(u_{j\bar{k}})$ is of maximal rank, it is well known (see e.g. [4]) that there is a $C^1$ foliation of $\mathbb{C} \times \bar{S}$ by holomorphic discs (with boundary) which are tangent to $dd^c u$. This foliation is also invariant under the mapping $(z, w) \mapsto (-z, w)$ and thus $\{0\} \times \bar{S}$ is one of the leaves. The neighboring leaves are graphs of functions defined on $\bar{S}$: there exists $f \in C^1(U \times \bar{S})$, where $U$ is a neighborhood of 0, $f(z, \cdot)$ holomorphic in $S$ and $\{(f(z, w), w) : w \in \bar{S}\}$ is the leaf passing through $(z, 0)$. Since this leaf is tangent to $dd^c u$, it follows that

$$u_{z\bar{z}}(f(z, w), w)\overline{f_w(z, w)} + u_{z\bar{w}}(f(z, w), w) = 0$$

which is equivalent to the fact that $u_z(f(z, w), w)$ is holomorphic in $w$. Set

$$\Phi(w) := \frac{d}{dt}\Big|_{t=0} f(t,w)$$

and

$$\Psi(w) := \frac{d}{dt}\Big|_{t=0} u_z(f(t,w),w) = u_{zz}(0,w)\Phi(w) + u_{z\bar z}(0,w)\overline{\Phi(w)}.$$

Then $\Phi$, $\Psi$ are holomorphic in $S$, $\Phi$ is $C^1$ on $\bar S$, and $\Psi$ is continuous on $\bar S$.

Since $u$ is independent of $\operatorname{Re} w$, we can write

$$\Psi(w) = \begin{cases} \overline{\Phi(w)}, & \operatorname{Im} w = 0 \\ P\,\overline{\Phi(w)} + Q\,\Phi(w), & \operatorname{Im} w = 1, \end{cases}$$

where $P = g_{z\bar z}(0) + \varphi_{z\bar z}(0) = 1 + \varphi_{z\bar z}(0) > 0$ and $Q = g_{zz}(0) + \varphi_{zz}(0) = \varphi_{zz}(0)$. Since $\Psi_{\bar w} = 0$,

$$u_{zz\bar w}(0,w)\Phi(w) + u_{z\bar z\bar w}(0,w)\overline{\Phi(w)} + u_{z\bar z}(0,w)\overline{\Phi'(w)} = 0.$$

On $\{\operatorname{Im} w = 0\}$ we thus have

$$\begin{cases} \Phi'(\sigma) = A\Phi(\sigma) + B\overline{\Phi(\sigma)} \\ \Phi(0) = 1, \end{cases}$$

where

$$A = -u_{z\bar z w}(0,0), \quad B = -u_{\bar z\bar z w}(0,0).$$

Therefore $\Phi$ on $\{\operatorname{Im} w = 0\}$ is of the form

$$\Phi(\sigma) = x e^{\lambda\sigma} + \bar y e^{\bar\lambda\sigma},$$

where

$$\begin{pmatrix} A & B \\ \bar B & \bar A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

and $x + \bar y = 1$. Note that $A \in i\mathbb{R}$ (because $u_\sigma = 0$), and thus either $\lambda \in \mathbb{R}$ or $\lambda \in i\mathbb{R}$.

By the Schwarz reflection principle and analytic continuation we obtain

$$\Phi(w) = x e^{\lambda w} + \bar y e^{\bar\lambda w}, \quad w \in \bar S.$$

Similarly, since

$$\Psi(\sigma) = \overline{\Phi(\sigma)} = \bar x e^{\bar\lambda\sigma} + y e^{\lambda\sigma},$$

we infer

$$\Psi(w) = \bar{x}e^{\bar{\lambda}w} + ye^{\lambda w}, \quad w \in \bar{S}.$$

Therefore, using the fact that $\Psi(w) = P\,\overline{\Phi(w)} + Q\,\Phi(w)$ on $\{\operatorname{Im} w = 1\}$, we get

$$\bar{x}e^{\bar{\lambda}(\sigma+i)} + ye^{\lambda(\sigma+i)} = P\left(\bar{x}e^{\bar{\lambda}(\sigma-i)} + ye^{\lambda(\sigma-i)}\right) + Q\left(xe^{\lambda(\sigma+i)} + \bar{y}e^{\bar{\lambda}(\sigma+i)}\right).$$

We have to consider two cases. If $\lambda \in \mathbb{R}$ then

$$e^{\lambda i} = Pe^{-\lambda i} + Qe^{\lambda i}.$$

This means that

$$P = e^{2\lambda i}(1-Q),$$

in particular

$$P = |1-Q|.$$

If $\lambda = i\mu \in i\mathbb{R}$ then we will get

$$\begin{cases} \bar{x}e^{\mu} = P\bar{x}e^{-\mu} + Q\bar{y}e^{\mu} \\ ye^{-\mu} = Pye^{\mu} + Qxe^{-\mu}. \end{cases}$$

Rewrite this as

$$\begin{cases} \bar{x}\left(e^{2\mu} - P\right) = \bar{y}Qe^{2\mu} \\ \bar{y}\left(e^{-2\mu} - P\right) = \bar{x}\bar{Q}e^{-2\mu}. \end{cases}$$

Since at least one of $x, y$ does not vanish, we will obtain

$$|Q|^2 = (e^{2\mu} - P)(e^{-2\mu} - P) \le (1-P)^2. \qquad \square$$

If $\varphi$ is a smooth compactly supported function in $\mathbb{C}$ then $\varphi \in \mathcal{H}$ provided that $g_{z\bar{z}} + \varphi_{z\bar{z}} > 0$. The following lemma shows that there are such functions satisfying (26) but not (27):

**Lemma 10.** *For every real $a$ and $\varepsilon > 0$ there exists smooth $\varphi$ with support in the unit disc, satisfying (26), and such that $\varphi_{zz}(0) = a$, $\varphi_{z\bar{z}}(0) = 0$, $|\varphi_{z\bar{z}}| \le \varepsilon$ in $\mathbb{C}$.*

*Proof.* We may assume that $a > 0$. The function we seek will be of the form

$$\varphi(z) = \operatorname{Re}(z^2)\chi(|z|^2),$$

where $\chi \in C^\infty(\mathbb{R}_+)$ is supported in the interval $(0, 1)$ and constant near 0. Then $\varphi_{zz}(0) = \chi(0)$, $\varphi_{z\bar{z}}(0) = 0$ and

$$\varphi_{z\bar{z}} = \operatorname{Re}(z^2)\left(3\chi'(|z|^2) + |z|^2\chi''(|z|^2)\right).$$

We are looking for $\chi$ of the form

$$\chi(t) = f(-\log t),$$

where $f \in C^\infty(\mathbb{R}_+)$ is supported in $(1, \infty)$ and equal to $a$ near $\infty$. We have, with $t = |z|^2$,

$$|\varphi_{z\bar{z}}| \le t\big|3\chi'(t) + t\chi''(t)\big| = \big|-2f'(-\log t) + f''(-\log t)\big|.$$

We can now easily arrange $f$ in such a way that $|f'|$ and $|f''|$ are arbitrarily small. $\square$

## 6 Metric Structure of $\mathcal{H}$

Although smooth geodesics in $\mathcal{H}$ do not always exist, one can make a geometric use of existence of $\varepsilon$-geodesics. The Riemannian structure gives a distance on $\mathcal{H}$:

$$d(\varphi_0, \varphi_1) = \inf\{l(\varphi) : \varphi \in C^\infty([0,1], \mathcal{H}), \varphi(0) = \varphi_0, \varphi(1) = \varphi_1\}, \quad \varphi_0, \varphi_1 \in \mathcal{H},$$

where

$$l(\varphi) = \int_0^1 |\dot\varphi| dt = \int_0^1 \sqrt{\frac{1}{V}\int_M \dot\varphi^2 \omega_\varphi^n}\, dt$$

(note that the family in the definition of $d$ is always nonempty, for example $\varphi(t) = (1-t)\varphi_0 + t\varphi_1$ is a smooth curve in $\mathcal{H}$ connecting $\varphi_0$ with $\varphi_1$). We will show the following result of Chen [18] (see also [15]):

**Theorem 11.** *$(\mathcal{H}, d)$ is a metric space.*

The only problem with this result is to show that $d(\varphi_0, \varphi_1) > 0$ if $\varphi_0 \neq \varphi_1$. The main tool in the proof will be existence of $\varepsilon$-geodesics. In fact, making use of results proved in Sects. 7–13 and the standard elliptic theory, we have the following existence result for $\varepsilon$-geodesics:

