DEFINING NONLINEAR ELLIPTIC OPERATORS FOR NON-SMOOTH FUNCTIONS

Zbigniew Błocki

Dedicated to Christer Kiselman on the occasion of his retirement

Abstract. We discuss the problem of defining basic nonlinear elliptic operators of second order (real and complex Monge-Ampère operators, more general Hessian operators) for natural classes of non-smooth functions associated with them (convex, plurisubharmonic, etc.) and survey recent developments in this area.

1. Introduction. One of the basic facts of the classical potential theory is that a Laplacian of an arbitrary, not necessarily smooth, subharmonic function can be well defined as a nonnegative Radon measure. The aim of this article is to describe the problem of defining, for natural classes of non-smooth functions, the most important nonlinear elliptic operators of second order: the real and complex Monge-Ampère operators and more general Hessian operators. The problem of defining them stems from the fact that we cannot multiply distributions. Our perspective here will be the real theory, so when discussing for example the complex Monge-Ampère operator we will treat it in fact as a real operator defined on real-valued functions on domains of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ (but of course it depends in a crucial way on the complex structure).

First, we will present a rather general definition of an elliptic operator of second order and associate with it a natural class of admissible functions. Since all the problems discussed here are of purely local character, we will not specify open sets in $\mathbb{R}^n$ where considered functions are defined (we may treat them as germs at the origin). For smooth u defined on an open subset of $\mathbb{R}^n$ we consider operators of the form

$$\mathcal{F}(u) = F(D^2 u),$$

where $D^2 u = (\partial^2 u / \partial x_i \partial x_j)$ is the (real) Hessian matrix of u and F is a smooth real-valued function defined on the space $\mathcal{S}$ of all real symmetric $n \times n$ matrices. We first define the cone of *admissible matrices*:

$$\mathcal{S}_{\mathcal{F}} := \{A \in \mathcal{S} : F(A+B) \geq 0 \text{ for all } B \in \mathcal{S}_+\},$$

where

$$\mathcal{S}_+ := \{A \in \mathcal{S} : A \geq 0\}.$$

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The operator $\mathcal{F}$ is called *elliptic* (in the weak sense) if

$$\left(\frac{\partial F}{\partial a_{ij}}(A)\right) \in \mathcal{S}_+, \quad A \in \mathcal{S}_\mathcal{F}.$$

The definition of the matrix $(\partial F/\partial a_{ij}(A))$ is a little bit ambiguous, for $\mathcal{S}$ is only a linear subspace of $\mathbb{R}^{n^2}$ and it is not a priori clear how one should extend functions defined only on $\mathcal{S}$. More precisely, we can say that $(\partial F/\partial a_{ij}(A))$ is the symmetric matrix uniquely determined by

$$tr\left[\left(\frac{\partial F}{\partial a_{ij}}(A)\right)B\right] = \frac{d}{dt}F(A+tB)\Big|_{t=0}, \quad B \in \mathcal{S}.$$

Note that $\mathcal{F}$ is elliptic if and only if

$$F(A+B) \geq F(A), \quad A \in \mathcal{S}_\mathcal{F}, \ B \in \mathcal{S}_+.$$

We will say that a smooth function u is *admissible for* $\mathcal{F}$ if its Hessian D^2u is an admissible matrix at every point.

If the set of admissible matrices $\mathcal{S}_\mathcal{F}$ is a convex cone then one can easily define a notion of a non-smooth admissible function. Namely, we may then write

$$\mathcal{S}_\mathcal{F} = \bigcap_{B \in \widehat{\mathcal{S}}} \{A \in \mathcal{S} : tr(AB) \geq 0\}, \tag{2}$$

where $\widehat{\mathcal{S}}$ is a subset of $\mathcal{S}_+$ (note that $\widehat{\mathcal{S}}$ is usually not uniquely determined), and we say that an upper-semicontinuous $u \in L^1_{loc}$ is *admissible* if

$$tr(D^2uB) = \sum_{i,j} b_{ij}\frac{\partial^2 u}{\partial x_i \partial y_j} \geq 0, \quad B = (b_{ij}) \in \widehat{\mathcal{S}}$$

(in the distributional sense). Moreover, every admissible u can be approximated pointwise from above by smooth admissible functions, because for $\rho \geq 0$ and $B \in \widehat{\mathcal{S}}$ we have $tr(D^2(u*\rho)B) = tr(D^2uB)*\rho \geq 0$ and thus the standard regularizations $u*\rho$ are also admissible.

