MINICOURSE ON PLURIPOTENTIAL THEORY

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Introduction

This is a brief introduction to pluripotential theory. We discuss the following topics in subsequent sections:

Section 1 contains the fundamentals of the theory due to Bedford-Taylor [5], [7] and some generalizations of Demailly [25]. In Section 2 we present characterization of the domain of definition for the complex Monge-Ampère operator from [14], [16]. In Section 3 we survey the Dirichlet problem for this operator, which is overall a very broad topic. Relative Bedford-Taylor capacity, pluripolar and negligible sets, as well as extremal functions are briefly discussed in Section 4. The Siciak extremal function and the pluricomplex Green function are treated there in a bigger detail. The latter is used in Section 5 for various applications to the Bergman kernel and metric. Sections 2 and 5 contain the most recent material (but already almost 10 years old).

Especially in the first two sections we present some proofs to get the reader acquainted with common techniques. Anybody interested in more details should see the expositions [25], [27], [12], [47] or [41]. We also give some easy exercises as well as open problems.

1. Definition of the complex Monge-Ampère operator

The complex Monge-Ampère for smooth functions defined on an open subset of $\mathbb{C}^n$ is given by

$$\det(u_{j\bar{k}}),$$

where we use the notation $u_{j\bar{k}} = \partial^2 u/\partial z_j \partial \bar{z}_k$. One would like to extend this (as a nonnegative measure) for non-smooth plurisubharmonic (psh) u, similarly as in analogous cases of Laplacian for subharmonic functions and the real Monge-Ampère operator for convex functions (see [54] for an exposition of the latter).

First observation is that, unlike in the these two cases, it is not always possible. Following Kiselman [40] consider the function

$$u(z) = (-\log|z_1|)^{1/n}(|z_2|^2 + \cdots + |z_n|^2 - 1).$$

It is psh near the origin, smooth away from $\{z_n = 0\}$ but

$$\det(u_{j\bar{k}}) = \frac{1 - \frac{1}{n} - |z_2|^2 - \cdots - |z_n|^2}{-4n|z_1|^2 \log|z_1|}$$

is not locally integrable near $\{z_n = 0\}$ (if $n \geq 2$). (The first example of this kind was constructed by Shiffman and Taylor [57].)

Bedford-Taylor's theory [5], [7] enables to define the Monge-Ampère operator for locally bounded psh functions. As the definition uses induction on the degree of nonlinearity, one needs to introduce positive currents. A *complex current* of bidegree (p, q) (or bidimension $(n-p, n-q)$) is a differential form

$$T = \sum_{\substack{|I|=p \\ |J|=q}}{}' T_{IJ}\, dz_I \wedge d\bar{z}_J$$

whose coefficients T_{IJ} are distributions. Equivalently, a current T of bidegree (p, q) in an open domain Ω in $\mathbb{C}^n$ (which we write $T \in \mathcal{D}'_{(p,q)}(\Omega)$) is a continuous functional on the space $\mathcal{D}_{(n-p,n-q)}(\Omega)$ of smooth complex forms of bidegree $(n-p, n-q)$ with compact support.

We say that a current T of bidegree (p, p) is *positive* (we will write $T \geq 0$) if it is real (that is $\bar{T} = T$) and for any $\alpha_1, \dots, \alpha_{n-p} \in \mathbb{C}_{(1,0)}$ one has

$$T \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_{n-p} \wedge \bar{\alpha}_{n-p} \geq 0.$$

Exercise 1. *Prove that a* $(1,1)$*-current*

$$\sum_{j,k} T_{jk}\, i\, dz_j \wedge d\bar{z}_k$$

is positive if and only if (T_{jk}) *is positive semi-definite.*

The following result is crucial:

Theorem 1.1. *Positive currents are of order 0 (that is their coefficients are complex measures).*