**Theorem 12.** *For $\varphi_0, \varphi_1 \in \mathcal{H}$ and $\varepsilon > 0$ there exists a unique $\varepsilon$-geodesic $\varphi$ connecting $\varphi_0$ with $\varphi_1$. Moreover, it depends smoothly on $\varphi_0, \varphi_1$, i.e. if $\varphi_0, \varphi_1 \in C^\infty([0,1], \mathcal{H})$ then there exists unique $\varphi \in C^\infty([0,1] \times [0,1], \mathcal{H})$ such that $\varphi(0,\cdot) = \varphi_0$, $\varphi(1,\cdot) = \varphi_1$, and $\varphi(\cdot, t)$ is an $\varepsilon$-geodesic for every $t \in [0,1]$. In addition,*

$$\Delta\varphi,\ |\nabla\dot\varphi|,\ \ddot\varphi \le C, \tag{28}$$

*(here $\Delta$ and $\nabla$ are taken w.r.t. $\omega$) where $C$ is independent of $\varepsilon$ (if $\varepsilon$ is small).*

We start with the following lemma:

**Lemma 13.** *For an $\varepsilon$-geodesic $\varphi$ connecting $\varphi_0, \varphi_1 \in \mathcal{H}$ we have*

$$\frac{1}{V}\int_M \dot{\varphi}^2\omega_\varphi^n \geq \mathcal{E}(\varphi_0,\varphi_1) - 2\varepsilon \sup_{[0,1]} ||\dot{\varphi}||,$$

*where*

$$\mathcal{E}(\varphi_0,\varphi_1) := \max\Big\{\frac{1}{V}\int_{\{\varphi_0>\varphi_1\}}(\varphi_0-\varphi_1)^2\omega_{\varphi_0}^n, \frac{1}{V}\int_{\{\varphi_1>\varphi_0\}}(\varphi_1-\varphi_0)^2\omega_{\varphi_1}^n\Big\}.$$

*In particular,*

$$l(\varphi)^2 \geq \mathcal{E}(\varphi_0,\varphi_1) - 2\varepsilon \sup_{[0,1]} ||\dot{\varphi}||.$$

*Proof.* For

$$E := \frac{1}{V}\int_M \dot{\varphi}^2\omega_\varphi^n$$

we have

$$\dot{E} = \frac{1}{V}\int_M (2\dot{\varphi}\ddot{\varphi} + \frac{1}{2}\dot{\varphi}^2\Delta\dot{\varphi})\omega_\varphi^n = \frac{2}{V}\int_M \dot{\varphi}(\ddot{\varphi} - \frac{1}{2}|\nabla\dot{\varphi}|^2)\omega_\varphi^n = \frac{2\varepsilon}{V}\int_M \dot{\varphi}\omega^n.$$

Thus $|\dot{E}| \leq 2\varepsilon \sup_{[0,1]} ||\dot{\varphi}||$ which implies that

$$E(t) \geq \max\{E(0), E(1)\} - 2\varepsilon \sup_{[0,1]} ||\dot{\varphi}||.$$

Since $\ddot{\varphi} \geq 0$,

$$\dot{\varphi}(0) \leq \varphi(1) - \varphi(0) \leq \dot{\varphi}(1).$$

For $z \in M$ with $\varphi_1(z) > \varphi_0(z)$ we thus have $\dot{\varphi}(z,1)^2 \geq (\varphi_1(z) - \varphi_0(z))^2$. Therefore

$$E(1) \geq \frac{1}{V}\int_{\{\varphi_1>\varphi_0\}}(\varphi_1-\varphi_0)^2\omega_{\varphi_1}^n.$$

Similarly

$$E(0) \geq \frac{1}{V}\int_{\{\varphi_0>\varphi_1\}}(\varphi_0-\varphi_1)^2\omega_{\varphi_0}^n$$

and the desired estimate follows. □

**Theorem 14.** *Suppose $\psi \in C^\infty([0,1],\mathcal{H})$ and $\tilde{\psi} \in \mathcal{H} \setminus \psi([0,1])$. For $\varepsilon > 0$ by $\varphi$ denote an element of $C^\infty([0,1]\times[0,1],\mathcal{H})$ uniquely determined by the following property: $\varphi(\cdot,t)$ is an $\varepsilon$-geodesic connecting $\tilde{\psi}$ with $\psi(t)$. Then for $\varepsilon$ sufficiently small*

$$l(\varphi(\cdot,0)) \leq l(\psi) + l(\varphi(\cdot,1)) + C\varepsilon,$$

*where $C > 0$ is independent of $\varepsilon$.*

*Proof.* Without loss of generality we may assume that $V = 1$. Set

$$l_1(t) := \int_0^t ||\dot\psi|| d\tilde t, \quad l_2(t) := l(\varphi(\cdot, t)).$$

It is enough to show that $l_1' + l_2' \geq -C\varepsilon$ on [0, 1]. We clearly have

$$l_1' = ||\dot\psi|| = \sqrt{\int_M \dot\psi^2 \omega_\psi^n}.$$

On the other hand,

$$l_2(t) = \int_0^1 \sqrt{E(s,t)}\, ds,$$

where

$$E = \int_M \varphi_s^2 \omega_\varphi^n$$

(using the notation $\varphi_s = \partial\varphi/\partial s$). We have

$$E_s = 2\int_M \varphi_s \nabla_{\varphi_s} \varphi_s \omega_\varphi^n = 2\varepsilon \int_M \varphi_s \omega^n$$

and

$$\begin{aligned} E_t &= \int_M \big(2\varphi_s\varphi_{st} + \frac{1}{2}\varphi_s^2 \Delta\varphi_t\big)\omega_\varphi^n \\ &= 2\int_M \varphi_s\big(\varphi_{st} - \frac{1}{2}\langle \nabla\varphi_s, \nabla\varphi_t\rangle\big)\omega_\varphi^n \\ &= 2\int_M \varphi_s \nabla_{\varphi_s} \varphi_t \omega_\varphi^n \\ &= 2\frac{\partial}{\partial s}\langle\langle \varphi_s, \varphi_t\rangle\rangle - 2\int_M \varphi_t \nabla_{\varphi_s}\varphi_s \omega_\varphi^n \\ &= 2\frac{\partial}{\partial s}\int_M \varphi_s\varphi_t\omega_\varphi^n - 2\varepsilon\int_M \varphi_t \omega^n. \end{aligned}$$

Therefore

$$\begin{aligned} l_2' &= \frac{1}{2}\int_0^1 E^{-1/2} E_t\, ds \\ &= \int_0^1 E^{-1/2}\frac{\partial}{\partial s}\int_M \varphi_s\varphi_t\omega_\varphi^n\, ds - \varepsilon\int_0^1 E^{-1/2}\int_M \varphi_t\omega^n\, ds, \end{aligned}$$

and the first term is equal to

$$\left[E^{-1/2}\int_M \varphi_s\varphi_t\omega_\varphi^n\right]_{s=0}^{s=1} + \frac{1}{2}\int_0^1 E^{-3/2}E_s\int_M \varphi_s\varphi_t\omega_\varphi^n\,ds$$
$$= \left(\int_M \eta^2\omega_\varphi^n\right)^{-1/2}\int_M \eta\dot\psi\omega_\varphi^n - \varepsilon\int_0^1 E^{-3/2}\int_M \varphi_s\omega^n\int_M \varphi_s\varphi_t\omega_\varphi^n\,ds,$$

where $\eta = \varphi_s(1,\cdot)$; we have used that $\varphi_t(0,\cdot) = 0$, $\varphi_t(1,\cdot) = \dot\psi$, and

$$E(1,\cdot) = \int_M \eta^2\omega_\varphi^n.$$

From the Schwarz inequality it now follows that $l_1' + l_2' \geq -R$, where

$$R = \varepsilon\int_0^1 E^{-1/2}\int_M \varphi_t\omega^n\,ds + \varepsilon\int_0^1 E^{-3/2}\int_M \varphi_s\omega^n\int_M \varphi_s\varphi_t\omega_\varphi^n\,ds.$$

By Lemma 13

$$E(s,t) \geq \mathcal{E}(\tilde\psi,\psi(t)) - 2\varepsilon\sup_{[0,1]}||\varphi_s(\cdot,t)||.$$