However, it is not at all clear how to define $\mathcal{F}(u)$ for non-smooth admissible u and as we will see it is not so simple (sometimes even not possible) even for the most basic operators.

2. The real Monge-Ampère operator. It is the model example of a nonlinear elliptic operator of second order:

$$M(u) = \det D^2u. \tag{3}$$

One can show that $\mathcal{S}_M = \mathcal{S}_+$ and that it is of the form (2), where by $\widehat{\mathcal{S}}$ we may take for example the whole $\mathcal{S}_+$. Hence, the admissible functions for M are precisely convex functions.

The usefulness of the Monge-Ampère operator in geometry is mainly caused by the fact that the Gauss curvature of a hypersurface of $\mathbb{R}^{n+1}$ which is a graph of a function u of n variables is given by

$$K = \frac{\det D^2 u}{(1 + |\nabla u|^2)^{n/2+1}}.$$

Defining M for non-smooth convex functions gives in particular the notion of Gauss curvature (as a measure) of an arbitrary convex hypersurface.

It is clear that the right hand-side of (3) cannot be directly defined in terms of distributions, since we cannot multiply them. The construction of the measure $M(u)$ for arbitrary convex u is due to A.D. Aleksandrov. The starting point is the following observation: if u, defined on a convex domain $\Omega \in \mathbb{R}^n$, is smooth and strongly convex (i.e. $D^2 u > 0$) then ∇u, treated as a mapping $\Omega \to \mathbb{R}^n$, is injective and diffeomorphic and its Jacobian is precisely $\det D^2 u$. For every Borel subset $E \subset \Omega$ we have therefore

$$\int_E \det D^2 u \, d\lambda = \lambda(\nabla u(E)) \tag{4}$$

(λ denotes the Lebesgue measure). Moreover, the set $\nabla u(E)$ (called *gradient image*) can be naturally defined also for non-smooth convex functions by means of affine supporting functions:

$$\nabla u(x) := \{y \in \mathbb{R}^n : u(x) + \langle \cdot - x, y \rangle \le u\}, \quad \nabla u(E) := \bigcup_{x \in E} \nabla u(x).$$

It follows from the properties of convex functions that at every x the set $\nabla u(x)$ is non-empty. If u is differentiable at x then of course $\nabla u(x)$ consists of one vector.

It remains to show that the right hand-side of (4) defines a measure on the σ-algebra of Borel sets. The key is the following result:

THEOREM 1 (ALEKSANDROV [1]). *For arbitrary convex u the set of $y \in \mathbb{R}^n$ that belong to gradient images of more than one point is of Lebesgue measure zero.*

One can namely show (see also [19]) that if $\widetilde{y} \in \nabla u(x) \cap \nabla u(\widetilde{x})$ for some $x \neq \widetilde{x}$, then the conjugate of u

$$v(y) := \sup_x (\langle x, y \rangle - u(x)), \quad y \in \mathbb{R}^n,$$

is not differentiable at $\widetilde{y}$. But it is well known that convex functions are differentiable almost everywhere.

An alternative, more analytic way of constructing the measure $M(u)$ will be presented below when we discuss the complex Monge-Ampère operator (see also [19]).

3. The complex Monge'a-Ampère operator. For smooth u defined on an open subset of $\mathbb{C}^n$ we set

$$M^c(u) := \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right). \tag{5}$$

This operator appears in many areas of complex analysis and geometry. In a spectacular way it was used by Yau [23] in the proof of the Calabi conjecture and in the construction of a Kähler-Einstein metric on compact Kähler manifolds with either negative or vanishing first Chern class. Its usefulness is caused by the fact that the Ricci curvature of a Kähler metric $(g_{j\bar{k}})$ is given by

$$R_{p\bar{q}} = -\frac{\partial^2}{\partial z_p \partial \overline{z}_q} \log \det(g_{j\bar{k}}),$$

and $(g_{j\bar{k}})$ is locally a complex Hessian of a certain smooth function.