Proof. Let $\{\beta_j\}$ be a basis of $\mathbb{C}_{(n-p,n-p)}$ whose elements are of the form

$$i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_{n-p} \wedge \bar{\alpha}_{n-p}, \quad \alpha_1, \dots, \alpha_{n-p} \in \mathbb{C}_{(1,0)}.$$

(To show that such a basis exists it is enough to prove that $dz_J \wedge d\bar{z}_K$ can be written as a linear combination of such forms. This follows easily from

$$\begin{aligned} 2dz_j \wedge d\bar{z}_k =& (dz_j + dz_k) \wedge (d\bar{z}_j + d\bar{z}_k) + i(dz_j + idz_k) \wedge (d\bar{z}_j - id\bar{z}_k) \\ &- (i+1)(dz_j \wedge d\bar{z}_j + dz_k \wedge d\bar{z}_k).) \end{aligned}$$

By $\{\beta_j'\}$ denote the dual basis in $\mathbb{C}_{(p,p)}$. Write

$$T = {\sum_{I,J}}' T_{IJ} dz_I \wedge d\bar{z}_J = \sum_j T_j \beta_j',$$

where $T_j d\lambda = T \wedge \beta_j \geq 0$. We see that T_{IJ} can be expressed as linear combinations of positive measures T_j, and thus are complex measures. □

Let T be a closed (that is $dT = 0$) positive current of bidegree (q,q), $q < n$. Since complex measures (that is distributions of order 0) can be multiplied by locally bounded functions, for any $u \in PSH \cap L^\infty_{loc}$ we can define

$$dd^c u \wedge T := dd^c(uT).$$

(Here $d^c = i(\bar{\partial} - \partial)$, so that $dd^c = 2i\partial\bar{\partial}$.)

Proposition 1.2. *$dd^c u \wedge T$ is a closed positive current.*

Proof. Closedness is clear. Since u is locally bounded, by the Lebesgue bounded convergence theorem we have weak convergence $u_\varepsilon T \to uT$, where $u_\varepsilon = u * \rho_\varepsilon$ is the standard regularization of psh functions. Therefore $dd^c(u_\varepsilon T) \to dd^c(uT)$ weakly. We clearly have $dd^c(u_\varepsilon T) = dd^c u_\varepsilon \wedge T$ in the usual sense. Since positive currents can be weakly approximated by smooth positive forms, it remains to show the following result:

Proposition 1.3. *If $\alpha \in \mathbb{C}_{(p,p)}$ and $\beta \in \mathbb{C}_{(1,1)}$ are positive then so is $\alpha \wedge \beta$.*

Proof. It is an easy consequence of the fact that after a change of variables we can write

$$\beta = \sum_j \lambda_j \, i \, dz_j \wedge d\bar{z}_j,$$

where $\lambda_j \geq 0$. □

The above result is false for $\beta \in \mathbb{C}_{(q,q)}$, it was originally shown in [34] (and independently in [4]). S. Dinew [30] constructed explicit positive $\alpha, \beta \in \mathbb{C}_{(2,2)}(\mathbb{C}^4)$ such that $\alpha \wedge \beta < 0$.

Problem 1. *Construct explicit* $\alpha \in \mathbb{C}_{(2,2)}(\mathbb{C}^4)$ *with* $\alpha^2 < 0$.

From now on assume that T is a closed positive current of bidegree (q, q). We can define inductively

$$dd^c u_1 \wedge \dots \wedge dd^c u_p \wedge T, \quad u_1, \dots, u_p \in PSH \cap L^\infty_{loc}$$

and $(dd^c u)^n$ is a positive measure for locally bounded psh u. If u is smooth then

$$(dd^c u)^n = n! 4^n \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) d\lambda.$$

Exercise 2. *Show that* $(dd^c \log_+ |z|)^n = (2\pi)^n d\sigma/\sigma(\mathbb{S})$ *where* $d\sigma$ *is the surface measure on the unit sphere* $\mathbb{S}$ *in* $\mathbb{C}^n$.