Since $\mathcal{E}(\tilde\psi,\psi(t))$ is continuous and positive for $t \in [0,1]$, it follows that for $\varepsilon$ sufficiently small

$$E \geq c > 0$$

and thus $R \leq C\varepsilon$. □

We are now in position to show that the geodesic distance is the same as $d$:

**Theorem 15.** *Let $\varphi^\varepsilon$ be an $\varepsilon$-geodesic connecting $\varphi_0, \varphi_1 \in \mathcal{H}$. Then*

$$d(\varphi_0,\varphi_1) = \lim_{\varepsilon\to 0+} l(\varphi^\varepsilon).$$

*Proof.* Let $\psi \in C^\infty([0,1],\mathcal{H})$ be an arbitrary curve connecting $\varphi_0, \varphi_1 \in \mathcal{H}$. We have to show that

$$\overline{\lim_{\varepsilon\to 0+}}\, l(\varphi^\varepsilon) \leq l(\psi).$$

Without loss of generality we may assume that $\varphi_1 \notin \psi([0,1))$. Extend $\varphi^\varepsilon$ to a function from $C^\infty([0,1]\times[0,1),\mathcal{H})$ in such a way that $\varphi^\varepsilon(0,\cdot) \equiv \varphi_1$, $\varphi^\varepsilon(1,\cdot) \equiv \psi$ on $[0,1)$ and $\varphi^\varepsilon(\cdot,t)$ is an $\varepsilon$-geodesic for $t \in [0,1)$. By Theorem 14 for $t \in [0,1)$ we have

$$l(\varphi^\varepsilon(\cdot,0)) \leq l(\psi|_{[0,t]}) + l(\varphi^\varepsilon(\cdot,t)) + C(t)\varepsilon.$$

Since clearly

$$\lim_{t\to 1-} l(\psi|_{[0,t]}) = l(\psi),$$

it remains to show that

$$\varlimsup_{t\to 1-}\varlimsup_{\varepsilon\to 0+} l(\varphi^\varepsilon(\cdot,t))=0.$$

But it follows immediately from the following:

**Lemma 16.** *For an $\varepsilon$-geodesic $\varphi$ connecting $\varphi_0,\varphi_1\in\mathcal{H}$ we have*

$$l(\varphi)\le\sqrt{V}\left(||\varphi_0-\varphi_1||_{L^\infty(M)}+\frac{\varepsilon}{2\lambda^n}\right),$$

*where $\lambda>0$ is such that $\omega_{\varphi_0}\ge\lambda\omega$, $\omega_{\varphi_1}\ge\lambda\omega$.*

*Proof.* Since $\ddot{\varphi}\ge 0$,

$$\dot{\varphi}(0)\le\dot{\varphi}\le\dot{\varphi}(1).$$

So to estimate $|\dot{\varphi}|$ we need to bound $\dot{\varphi}(0)$ from below and $\dot{\varphi}(1)$ from above. The function

$$v(\zeta)=2b\log^2|\zeta|+(a-2b)\log|\zeta|-a$$

satisfies $v_{\zeta\bar{\zeta}}=b|\zeta|^{-2}$, $v=-a$ on $|\zeta|=1$, and $v=0$ on $|\zeta|=e$. We want to choose $a,b$ so that $\varphi_1+v\le\varphi$ on $\tilde{M}:=M\times\{1\le|\zeta|\le e\}$.

On one hand, if $a:=||\varphi_0-\varphi_1||_{L^\infty(M)}$ then $\varphi_1+v\le\varphi$ on $\partial\tilde{M}$. On the other one we have (if $b>0$)

$$(\omega+dd^c(\varphi_1+v))^{n+1}\ge\big(\lambda\omega+\frac{b}{|\zeta|^2}dd^c|\zeta|^2\big)^{n+1}=\frac{b\lambda^n}{|\zeta|^2}(\omega+dd^c|\zeta|^2)^{n+1}.$$

Therefore, by (21) if $b:=\varepsilon/4\lambda^n$ we will get $\omega^{n+1}_{\varphi_1+v}\ge\omega^{n+1}_\varphi$ and $\varphi_1+v\le\varphi$ on $\tilde{M}$ by comparison principle. We will obtain

$$\dot{\varphi}(1)\le\frac{d}{dt}\left(2bt^2+(a-2b)t-a\right)\Big|_{t=1}=||\varphi_0-\varphi_1||_{L^\infty(M)}+\frac{\varepsilon}{2\lambda^n}.$$

Similarly we can show the lower bound for $\dot{\varphi}(0)$ and the estimate follows from the definition of $l(\varphi)$. □

Combining Theorem 14 with Lemma 13 we get the following quantitative estimate from which Theorem 11 follows:

**Theorem 17.** *For $\varphi_0,\varphi_1\in\mathcal{H}$ we have*

$$d(\varphi_0,\varphi_1)\ge\sqrt{\max\Big\{\int_{\{\varphi_0>\varphi_1\}}(\varphi_0-\varphi_1)^2\omega^n_{\varphi_0},\int_{\{\varphi_1>\varphi_0\}}(\varphi_1-\varphi_0)^2\omega^n_{\varphi_1}\Big\}}.$$ □

## 7 Monge–Ampère Equation, Uniqueness

We assume that $M$ is a compact complex manifold with smooth boundary (which may be empty) with a Kähler form $\omega$. Our goal will be to prove the following two results:

**Theorem 18 (Yau [47]).** *Assume that $M$ has no boundary. Then for $f \in C^\infty(M)$, $f > 0$ such that $\int_M f\omega^n = V$ there exists, unique up to an additive constant, $\varphi \in C^\infty(M)$ with $\omega + dd^c\varphi > 0$ satisfying the complex Monge–Ampère equation*

$$(\omega + dd^c\varphi)^n = f\omega^n. \tag{29}$$

**Theorem 19.** *Assume that $M$ has smooth nonempty boundary. Take $f \in C^\infty(M)$, $f > 0$, and let $\psi \in C^\infty(M)$ be such that $\omega + dd^c\psi > 0$ and $(\omega + dd^c\psi)^n \geq f\omega^n$. Then there exists $\varphi \in C^\infty(M)$, $\omega + dd^c\varphi > 0$, satisfying (29) and $\varphi = \psi$ on $\partial M$.*

Theorem 19 can be rephrased as follows: the Dirichlet problem

$$\begin{cases} \varphi \in C^\infty(M) \\ \omega + dd^c\varphi > 0 \\ (\omega + dd^c\varphi)^n = f\omega^n \\ \varphi = \psi \text{ on } \partial M \end{cases}$$

has a solution provided that it has a smooth subsolution. It is a combination of the results proved in several papers [1, 15, 16, 18, 27, 47].

We will give a proof of Theorem 19 under additional assumption that the boundary of $M$ is *flat*, that is near every boundary point, after a holomorphic change of coordinates, the boundary is of the form $\{\operatorname{Re} z^n = 0\}$. We will use this assumption only for the boundary estimate for second derivatives (see Theorem 27 below), but the result is also true without it (see [27]).

This extra assumption is satisfied in the geodesic equation case, then $M$ is of the form $M' \times \bar{D}$, where $M'$ is a manifold without boundary and $D$ is a bounded domain in $\mathbb{C}$ with smooth boundary. This will immediately give existence of smooth $\varepsilon$-geodesics. (Note that by (20) the geodesic equation is covered here.)

The uniqueness in Theorems 18 and 19 is in fact very simple: if $\varphi, \tilde{\varphi}$ are the solutions then

$$0 = \omega_\varphi^n - \omega_{\tilde{\varphi}}^n = dd^c(\varphi - \tilde{\varphi}) \wedge T,$$

where

$$T = \sum_{p=0}^{n-1} \omega_\varphi^p \wedge \omega_{\tilde{\varphi}}^{n-p-1}.$$

Since $T > 0$, we will get $\varphi - \tilde{\varphi} = const$ in the first case and $\varphi = \tilde{\varphi}$ in the second one.

This argument does not work anymore if we allow the solutions to be degenerate, that is assuming only that $\omega_\varphi \geq 0$, $\omega_{\tilde{\varphi}} \geq 0$. In fact, much more general results hold here. We will allow continuous solutions given by the Bedford Taylor theory [5] (see also [8]) – then $\omega_\varphi^n$ is a measure.