First note that M^c is a real operator in the sense that if u is real-valued then so is $M^c(u)$. We may therefore consider notions defined in the introduction. Every real symmetric $2n \times 2n$ matrix we write in the form

$$A = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix},$$

where P, Q, R are real $n \times n$ matrices such that P and R are symmetric. Then

$$F(A) = 4^{-n} \det[P + R + i(Q - Q^t)].$$

One can show that $\mathcal{S}_{M^c}$ consists of A with

$$P + R + i(Q - Q^t) \geq 0,$$

which is equivalent to

$$\begin{pmatrix} P + R & Q^t - Q \\ Q - Q^t & P + R \end{pmatrix} \geq 0.$$

The set $\mathcal{S}_{M^c}$ is of the form (2): by $\widehat{\mathcal{S}}$ we may take the set of all nonnegative hermitian matrices $X + iY$, which we identify with matrices of the form

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathcal{S}_+.$$

Admissible functions for M^c are thus characterized by the condition

$$\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \right) \geq 0,$$

that is we get precisely the class plurisubharmonic functions.

To define the operator M^c for non-smooth plurisubharmonic functions is especially important in the pluripotential theory, which is a counterpart of the classical potential theory in several complex variables - see e.g. [2], [13], [18]. The complex Monge-Ampère operator in many respects behaves similarly as the real one and one could expect to define $M^c(u)$ as a nonnegative measure for every non-smooth admissible u. It turns out not to be the case. It was first shown by Shiffman and Taylor (see [20]). A simpler example was proposed by Kiselman [17]: the function

$$u(z) = (-\log |z_1|)^{1/n} (|z_2|^2 + \cdots + |z_n|^2 - 1)$$

is plurisubharmonic in a neighborhood of the origin, smooth away from the set $\{z_1 = 0\}$, but $M^c(u)$ is not integrable near $\{z_1 = 0\}$.

We see therefore that the real and complex Monge-Ampère operators have some crucial differences. As described in [6], attempts to apply the real methods in the complex case too closely may sometimes fail.

The complex Monge-Ampére operator can however be defined for certain crucial non-smooth plurisubharmonic functions, which is sufficient in most applications in pluripotential theory. It was achieved by Bedford and Taylor: in [3] for continuous and in [5] for locally bounded functions. Contrary to the geometric construction of Alexandrov (which cannot be repeated in the complex case), the construction of Bedford and Taylor is analytic. Its main tool is the theory of complex currents created mostly by Lelong. We will now describe the main ideas behind this construction.

For $p, q = 0, 1, \dots, n-1$ we consider complex forms of bidegree (p, q):

$$T = \sum_{J,K} T_{JK} dz_J \wedge d\overline{z}_K,$$

where the summation is over indices of the form $J = (j_1, \dots, j_p)$, $K = (k_1, \dots, k_q)$, where $1 \le j_1 < \dots < j_p \le n$, $1 \le k_1 < \dots < k_q \le n$, and $dz_J = dz_{j_1} \wedge \dots \wedge dz_{j_p}$, $d\overline{z}_K = d\overline{z}_{k_1} \wedge \dots \wedge d\overline{z}_{k_q}$ (see e.g. [15]). If the coefficients T_{JK} are distributions then we call T a *current*, if they are of order zero (that is they are complex measures) then we will say that T is a current *of order zero*. For a current T we define the operators ∂ and $\overline{\partial}$:

$$\partial T = \sum_{J,K} \sum_{j=1}^{n} \frac{\partial T_{JK}}{\partial z_j} dz_j \wedge dz_J \wedge d\overline{z}_K,$$

$$\overline{\partial} T = \sum_{J,K} \sum_{j=1}^{n} \frac{\partial T_{JK}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_J \wedge d\overline{z}_K.$$

The obtained current ∂T is bidegree $(p+1, q)$ and $\overline{\partial} T$ is of bidegree $(p, q+1)$. Note that $\partial + \overline{\partial} = d$. Since $d^2 = 0$, it follows that $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$. If we now set $d^c := i(\overline{\partial} - \partial)$ then $dd^c = 2i\partial\overline{\partial}$. One can easily check that for smooth u

$$(dd^c u)^n = dd^c u \wedge \dots \wedge dd^c u = 4^n n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) d\lambda.$$

Therefore, the complex Monge-Ampère operator can be written in terms of the operators d, d^c, which is very useful when integrating by parts.