**Theorem 20 ([12]).** *Assume that $M$ has no boundary. If $\varphi, \tilde{\varphi} \in C(M)$ are such that $\omega_\varphi \geq 0$, $\omega_{\tilde{\varphi}} \geq 0$ and $\omega_\varphi^n = \omega_{\tilde{\varphi}}^n$ then $\varphi - \tilde{\varphi} = const.$*

*Proof.* Assume $n = 2$, the general case is similar, for details see [12]. Write

$$0 = \omega_\varphi^2 - \omega_{\tilde\varphi}^2 = dd^c\rho \wedge (\omega_\varphi + \omega_{\tilde\varphi}),$$

where $\rho = \varphi - \tilde\varphi$. Therefore

$$0 = -\int_M \rho dd^c\rho \wedge (\omega_\varphi + \omega_{\tilde\varphi}) = \int_M d\rho \wedge d^c\rho \wedge (\omega_\varphi + \omega_{\tilde\varphi}),$$

and thus

$$d\rho \wedge d^c\rho \wedge \omega_\varphi = d\rho \wedge d^c\rho \wedge \omega_{\tilde\varphi} = 0. \tag{30}$$

We have to show that

$$d\rho \wedge d^c\rho \wedge \omega = 0. \tag{31}$$

By (30)

$$\begin{aligned}\int_M d\rho \wedge d^c\rho \wedge \omega &= -\int_M d\rho \wedge d^c\rho \wedge dd^c\varphi \\ &= \int_M d\rho \wedge d^c\varphi \wedge dd^c\rho = \int_M d\rho \wedge d^c\varphi \wedge (\omega_\varphi - \omega_{\tilde\varphi}).\end{aligned}$$

By the Schwarz inequality and (30) again

$$\left|\int_M d\rho \wedge d^c\varphi \wedge \omega_\varphi\right|^2 \le \int_M d\rho \wedge d^c\rho \wedge \omega_\varphi \int_M d\varphi \wedge d^c\varphi \wedge \omega_\varphi = 0.$$

Similarly we show that

$$\int_M d\rho \wedge d^c\varphi \wedge \omega_{\tilde\varphi} = 0$$

and (31) follows. □

**Theorem 21 ([15]).** *Let $M$ have nonempty boundary. Assume that $\varphi, \tilde\varphi \in C(M)$ are such that $\omega_\varphi \ge 0$, $\omega_{\tilde\varphi} \ge 0$, $\omega_\varphi^n \ge \omega_{\tilde\varphi}^n$ and $\varphi \le \tilde\varphi$ on $\partial M$. Then $\varphi \le \tilde\varphi$ in $M$.*

*Proof.* For $\varepsilon > 0$ set $\varphi_\varepsilon := \max\{\varphi - \varepsilon, \tilde\varphi\}$, so that $\varphi_\varepsilon = \tilde\varphi$ near $\partial M$. Since for continuous plurisubharmonic functions we have

$$(dd^c \max\{u, v\})^n \ge \chi_{\{u\ge v\}}(dd^c u)^n + \chi_{\{u<v\}}(dd^c v)^n$$

(it is a very simple consequence of the continuity of the Monge–Ampère operator, see e.g. Theorem 3.8 in [8]), it follows that $\omega_{\varphi_\varepsilon}^n \ge \omega_{\tilde\varphi}^n$. Therefore, without loss of generality, we may assume that $\varphi \ge \tilde\varphi$ in $M$, $\varphi = \tilde\varphi$ near $\partial M$, and we have to show that $\varphi = \tilde\varphi$ in $M$.

Assume again $n = 2$. Then, since $\rho := \varphi - \tilde\varphi$ vanishes near $\partial M$, we have

$$0 \le \int_M \rho(\omega_\varphi^2 - \omega_{\tilde\varphi}^2) = -\int_M d\rho \wedge d^c\rho \wedge (\omega_\varphi + \omega_{\tilde\varphi}).$$

We thus get (30) and the rest of the proof is the same as that of Theorem 20. □

Assuming Theorem 19 and estimates proved in Sects. 8–13, we get Theorem 12. From the comparison principle it follows that $\varepsilon$-geodesics converge uniformly to a weak geodesic which is almost $C^{1,1}$ (that is it satisfies (28)). It is an open problem if it has to be fully $C^{1,1}$ (it was shown in [15] in case the bisectional curvature is nonnegative).

## 8 Continuity Method

In order to prove existence in Theorems 18 and 19 we fix an integer $k \geq 2$ and $\alpha \in (0, 1)$. Let $f_0$ denote the r.h.s. of the equation for which we already know the solution: $f_0 = 1$ in the first case and $f_0 = \omega_\psi^n/\omega^n$ in the second one. For $t \in [0, 1]$ set

$$f_t := (1 - t) f_0 + t f.$$

By $S$ denote the set of those $t \in [0, 1]$ for which the problem

$$\begin{cases} \varphi_t \in C^{k+2,\alpha}(M) \\ \omega + dd^c\varphi_t > 0 \\ (\omega + dd^c\varphi_t)^n = f_t\omega^n \\ \int_M \varphi_t\, \omega^n = 0, \end{cases}$$

resp.

$$\begin{cases} \varphi_t \in C^{k+2,\alpha}(M) \\ \omega + dd^c\varphi_t > 0 \\ (\omega + dd^c\varphi_t)^n = f_t\omega^n \\ \varphi_t = \psi \text{ on } \partial M \end{cases}$$

has a solution (by the previous section it has to be unique). We clearly have $0 \in S$ and we have to show that $1 \in S$. For this it will be enough to prove that $S$ is open and closed.

**Openness.** The Monge–Ampère operator we treat as the mapping

$$\mathcal{M} : \mathcal{A} \ni \varphi \longmapsto \frac{\omega_\varphi^n}{\omega^n} \in \mathcal{B},$$

where

$$\mathcal{A} := \{\varphi \in C^{k+2,\alpha}(M) : \omega_\varphi > 0, \int_M \varphi\, \omega^n = 0\}$$

$$\mathcal{B} := \{\tilde{f} \in C^{k,\alpha}(M) : \int_M \tilde{f}\omega^n = \int_M \omega^n\},$$

resp.

$$\mathcal{A} := \{\varphi \in C^{k+2,\alpha}(M) : \omega_\varphi > 0, \varphi = \psi \text{ on } \partial M\}$$
$$\mathcal{B} := C^{k,\alpha}(M).$$

Then $\mathcal{A}$ is an open subset of the Banach space

$$\mathcal{E} := \{\eta \in C^{k+2,\alpha}(M) : \int_M \eta\omega^n = 0\},$$

resp. a hyperplane in the Banach space $C^{k+2,\alpha}(M)$ with the tangent space

$$\mathcal{E} := \{\eta \in C^{k+2,\alpha}(M) : \varphi = 0 \text{ on } \partial M\}.$$

On the other hand, $\mathcal{B}$ is a hyperplane of the Banach space $C^{k,\alpha}(M)$ with the tangent space

$$\mathcal{F} := \{\tilde{f} \in C^{k,\alpha}(M) : \int_M \tilde{f}\omega^n = 0\},$$

resp. $\mathcal{B}$ is a Banach space itself and $\mathcal{F} := \mathcal{B}$. We would like to show that for every $\varphi \in \mathcal{A}$ the differential

$$d_\varphi\mathcal{M} : \mathcal{E} \to \mathcal{F}$$

is an isomorphism. But since

$$d_\varphi\mathcal{M}.\eta = \frac{1}{2}\Delta\eta,$$

where the Laplacian is taken w.r.t. $\omega_\varphi$, it follows from the standard theory of the Laplace equation on Riemannian manifolds. Therefore $\mathcal{M}$ is locally invertible, in particular $\mathcal{M}(\mathcal{A})$ is open in $\mathcal{B}$ and thus $S$ is open in [0, 1].

**Closedness.** Assume that we knew that

$$||\varphi_t||_{k+2,\alpha} \le C, \quad t \in S, \tag{32}$$

for some uniform constant $C$, where $||\cdot||_{k,\alpha} = ||\cdot||_{C^{k,\alpha}(M)}$. Then by the Arzela–Ascoli theorem every sequence in $\{\varphi_t : t \in S\}$ would contain a subsequence whose derivatives of order at most $k+1$ converged uniformly.