For a locally bounded plurisubharmonic u and $k = 1, \dots, n$, Bedford and Taylor [5] inductively defined the current $(dd^c u)^k := dd^c(u(dd^c)^{k-1})$ and showed that it is of order zero. Note that we can multiply locally bounded functions by currents of order zero (because their coefficients are complex measures) and we will of course get a current of order zero. One of the basic results is the continuity of the operator $(dd^c)^k$ for decreasing sequences:

THEOREM 2 (BEDFORD, TAYLOR [5]). *If u_j is a sequence of plurisubharmonic functions decreasing to $u \in L^\infty_{loc}$ then $(dd^c u_j)^k$ converges weakly (that is in the weak* toopology) to $(dd^c u)^k$.*

Every $z \in \mathbb{C}^n$ can be written in the form $z = x + iy$, where $x, y \in \mathbb{R}^n$. If the function $u(z) = u(x + iy)$ does not depend on y then it is plurisubharmonic (as a function of z) if and only if it is convex (as a function of x). In other words, we may treat convex functions as continuous plurisubharmonic functions. The Bedford-Taylor definition thus gives in particular another definition of the real Monge-Ampère operator. As shown in [19], both definitions are equivalent.

4. The domain of definition of the complex Monge-Ampère operator. As we have already noticed, $M^c(u)$ cannot be well defined as a measure for arbitrary plurisubharmonic u (but it can be if u is for example locally bounded). Thus the question arises to determine when $M^c(u)$ can be well defined. It was first studied in depth by Cegrell in [12] where he introduced a class $\mathcal{E}$ (defined globally on sufficiently regular domains in $\mathbb{C}^n$) and proved its relation to this problem. Another, local approach was proposed in [7] (for $n = 2$) and [9] (for arbitrary n).

In order to explain what we precisely mean by the (local) domain of definition $\mathcal{D}$ of M^c, let us first analyze two examples due to Cegrell. In [10] he showed that there exists a sequence u_j of smooth plurisubharmonic functions converging weakly (and thus in L^p_{loc} for every $p < \infty$, see e.g. [16]) to a smooth plurisubharmonic u such that $M^c(u_j)$ does not converge even weakly to $M^c(u)$. This shows in particular that the monotone convergence appearing in Bedford-Taylor's Theorem 2 is crucial and cannot be replaced with weak convergence (and equivalently with L^p_{loc} convergence).

On the other hand, following [11] consider the unbounded plurisubharmonic function

$$u(z) = 2 \log |z_1 \dots z_n|$$

and two sequences of smooth plurisubharmonic functions decreasing to u

$$\begin{aligned} u_j(z) &= \log(|z_1 \dots z_n|^2 + 1/j), \\ v_j(z) &= \log(|z_1|^2 + 1/j) + \dots + \log(|z_n|^2 + 1/j). \end{aligned}$$

One can check that $M^c(u_j)$ converges weakly to 0 but $M^c(v_j)$ to $\pi^n \delta_0$, where by δ_0 we denote the point mass.

In view of the above examples the following definition of the subclass $\mathcal{D}$ of the class of plurisubharmonic functions is completely natural: we say that a plurisubharmonic function u belongs to $\mathcal{D}$ if there exists a nonnegative Radon measure μ such that for every sequence of smooth plurisubharmonic functions u_j decreasing to u the sequence $M^c(u_j)$ converges weakly to μ. Note that this definition is of a purely local character (we really consider germs of functions) and thus the sequence u_j may be defined on a smaller open set than u. For $u \in \mathcal{D}$ we set $M^c(u) = \mu$ (it is obvious that for a given u there can be at most one such a measure μ).

From the first of the above Cegrell examples it is clear that we cannot replace the monotone convergence in the definition of $\mathcal{D}$ with the weak convergence. From the second example it follows that we must consider all approximating sequences u_j and it is not enough to check the convergence only for one sequence. One can can easily show (see [7]) that $\mathcal{D}$ is the maximal subclass of the class of plurisubharmonic functions where the operator M^c can be defined, so that (5) holds for smooth functions and M^c is continuous for decreasing sequences.