The proof of existence of solutions in Theorems 18 and 19 is therefore reduced to (32) for all $k$ big enough. The first step (but historically the latest in the Calabi–Yau case) is the $L^\infty$-estimate, this is done in Sect. 9. The gradient and second derivative estimates are presented in Sects. 10–12. They are all very specific for the complex Monge–Ampère equation and most of them (except for Theorem 25) are applicable also in the degenerate case, that is they do not depend on a lower positive bound for $f$. Finally, in Sect. 13, we make use of the general Evans–Krylov theory for nonlinear elliptic equations of second order (see e.g. [26], in the boundary case it is due to Caffarelli et al. [16]). This gives a $C^{2,\alpha}$ bound and then higher order estimates follow from the standard Schauder theory of linear elliptic equations of second order with variable coefficients.

## 9 $L^\infty$-Estimate

If $\partial M \neq \emptyset$ then by the comparison principle, Theorem 21, for any $\varphi \in C(M)$ with $\omega_\varphi \geq 0$, $\omega_\varphi^n \leq \omega_\psi^n$, $\varphi = \psi$ on $\partial M$, we have

$$\psi \leq \varphi \leq \max_{\partial M} \psi,$$

so we immediately get the $L^\infty$-estimate in the second case. The case $\partial M = \emptyset$ is more difficult and historically turned out to be the main obstacle in proving the Calabi conjecture. Its proof making use of Moser's iteration was in fact the main contribution of Yau [47] (see also [31] for some simplifications).

**Theorem 22.** *Assume $\partial M = \emptyset$. Take $\varphi \in C(M)$ with $\omega_\varphi \geq 0$, satisfying the Monge–Ampère equation $\omega_\varphi^n = f\omega^n$. Then*

$$osc\, \varphi \leq C(M, \omega, ||f||_\infty).$$

*Proof.* It will be convenient to assume that $V = \int_M \omega^n = 1$ and that $\max_M \varphi = -1$, so that $||\varphi||_p \leq ||\varphi||_q$ for $p \leq q$ (we use the notation $||\cdot||_p = ||\cdot||_{L^p(M)}$). Write

$$(f-1)\omega^n = dd^c\varphi \wedge T,$$

where

$$T = \sum_{p=0}^{n-1} \omega_\varphi^p \wedge \omega^{n-p-1}.$$

Note that $T \geq \omega^{n-1}$. Then for $p \geq 2$

$$\begin{aligned}
\int_M (-\varphi)^{p-1}(f-1)\omega^n &= \int_M (-\varphi)^{p-1} dd^c\varphi \wedge T \\
&= -\int_M d(-\varphi)^{p-1} \wedge d^c\varphi \wedge T \\
&= (p-1)\int_M (-\varphi)^{p-2} d\varphi \wedge d^c\varphi \wedge T \\
&\geq (p-1)\int_M (-\varphi)^{p-2} d\varphi \wedge d^c\varphi \wedge \omega^{n-1} \\
&= \frac{4(p-1)}{p^2}\int_M d(-\varphi)^{p/2} \wedge d^c(-\varphi)^{p/2} \wedge \omega^{n-1} \\
&\geq \frac{c_n}{p} ||\nabla((-\varphi))^{p/2}||_2^2.
\end{aligned} \tag{33}$$

By the Sobolev inequality

$$||\varphi||_{pn/(n-1)}^{p/2} = ||(-\varphi)^{p/2}||_{2n/(n-1)} \le C(M)\left(||(-\varphi)^{p/2}||_2 + ||\nabla((-\varphi)^{p/2})||_2\right).$$

Combining this with (33) we will get

$$||\varphi||_{pn/(n-1)} \le (Cp)^{1/p}||\varphi||_p.$$

Setting

$$p_0 := 2, \quad p_{k+1} = np_k/(n-1),\ k = 1, 2, \dots,$$

we will get

$$||\varphi||_\infty = \lim_{k\to\infty} ||\varphi||_{p_k} \le \tilde{C}\,||\varphi||_2$$

and it remains to use the following elementary estimate:

**Proposition 23.** *Assume that $\partial M = \emptyset$ and let $\varphi \in C(M)$ be such that $\omega_\varphi \ge 0$, $\max_\varphi M = 0$. Then for any $p < \infty$*

$$||\varphi||_p \le C(M, p).$$

*Proof.* It will easily follow from local properties of plurisubharmonic functions. For $p = 1$ we can use the following result: if $u$ is a negative subharmonic function in the ball $B(0, 3R)$ in $\mathbb{C}^n$ then

$$||u||_{L^1(B(0,R))} \le C(n, R) \inf_{B(0,R)} (-u).$$

After covering $M$ with finite number of balls of radius $R$, a simple procedure starting at the point where $\varphi = 0$ will give us the required estimate for $||\varphi||_1$. The case $p > 1$ is now an immediate consequence of the following fact: if $u$ is a negative plurisubharmonic function in $B(0, 2R)$ then

$$||u||_{L^p(B(0,R))} \le C(n, p, R)\,||u||_{L^1(B(0,2R))}. \qquad \square$$

## 10 Interior Second Derivative Estimate

It turns out that in case of Theorem 18 one can bypass the gradient estimate. The interior estimate for the second derivative which will be needed in the proofs of both cases was shown independently by Aubin [1] and Yau [47]. We will show the following version from [14]:

**Theorem 24.** *Assume that $\varphi \in C^4(M)$ satisfies $\omega_\varphi > 0$ and $\omega_\varphi^n = f\omega^n$. Then*

$$\Delta\varphi \le C, \tag{34}$$

*where $C$ depends only on $n$, on upper bounds for $f$, the scalar curvature of $M$, $\operatorname{osc}\varphi$ and* $\sup_{\partial M}\Delta\varphi$ *(if $\partial M = \emptyset$ then this is void), and on lower bounds for $f^{1/(n-1)}\Delta(\log f)$ and the bisectional curvature of $M$.*

*Proof.* By $C_1, C_2, \ldots$ we will denote constants depending only on the required quantities. Set

$$\alpha := \log(\Delta\varphi + 2n) - A\varphi$$

(note that $\Delta\varphi > -2n$), where $A$ under control will be specified later. We may assume that $\alpha$ attains maximum at $y$ in the interior of $M$, otherwise we are done. Let $g$ be a local potential for $\omega$ near $y$ and set $u := g + \varphi$. We choose normal coordinates at $y$ (so that $g_{j\bar{k}} = \delta_{jk}$, $g_{j\bar{k}l} = 0$ at $y$), so that in addition the matrix $(u_{j\bar{k}})$ is diagonal at $y$. Then at $y$

$$\alpha_{p\bar{p}} = \frac{(\Delta u)_{p\bar{p}}}{\Delta u} - \frac{|(\Delta u)_p|^2}{(\Delta u)^2} + A - Au_{p\bar{p}}$$

$$(\Delta u)_p = 2\sum_j u_{j\bar{j}p}$$

$$(\Delta)_{p\bar{p}} = 2\sum_j u_{j\bar{j}p\bar{p}} + 2\sum_j R_{j\bar{j}p\bar{p}}u_{j\bar{j}}.$$

(by (9)). The equation $\omega_\varphi^n = f\omega^n$ now reads

$$\det(u_{p\bar{q}}) = f\det(g_{p\bar{q}}). \tag{35}$$

Differentiating w.r.t. $z^j$ and $\bar{z}^j$ we get

$$u^{p\bar{q}}u_{p\bar{q}j} = (\log f)_j + g^{p\bar{q}}g_{p\bar{q}j} \tag{36}$$

and

$$u^{p\bar{q}}u_{p\bar{q}j\bar{j}} = (\log f)_{j\bar{j}} + u^{p\bar{t}}u^{s\bar{q}}u_{p\bar{q}j}u_{s\bar{t}\bar{j}} + g^{p\bar{q}}g_{p\bar{q}j\bar{j}} - g^{p\bar{t}}g^{s\bar{q}}g_{p\bar{q}j}g_{s\bar{t}\bar{j}}. \tag{37}$$

Therefore at $y$

$$\sum_p \frac{u_{j\bar{j}p\bar{p}}}{u_{p\bar{p}}} = (\log f)_{j\bar{j}} + \sum_{p,q}\frac{|u_{p\bar{q}j}|^2}{u_{p\bar{p}}u_{q\bar{q}}} - \sum_p R_{j\bar{j}p\bar{p}}$$

and, since $\alpha_{p\bar{p}} \le 0$,

$$0 \geq \sum_p \frac{\alpha_{p\bar{p}}}{u_{p\bar{p}}} = \frac{1}{\Delta u}\left(\Delta(\log f) + 2\sum_{j,p,q}\frac{|u_{p\bar{q}j}|^2}{u_{p\bar{p}}u_{q\bar{q}}} - S + 2\sum_{j,p}\frac{R_{j\bar{j}p\bar{p}}u_{j\bar{j}}}{u_{p\bar{p}}}\right)$$
$$- \frac{4}{(\Delta u)^2}\sum_p \frac{|\sum_j u_{j\bar{j}p}|^2}{u_{p\bar{p}}} + A\sum_p \frac{1}{u_{p\bar{p}}} - An.$$

By the Schwarz inequality

$$\Big|\sum_j u_{j\bar{j}p}\Big|^2 \leq \frac{\Delta u}{2}\sum_q \frac{|u_{q\bar{q}p}|^2}{u_{q\bar{q}}}$$

and therefore we can get rid of the terms with third derivatives. We also have

$$\Delta(\log f) \geq -\frac{C_1}{f^{1/(n-1)}},$$

$$2\sum_{j,p}\frac{R_{j\bar{j}p\bar{p}}u_{j\bar{j}}}{u_{p\bar{p}}} \geq -C_2\Delta u\sum_p \frac{1}{u_{p\bar{p}}}$$

(by (12)), and

$$\sum_p \frac{1}{u_{p\bar{p}}} \geq \left((n-1)\sum_p \frac{u_{p\bar{p}}}{u_{1\bar{1}}\dots u_{n\bar{n}}}\right)^{1/(n-1)} \geq \left(\frac{\Delta u}{f}\right)^{1/(n-1)}$$

(we may assume $n \geq 2$). Therefore, choosing $A := C_2 + 1$, at $y$ we get

$$-\frac{C_1}{f^{1/(n-1)}\Delta u} - \frac{S}{\Delta u} + \left(\frac{\Delta u}{f}\right)^{1/(n-1)} - C_3 \geq 0.$$

Multiplying by $f^{1/(n-1)}\Delta u$ we will get at $y$

$$(\Delta u)^{n/(n-1)} - C_4\Delta u - C_5 \leq 0,$$

and thus

$$\Delta u(y) \leq C_6.$$

Therefore $\alpha \leq \alpha(y) \leq C_7$ and we get (34). □

An upper bound for $\Delta\varphi$ for functions satisfying $\omega_\varphi \geq 0$ easily gives a bound mixed complex derivatives of $\varphi$

$$|\varphi_{j\bar{k}}| \leq C.$$

However, it does not imply the full estimate for the second derivative of $\varphi$:

*Example.* Set $S := \{re^{it} : 0 \le r \le 1,\ \pi/4 \le t \le 3\pi/4\}$ and

$$u(z) := \frac{2}{\pi}\int_S \log|z-\zeta|d\lambda(\zeta).$$

Then $u_{z\bar z} = \chi_S \in L^\infty(\mathbb{C})$, and thus $u \in W^{2,p}_{loc}(\mathbb{C})$ for every $p < \infty$ (which implies that $u \in C^{1,\alpha}(\mathbb{C})$ for every $\alpha < 1$ by Morrey's embedding theorem). However

$$u_{xx}(0) = \frac{2}{\pi}\int_S \frac{y^2-x^2}{(x^2+y^2)^2}d\lambda(z) = 4\int_0^1\int_{\pi/4}^{3\pi/4}\frac{\sin^2 t-\cos^2 t}{r^2}\,dt\,dr = \infty,$$

and $u \notin W^{2,\infty}_{loc}(\mathbb{C}) = C^{1,1}(\mathbb{C})$.

The following estimate will enable to apply the real Evans–Krylov theory (see Sect. 13) directly, without reproving its complex version.

**Theorem 25 ([15]).** *Assume that $\varphi \in C^4(M)$, $\omega_\varphi > 0$, $\omega_\varphi^n = f\omega^n$. Then*

$$|\nabla^2\varphi| \le C, \tag{38}$$

*where $C$ depends on $n$, on upper bounds for $|R|$, $|\nabla R|$, $||\varphi||_{C^{0,1}(M)}$, $\Delta\varphi$, $\sup_{\partial M}|\nabla^2\varphi|$, $||f||_{C^{1,1}(M)}$ and a lower (positive) bound for $f$ on $M$.*

*Proof.* We have to estimate the eigenvalues of the mapping $X \mapsto \nabla_X\nabla\varphi$. Since their sum is under control from below (by $-2n$), it will be enough to get an upper bound. The maximal eigenvalue is given by

$$\beta = \max_{X\in TM\setminus\{0\}}\frac{\langle\nabla_X\nabla\varphi, X\rangle}{|X|^2}.$$

This is a continuous function on $M$ (but not necessarily smooth). Locally we have

$$\begin{aligned}\nabla_{\partial_j}\nabla\varphi &= \partial_j(g^{p\bar q}\varphi_p)\partial_{\bar q} + \partial_j(g^{p\bar q}\varphi_{\bar q})\partial_p + g^{p\bar q}\varphi_{\bar q}\Gamma^s_{jp}\partial_s\\ &= g^{p\bar q}\varphi_{j\bar q}\partial_p + (g^{p\bar q}\varphi_p)_j\partial_{\bar q}.\end{aligned}$$

Therefore for a real vector field $X = X^j\partial_j + \bar X^k\partial_{\bar k}$

$$\begin{aligned}\langle\nabla_X\nabla\varphi, X\rangle &= 2\mathrm{Re}\, X^j\big(\bar X^k\varphi_{j\bar k} + X^l g_{l\bar q}(g^{p\bar q}\varphi_p)_j\big)\\ &= D_X^2\varphi + 2\mathrm{Re}\big(X^jX^lg^{p\bar q}g_{j\bar q l}\varphi_p\big),\end{aligned}$$

where $D_X$ denotes Euclidean directional derivative in direction $X$.

Set

$$\alpha := \beta + \frac{1}{2}|\nabla\varphi|^2.$$

We may assume that $\alpha$ attains maximum $y$ in the interior of $M$. Near $y$ we choose normal coordinates ($g_{j\bar{k}} = \delta_{jk}$, $g_{j\bar{k}l} = g_{j\bar{k}lm} = 0$ at $y$) so that in addition the matrix $(\varphi_{j\bar{k}})$ is diagonal at $y$. Take fixed $X = (X^1, \dots, X^N) \in \mathbb{C}^N$ such that at $y$ one has $|X|^2 (= 2g_{j\bar{k}} X^j \bar{X}^k) = 1$. Near $y$ define

$$\tilde{\beta} := \frac{\langle \nabla_X \nabla\varphi, X\rangle}{|X|^2}$$

and

$$\tilde{\alpha} := \tilde{\beta} + |\nabla\varphi|^2.$$

Then $\tilde{\beta} \le \beta$, $\tilde{\beta}(y) = \beta(y)$ and $\tilde{\alpha} \le \alpha \le \alpha(y) = \tilde{\alpha}(y)$, so that $\tilde{\alpha}$ (which is defined locally) also has a maximum at $y$, the same as that of $\alpha$. The advantage of $\tilde{\alpha}$ is that it is smooth (this argument goes back to [11]). It remains to estimate $\tilde{\beta}(y)$ from above.

The function $u := \varphi + g$ solves (35). Similarly as with (37) we will get at $y$

$$\sum_p \frac{D_X^2 \varphi_{p\bar{p}}}{u_{p\bar{p}}} \ge D_X^2(\log f) + \sum_p D_X^2 g_{p\bar{p}} - \sum_p \frac{D_X^2 g_{p\bar{p}}}{u_{p\bar{p}}}.$$

Since $f$ is under control from below, we have $D_X^2(\log f) \ge -C_1$ and by Theorem 24

$$\frac{1}{C_2} \le u_{p\bar{p}} \le C_3.$$

This, together with the fact that $|R|$ is under control, implies that

$$\sum_p \frac{D_X^2 \varphi_{p\bar{p}}}{u_{p\bar{p}}} \ge -C_4. \tag{39}$$

Using the fact that $|X| = 1$ and $(|X|^2)_p = 0$ at $y$, combined with (36), at $y$ we will get

$$\begin{aligned} \tilde{\beta}_{p\bar{p}} &= D_X^2 \varphi_{p\bar{p}} + 2\mathrm{Re} \sum_l X^j X^k g_{j\bar{l}k\bar{p}p} \varphi_l \\ &\quad + 2\mathrm{Re} \sum_l X^j X^k g_{j\bar{l}k\bar{p}} \varphi_{lp} - X^j \bar{X}^k g_{j\bar{k}p\bar{p}} D_X^2 \varphi \\ &\ge D_X^2 \varphi_{p\bar{p}} - C_5 - C_6 \tilde{\beta}, \end{aligned} \tag{40}$$

where we used in addition that $|\nabla R|$ is under control.