It turns out that one can completely characterize the class $\mathcal{D}$. Let us first consider the case $n = 2$. As noticed already by Bedford and Taylor in [3] (see also [4]), if u

is smooth in $\Omega\subset\mathbb{C}^2$ then integration by parts gives

$$\int\varphi(dd^cu)^2=-\int du\wedge d^cu\wedge dd^c\varphi,\quad \varphi\in C_0^\infty(\Omega).$$

It follows that that the complex Monge-Ampère operator in $\mathbb{C}^2$ can be well defined for functions belonging to the Sobolev space $W^{1,2}_{loc}$. It is not obvious however whether, firstly, so defined operator is continuous for decreasing sequences of functions and, secondly, these are all plurisubharmonic functions for which the operator can be well defined. Both questions were answered in the affirmative in [7] and we thus have the following precise description of the domain of definition of the complex Monge-Ampère operator in $\mathbb{C}^2$:

THEOREM 3 ([7]). *For $n=2$ the class $\mathcal{D}$ consists precisely of plurisubharmonic functions belonging to $W^{1,2}_{loc}$.*

The characterization of $\mathcal{D}$ for $n\geq 3$ turns out to be more complicated, both in terms of the statement of the result as well as its proof.

THEOREM 4 ([9]). *For a negative plurisubharmonic u the following are equivalent*

i) $u\in\mathcal{D}$;

ii) **For all** *sequences of smooth plurisubharmonic functions u_j decreasing to u the sequence $(dd^cu_j)^n$ is weakly bounded;*

iii) **For all** *sequences of smooth plurisubharmonic functions u_j decreasing to u the sequences*

$$|u_j|^{n-2-p}du_j\wedge d^cu_j\wedge(dd^cu_j)^p\wedge\omega^{n-p-1},\quad p=0,1,\dots,n-2,\tag{6}$$

($\omega:=dd^c|z|^2$ is the Kähler form in $\mathbb{C}^n$) are weakly bounded;

iv) **There exists** *a sequence of smooth plurisubharmonic functions u_j decreasing to u such that the sequences (6) are weakly bounded.*

From the equivalence of i) and ii) it follows that if there exists a decreasing approximating sequence whose Monge-Ampère measures are not weakly convergent then we can find another sequence whose Monge-Ampère measures are not even weakly bounded. The importance of condition iii) is that, contrary to ii) (by the Cegrell example from [11]), we can replace the quantifier **for all** with **there exists**. This means that for a given u it is enough to check weak boundedness of (6) for just one approximating sequence, for example for the regularizations of u.

The following example shows that the condition of local weak boundedness of (6) is in fact optimal: for $q=1,\dots,n-1$ let

$$u(z):=\log(|z_1|^2+\dots+|z_q|^2),\quad u_j(z):=\log(|z_1|^2+\dots+|z_q|^2+1/j).$$

Then (6) is locally weakly bounded for $p\neq q-1$ (it vanishes for $p\geq q$), but not for $p=q-1$.

Theorem 4 and its proof can be used to prove the following important property of $\mathcal{D}$ which does not seem to be an easy consequence of the definition:

THEOREM 5 ([9]). *If $u\in\mathcal{D}$ and v is plurisubharmonic and such that $u\leq v$ then $v\in\mathcal{D}$.*

One of the steps in the proof of Theorems 3 and 4 was to show that actually both classes $\mathcal{D}$ and $\mathcal{E}$ coincide (only then does it follow that to belong to $\mathcal{E}$ is in fact a local property) and then use a result from [12] on the continuity of M^c on $\mathcal{E}$.

5. The real Hessian operator. For $m = 1, \dots, n$ we consider the elementary symmetric functions

$$S_m(\lambda) = \sum_{1 \le i_1 < \dots < i_m \le n} \lambda_{i_1} \dots \lambda_{i_m}, \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n,$$

$$S_m(A) = S_m(\lambda(A)), \quad A \in \mathcal{S},$$

where $\lambda(A) = (\lambda_1, \dots, \lambda_n)$ are the eigenvalues of the matrix A. The functions S_m are determined by

$$(\lambda_1 + t) \dots (\lambda_n + t) = \sum_{m=0}^{n} S_m(\lambda) t^{n-m}, \quad \det(A + tI) = \sum_{m=0}^{n} S_m(A) t^{n-m}, \quad t \in \mathbb{R},$$

where we set $S_0 := 1$. The real Hessian operator is defined by

$$H_m(u) = S_m(D^2 u). \tag{7}$$

We get $H_1 = \Delta$ and $H_n = M$.