Near $y$ we have

$$\frac{1}{2}(|\nabla\varphi|^2)_p = (g^{j\bar{k}})_p\varphi_j\varphi_{\bar{k}} + g^{j\bar{k}}\varphi_{jp}\varphi_{\bar{k}} + g^{j\bar{k}}\varphi_j\varphi_{p\bar{k}}.$$

Therefore at $y$

$$\frac{1}{2}(|\nabla\varphi|^2)_{p\bar{p}} = \sum_{j,k} R_{j\bar{k}p\bar{p}}\varphi_j\varphi_{\bar{k}} + 2\mathrm{Re}\sum_j \varphi_{jp\bar{p}}\varphi_{\bar{j}} + \sum_j |\varphi_{jp}|^2 + \varphi_{p\bar{p}}^2.$$

Since

$$2\mathrm{Re}\sum_{j,p}\frac{\varphi_{jp\bar{p}}\varphi_{\bar{j}}}{u_{p\bar{p}}} = 2\mathrm{Re}\sum_j(\log f)_j\varphi_{\bar{j}} \geq -C_7$$

and

$$\sum_{j,p}\frac{|\varphi_{jp}|^2}{u_{p\bar{p}}} \geq \frac{1}{C_8}\tilde{\beta}^2 - C_9,$$

from (39), (40) it follows that at $y$

$$0 \geq \sum_p \frac{\alpha_{p\bar{p}}}{u_{p\bar{p}}} \geq \frac{1}{C_8}\tilde{\beta}^2 - C_{10}\tilde{\beta} - C_{11}.$$

□

Since the estimate depends on a lower bound for $f$, Theorem 25 cannot be used in the degenerate case. It is an open problem if one can get rid of this dependence, this would in particular imply full $C^{1,1}$-regularity of weak geodesics obtained by Chen [18]. This was shown only under additional assumption that $M$ has nonnegative bisectional curvature, see [15].

## 11 Gradient Estimate

If $\partial M = \emptyset$ then Theorem 24 gives an a priori estimate for the Laplacian, and thus also for the gradient. However, if $\partial M \neq \emptyset$ then a direct gradient estimate is necessary because the boundary estimates from Sect. 12 depend on it.

The estimate for $|\nabla\varphi|$ on $\partial M$ follows easily from the comparison principle: if $h \in C^\infty(M)$ is harmonic in the interior of $M$ with $h = \psi$ on $\partial M$ then

$$\psi \leq \varphi \leq h$$

in $M$. Therefore on $\partial M$ we have

$$|\nabla\varphi| \leq \max\{|\nabla\psi|, |\nabla h|\}.$$

We have the following interior gradient bound from [14] (see also [29, 30]).

**Theorem 26.** *Let $\varphi \in C^3(M)$ be such that $\omega_\varphi > 0$ and $\omega_\varphi^n = f\omega^n$. Then*

$$|\nabla\varphi| \le C, \tag{41}$$

*where $C$ depends on $n$, on upper bounds for $\operatorname{osc}\varphi$, $\sup_{\partial M}|\nabla\varphi|$, $f$, $|\nabla(f^{1/n})|$ and on a lower bound for the bisectional curvature of $M$.*

*Proof.* We may assume that $\inf_M \varphi = 0$ and $C_0 := sup_M\varphi = \operatorname{osc}\varphi$. Set

$$\beta = \frac{1}{2}|\nabla\varphi|^2$$

and

$$\alpha := \log\beta - \gamma\circ\varphi,$$

where $\gamma \in C^\infty([0, C_0])$ with $\gamma' \ge 0$ will be determined later. We may assume that $\gamma$ attains maximum at $y$ in the interior of $M$. Near $y$ write $u = \varphi + g$, where $g$ is a local potential for $\omega$. Similarly as before, we may assume that at $y$ we have $g_{j\bar{k}} = \delta_{j\bar{k}}$, $g_{j\bar{k}l} = 0$ and $(u_{j\bar{k}})$ is diagonal.

At $y$ we will get

$$\begin{aligned}
\beta &= \sum_j |\varphi_j|^2\\
\beta_p &= \sum_j \varphi_{jp}\varphi_{\bar{j}} + \varphi_p(u_{p\bar{p}} - 1)\\
\beta_{p\bar{p}} &= \sum_{j,k} R_{j\bar{k}p\bar{p}}\varphi_j\varphi_{\bar{k}} + 2\mathrm{Re}\sum_j u_{p\bar{p}j}\varphi_{\bar{j}} + \sum_j |\varphi_{jp}|^2 + \varphi_{p\bar{p}}^2.
\end{aligned}$$

and

$$\alpha_{p\bar{p}} = \frac{\beta_{p\bar{p}}}{\beta} - \big((\gamma')^2 + \gamma''\big)|\varphi_p|^2 - \gamma'\varphi_{p\bar{p}},$$

where for simplicity we denote $\gamma'\circ\varphi$ just by $\gamma'$ (and similarly for $\gamma''$). By (36)

$$\sum_p \frac{u_{p\bar{p}j}}{u_{p\bar{p}}} = (\log f)_j.$$

Since

$$\frac{1}{\beta}\sum_{j,k,p}\frac{R_{j\bar{k}p\bar{p}}\varphi_j\varphi_{\bar{k}}}{u_{p\bar{p}}} \ge -C_1\sum_p \frac{1}{u_{p\bar{p}}}$$

and (we may assume that $\beta(y) \ge 1$)

$$\frac{2}{\beta}\mathrm{Re}\,\sum_j (\log f)_j\varphi_{\bar{j}} \geq -2|\nabla(\log f)| \geq -\frac{C_2}{f^{1/n}} \geq -C_2\sum_p \frac{1}{u_{p\bar{p}}},$$

we will obtain at $y$

$$0 \geq \sum_p \frac{\alpha_{p\bar{p}}}{u_{p\bar{p}}} \geq (\gamma' - C_3)\sum_p \frac{1}{u_{p\bar{p}}} + \frac{1}{\beta}\sum_{j,p}\frac{|\varphi_{jp}|^2}{u_{p\bar{p}}} - [(\gamma')^2 + \gamma'']\sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - n\gamma'.$$

We have to estimate the term

$$\frac{1}{\beta}\sum_{j,p}\frac{|\varphi_{jp}|^2}{u_{p\bar{p}}}$$

from below. For this we will use that fact that $\alpha_p = 0$ at $y$. Therefore $\beta_p = \gamma'\beta\varphi_p$, that is

$$\sum_j \varphi_{jp}\varphi_{\bar{j}} = (\gamma'\beta - u_{p\bar{p}} + 1)\varphi_p.$$

By the Schwarz inequality

$$\Big|\sum_j \varphi_{jp}\varphi_{\bar{j}}\Big|^2 \leq \beta\sum_j |\varphi_{jp}|^2,$$

hence

$$\frac{1}{\beta}\sum_{j,p}\frac{|\varphi_{jp}|^2}{u_{p\bar{p}}} \geq \frac{1}{\beta^2}\sum_p \frac{(\gamma'\beta + 1 - u_{p\bar{p}})^2|\varphi_p|^2}{u_{p\bar{p}}} \geq (\gamma')^2\sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - 2\gamma' - \frac{2}{\beta}.$$

This gives

$$0 \geq (\gamma' - C_3)\sum_p \frac{1}{u_{p\bar{p}}} - \gamma''\sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - (n+2)\gamma' - \frac{2}{\beta}.$$

We now set $\gamma(t) = -t^2/2 + (C_0 + C_3 + 1)t$, so that $\gamma'' = -1$ and $\gamma' \geq 1$ in $[0, C_0]$. We will get

$$\sum_p \frac{1}{u_{p\bar{p}}} + \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} \leq C_5.$$

Therefore $u_{p\bar{p}} \leq C_6$ and $\beta \leq C_7$ at $y$, and we easily arrive at (41). $\square$

## 12 Boundary Second Derivative Estimate

In this section we want to show the a priori estimate

$$|\nabla^2\varphi| \le C$$

on $\partial M$. It is due to Caffarelli et al. [16] if the boundary is strongly pseudoconvex and to B. Guan [27] in the general case. We will prove the following local result which is applicable to the case of flat boundary:

**Theorem 27.** *Write $B_R = B(0,R)$ and $B_R^- = B_R \cap \{x^n \le 0\}$. Let $u, v \in C^3(B_R^-)$ be such that $(u_{j\bar k}) > 0$, $(v_{j\bar k}) \ge \lambda(\delta_{jk})$ for some $\lambda > 0$,*

$$\det(u_{j\bar k}) = f \le \det(v_{j\bar k}).$$

*Assume moreover that $u \ge v$ on $B_R^-$ and $u = v$ on $\{x^n = 0\}$. Then*

$$|D^2u(0)| \le C,$$

*where $C$ depends on $n$, on upper bounds for $||v||_{C^{2,1}(B_R^-)}$, $||f^{1/n}||_{C^{0,1}(B_R^-)}$, $||u||_{C^{0,1}(B_R^-)}$, and on lower bounds for $\lambda$, $R$.*

*Proof.* If $s,t$ are tangential directions to $\{x^n = 0\}$ then $u_{st}(0) = v_{st}(0)$, so $|u_{st}(0)|$ is under control. The main step in the proof is to estimate the tangential-normal derivative $u_{tx^n}(0)$. Set $r := R/2$ and

$$\tilde w := -(u - v) + 2A_1 x^n(r + x^n),$$

where $A_1 > 0$ under control will be determined later. We have $\tilde w \le 0$ in $B_r^-$ and

$$u^{j\bar k}\tilde w_{j\bar k} = -n + u^{j\bar k}v_{j\bar k} + A_1 u^{n\bar n} \ge -n + \lambda \sum u^{j\bar j} + A_1 u^{n\bar n}.$$

By $\lambda_1 \le \cdots \le \lambda_n$ denote the eigenvalues of $(u_{j\bar k})$. Then $\sum u^{j\bar j} = \sum 1/\lambda_j$ and $u^{n\bar n} \ge 1/\lambda_n$. Since $\lambda_1 \dots \lambda_n = f$, by the inequality between geometric and arithmetic means we will obtain

$$\begin{aligned} u^{j\bar k}\tilde w_{j\bar k} &\ge -n + \frac{\lambda}{2}\sum u^{j\bar j} + \frac{\lambda}{2}\sum \frac{1}{\lambda_j} + \frac{A_1}{\lambda_n} \\ &\ge -n + \frac{\lambda}{2}\sum u^{j\bar j} + \frac{n(\lambda/2)^{1-1/n}A_1^{1/n}}{f^{1/n}} \\ &\ge \frac{\lambda}{2}\Big(1 + \sum u^{j\bar j}\Big) \end{aligned} \tag{42}$$

for $A_1$ sufficiently large.

Further define

$$w := \pm(u-v)_t + \frac{1}{2}(u-v)^2_{y^n} - A_2|z|^2 + A_3\tilde{w},$$

where positive $A_2, A_3$ under control will be determined later. Since $(u-v)_t = (u-v)_{y^n} = 0$ on $\{x^n = 0\}$, we have $w \le 0$ on $\{x^n = 0\}$. We also have

$$\left|\pm(u-v)_t + \frac{1}{2}(u-v)^2_{y^n}\right| \le C_1$$

and thus for $A_2$ sufficiently large $w \le 0$ on $\partial B_r \cap \{x^n \le 0\}$. By (36)

$$\begin{aligned} u^{j\bar{k}}\big(\pm(u-v)_t + \frac{1}{2}(u-v)^2_{y^n}\big)_{j\bar{k}} &= \pm(\log f)_t \mp u^{j\bar{k}} v_{tj\bar{k}} + (u-v)_{y^n}(\log f)_{y^n} \\ &\quad + u^{j\bar{k}}(u-v)_{y^n j}(u-v)_{y^n\bar{k}} \\ &\ge -C_2(1+\sum u^{j\bar{j}}), \end{aligned}$$

where the last inequality follows from

$$f^{-1/n} \le \frac{1}{n}\sum u^{j\bar{j}}.$$

Therefore, from (42) we get

$$u^{j\bar{k}} w_{j\bar{k}} \ge 0$$

if $A_3$ is chosen sufficiently large. Now from the maximum principle we obtain $w \le 0$ in $B_r^-$ and thus

$$|(u-v)_{tx^n}(0)| \le A_3\big((u-v)_{x^n}(0) + 2A_1 r\big),$$

so

$$|u_{tx^n}(0)| \le C_3.$$

It remains to estimate the normal-normal derivative $u_{x^n x^n}(0)$. At the origin we can now write

$$f = \det(u_{j\bar{k}}) = u_{n\bar{n}}\det(u_{j\bar{k}})_{j,k\le n-1} + \mathcal{R} = u_{n\bar{n}}\det(v_{j\bar{k}})_{j,k\le n-1} + \mathcal{R},$$

where $|\mathcal{R}|$ is under control. Therefore

$$0 \le u_{n\bar{n}}(0) \le \frac{C_4}{\lambda^{n-1}}$$

and the normal-normal estimate follows. □

## 13 Higher Order Estimates

We will make use of a general (real) theory of nonlinear elliptic equations of second order. They are of the form

$$F(D^2u, Du, u, x) = 0, \tag{43}$$

where

$$F : \mathbb{R}^{m^2} \times \mathbb{R}^m \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$$

($\Omega$ is a domain in $\mathbb{R}^m$) satisfies two basic assumptions:

$$F \text{ is concave in } D^2u \tag{44}$$

and elliptic, that is

$$\frac{\partial F}{\partial u_{x_s x_t}} \zeta^s \zeta^t \ge \lambda |\zeta|^2, \quad \zeta \in \mathbb{R}^m, \tag{45}$$

for some $\lambda > 0$.

If by $\mathcal{M}_+$ we denote the set of Hermitian positive matrices then, as one can show (see e.g. [13, 24]),

$$(\det A)^{1/n} = \frac{1}{n} \inf\{\text{trace}\,(AB) : B \in \mathcal{M}_+,\ \det B = 1\}, \quad A \in \mathcal{M}_+.$$

Moreover, one can also easily prove the following formula for the minimal eigenvalue of $(u_{j\bar{k}}) > 0$

$$\lambda_{min}\left(\frac{\partial \det(u_{j\bar{k}})}{\partial u_{x_s x_t}}\right) = \frac{\det(u_{j\bar{k}})}{4\lambda_{max}(u_{j\bar{k}})},$$

(see e.g. [9]). (Here $x_s$ denote real variables in $\mathbb{C}^n$, $s = 1, \dots, 2n$.)

By Theorem 24 we can assume that

$$\frac{1}{C}|\zeta|^2 \le u_{j\bar{k}} \zeta^j \bar{\zeta}^k \le C|\zeta|^2, \quad \zeta \in \mathbb{C}^n. \tag{46}$$

Therefore, if we define $F$ as

$$F(D^2u, z) := (\det(u_{j\bar{k}}))^{1/n} - f(z)$$

for functions (or rather matrices) satisfying (46) and extend it in a right way to the set of all symmetric real $2n \times 2n$-matrices, then $F$ satisfies (44) and (45).

**Theorem 28 ([22, 23, 32, 38, 46]).** *Assume that $u \in C^3(\Omega)$ solves (43), where $F$ is $C^2$ and satisfies (44) and (45). Then for $\Omega' \Subset \Omega$ there exists $\alpha \in (0,1)$ depending only on upper bounds for $||u||_{C^{1,1}(\Omega)}$, $||F||_{C^{1,1}(\Omega)}$ and a lower bound for $\lambda$, and $C$ depending in addition on a lower bound for dist $(\Omega', \partial\Omega)$, such that*

$$||u||_{C^{2,\alpha}(\Omega')} \le C.$$

**Theorem 29 ([16, 32]).** *Assume that $u$, defined in $B_R^+ := B(0,R) \cap \{x^m \ge 0\}$, solves (43) with $F$ satisfying (44) and (45) and $u = \psi$ on $B(0,R) \cap \{x^m = 0\}$. Then there exists $\alpha \in (0,1)$ and $C$, depending only on $m$, $\lambda$, $R$, $||u||_{C^{1,1}}$, $||F||_{C^{1,1}}$ and $||\psi||_{C^{3,1}}$, such that*

$$||u||_{C^{2,\alpha}(B_{R/2}^+)} \le C.$$

Now the standard Schauder theory applied to (the linearization of) $F$ gives the required a priori estimate (32).

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