One can show that

$$\mathcal{S}_{H_m} = \{A \in \mathcal{S} : S_j(A) \ge 0, \ j = 1, \dots, m\},$$

that H_m jest is an elliptic operator, and that $\mathcal{S}_{H_m}$ satisfies (2) with

$$\widehat{\mathcal{S}} = \{(\partial S_m / \partial a_{ij}(A)) \in \mathcal{S}_+ : A \in \mathcal{S}_{H_m}\}.$$

(The necessary multi-linear algebra is provided by [14].) We thus have admissible functions for H_m, they are called *m-convex.* Of course, 1-convex means subharmonic and n-convex is equivalent to convex.

The operator H_m for m-convex functions was defined by Trudinger and Wang. First in [21] they did it for continuous m-convex functions and showed the continuity for uniformly convergent sequences. Moreover, they showed that for $m > n/2$ all m-convex functions are continuous and that the weak convergence (which for subharmonic functions is equivalent to convergence in L^p_{loc} for $p < n/(n-2)$) implies the local uniform convergence. However, for $m \le n/2$ there exist discontinuous m-convex functions. In general, we have the following deep result:

THEOREM 6 (TRUDINGER, WANG [22]). *For every m-convex u one can uniqely define a measure $H_m(u)$ so that (7) holds for smooth functions and the operator is continuous for weakly convergent sequences.*

We have already seen that an analogous result is false for the complex Monge-Ampère operator - by the Cegrell example [10] it does not even hold for smooth plurisubharmonic functions. Interestingly, both the Bedford- Taylor analytic methods [3], [5], and the Cegrell example [10] turned out to be inspiring for Trudinger and Wang, although their result is purely real and not true in the complex case.

The geometric construction of Alexandrov is of no use for the Hessian operator (even the definition of m-convex functions is analytic and there is no equivalent geometric definition as is the case for $m = n$). In the proof of Theorem 6 successive integration by parts was used, partly similar to the one from [5].

Important examples of m-convex functions are the fundamental solutions for the operator H_m (i.e. $H_m(E) = \delta_0$):

$$E(x) = \lambda_n^{-1/m} \binom{n}{m}^{-1/m} \begin{cases} \frac{m}{2m-n}|x|^{(2m-n)/m}, & m \neq n/2, \\ \log|x|, & m = n/2, \end{cases}$$

where λ_n denotes the volume of the unit ball in $\mathbb{R}^n$. Note that E is not bounded if $m \le n/2$. One has

$$E \in W^{1,q}_{loc} \Leftrightarrow q < \frac{nm}{n-m}$$

and, for $m \le n/2$,

$$E \in L^p_{loc} \Leftrightarrow p < \frac{nm}{n-2m}.$$

It follows from [22] that in both cases the fundamental solution is an extremal example:

THEOREM 7 (TRUDINGER, WANG [22]). *Every m-convex function is in $W^{1,q}_{loc}$ for every $q < nm/(n-m)$.*

From the Sobolev embedding theorem we then immediately obtain:

COROLLARY 8. *If $m \le n/2$ then m-convex functions are in L^p_{loc} for every $p < nm/(n-2m)$.*

6. The complex Hessian operator. It is defined in the same way as in the real case:

$$H^c_m(u) = S_m\left(\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right)\right), \quad m = 1, \dots, n,$$

for functions u on open subsets of $\mathbb{C}^n$ (of course S_m is real for hermitian matrices, since their eigenvalues are real). The operator H^c_m can also be expressed in terms of the operators d and d^c:

$$(dd^c u)^m \wedge \omega^{n-m} = 4^{n-m/2} m!\, S_m\left(\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right)\right) d\lambda.$$

By $\mathcal{H}_m$ denote the set of all $(1,1)$-forms with constant coefficients

$$\beta = \sum_{j,k=1}^n a_{jk} i\, dz_j \wedge d\overline{z}_k, \quad a_{jk} \in \mathbb{C},$$

such that the matrix (a_{jk}) is hermitian (i.e. $\overline{\beta} = \beta$) and $\beta^j \wedge \omega^{n-j} \ge 0$ for $j = 1, \dots, m$. One can show (see [8], the proofs are based on [14]) that

$$\beta_1 \wedge \dots \wedge \beta_m \wedge \omega^{n-m} \ge 0, \quad \beta_1, \dots, \beta_m \in \mathcal{H}_m,$$

and that u is admissible for H_m^c if and only if

$$(dd^c u)^j \wedge \omega^{n-j} \geq 0, \quad j = 1, \dots, m,$$

which is equivalent to

$$dd^c u \wedge \beta_1 \wedge \cdots \wedge \beta_{m-1} \wedge \omega^{n-m} \geq 0, \quad \beta_1, \dots, \beta_{m-1} \in \mathcal{H}_m.$$

Similarly as we did for the complex Monge-Ampère operator we could state these conditions in terms of the real matrix $D^2 u$ and define the sets $\mathcal{S}_{H_m^c}$ and $\widehat{\mathcal{S}}$.

Admissible functions for H_m^c we will call *m-subharmonic*. (Perhaps a more logical term *m-plurisubharmonic* is already in use and denotes a completely different class of functions.) Again, 1-subharmonic means subharmonic and n-subharmonic means plurisubharmonic. Note that, similarly as before, if $u(x + iy)$ does not depend on y then u is m-convex (with respect to $x \in \mathbb{R}^n$) if and only if it is m-subharmonic (with respect to $z = x+iy \in \mathbb{C}^n$). The m-convex functions can be therefore treated as special cases of m-subharmonic functions.

We have the following fundamental solution for H_m^c:

$$E^c(z) = \lambda_{2n}^{-1/m} \binom{n}{m}^{-1/m} \begin{cases} -\frac{m}{n-m}|z|^{-2(n-m)/m}, & m < n, \\ \log|z|, & m = n. \end{cases}$$

One can easily check that

$$E^c \in W^{1,q}_{loc} \Leftrightarrow q < \frac{2nm}{2n-m}, \quad E^c \in L^p_{loc} \Leftrightarrow p < \frac{nm}{n-m},$$

and we can ask if the optimal results, similar to Theorem 7 and Corollary 8, hold in this case as well.

On one hand, we cannot expect a result similar to Theorem 7, that is that all m-subharmonic functions are in $W^{1,q}_{loc}$ for every $q < \frac{2nm}{2n-m}$. For $\log|z_1|$ is plurisubharmonic (and thus m-subharmonic for every m) but $\log|z_1| \notin W^{1,2}_{loc}$. In general, gradient estimates similar to Theorem 7 do not hold in the complex case.

On the other hand, the following conjecture seems plausible:

Conjecture. *Every m-subharmonic function is in L^p_{loc} for all $p < nm/(n-m)$.*

First note that a necessary (but not sufficient if $1 < m < n$) condition for a function to be m-subharmonic is that it is subharmonic on every $n - m + 1$-dimensional complex subspace. Integrating along such subspaces one then easily concludes that m-subharmonic functions are in L^p_{loc} for every

$$p < \frac{2(n-m+1)}{2(n-m+1)-2} = \frac{n-m+1}{n-m}$$

(note that $2k$ is the real dimension of a k-dimensional complex subspace). This trivial estimate was improved in [8] for $p < n/(n-m)$.

It is perhaps interesting to compare the real Hessian operator H_n in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with the complex Monge-Ampère operator M^c and the corresponding classes of admissible functions. We obtain two nonlinear logarithmic *potential theories*, real and complex, that are very different from each other. In particular, Theorem 6 holds for the real one and fails, even in a much weaker form, for the complex one. The latter depends heavily on the choice of a complex structure on $\mathbb{R}^{2n}$, whereas the real one is independent of it.

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Jagiellonian University, Institute of Mathematics, Reymonta 4, 30-059 Kraków, Poland e-mail: umblocki@cyf-kr.edu.pl