Theorem 1.4 (Chern-Levine-Nirenberg Inequality [24])**.** *Assume that* K *is compact in open* Ω *in* $\mathbb{C}^n$. *Then for a closed closed positive current* T *in* Ω *and* $u_1, \dots, u_p \in PSH \cap L^\infty(\Omega)$ *we have*

$$||dd^c u_1 \wedge \dots \wedge dd^c u_p \wedge T||_K \leq C ||u_1||_{L^\infty(\Omega)} \dots ||u_p||_{L^\infty(\Omega)} ||T||_\Omega,$$

where C *depends only on* K *and* Ω *and* $||T||_E = \sum'_{I,J} ||T_{IJ}||_E$ *is a total variation of the current* T *over the set* E.

For the proof we will need a preparatory result:

Proposition 1.5. *Let* $T = \sum'_{I,J} T_{IJ}\, dz_I \wedge d\bar{z}_J$ *be a positive current of bidegree* (p, p). *Then*

$$|T_{IJ}| \leq c_n\, T \wedge \omega^{n-p},$$

where $\omega = \sum_j \frac{i}{2}\, dz_j \wedge d\bar{z}_j$.

Proof. Let $\{\omega_{IJ}\}$ be a basis in $\mathbb{C}_{(n-p,n-p)}$ dual to $\{dz_I \wedge d\bar{z}_J\}$. Write

$$\omega_{IJ} = \sum_j c^j_{IJ}\, \beta_j,$$

where $\{\beta_j\}$ is chosen as in the proof of Theorem 1.1. Then

$$|T_{IJ}| = |T \wedge \omega_{IJ}| = |\sum_j c^j_{IJ}\, T \wedge \beta_j| \leq c_n T \wedge \omega^{n-p}$$

for $\beta_j \leq c' \omega^{n-p}$ by Proposition 1.3. □

Proof of Theorem 1.4. We may assume that $p = 1$. Let $\varphi \in C_0^\infty(\Omega)$ be nonnegative and such that $\varphi = 1$ on K. By Proposition 1.5

$$\begin{aligned}||dd^c u \wedge T||_K &\le c_n \int_K dd^c u \wedge T \wedge \omega^{n-p-1} \\ &\le c_n \int_\Omega \varphi\, dd^c u \wedge T \wedge \omega^{n-p-1} \\ &= c_n \int_\Omega u\, dd^c \varphi \wedge T \wedge \omega^{n-p-1}\end{aligned}$$

and the estimate follows. □

The following approximation result is due to Bedford and Taylor [5]:

Theorem 1.6. *For $k = 0, 1, \dots, p$, where $p + q \le n$, let $\{u_k^j\}$ be a sequence of psh functions decreasing to a locally bounded psh u_k as $j \to \infty$. Then we have weak convergence*

$$u_0^j\, dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j \wedge T \longrightarrow u_0\, dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T.$$

Proof. Suppose that u_k^j and T are defined in a neighborhood of $\bar{B}$, where $B = B(z_0, r)$. We may assume that for some positive constant M we have $-M \le u_k^j \le -1$ in a neighborhood of $\bar{B}$. If we take $B' \Subset B$ and $\psi(z) := |z - z_0|^2 - r^2$ then for A big enough $\max\{u_k^j, A\psi\} = u_k^j$ on B' and $\max\{u_k^j, A\psi\} = A\psi$ in a constant neighborhood of ∂B. We may therefore assume that $u_k^j = u_k = A\psi$ in a neighborhood of ∂B.

The further proof is by induction with respect to p. The theorem is obviously true if $p = 0$. Let $p \ge 1$ and assume it holds for $p - 1$. It follows that

$$S^j := dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j \wedge T \longrightarrow dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T =: S$$

weakly and we have to show that $u_0^j S^j \to u_0 S$ weakly. Note it is very simple if all involved functions are continuous: then the convergence $u_k^j \to u_k$ is uniform and we may write

$$u_0^j S^j - u_0 S = (u_0^j - u_0) S^j + u_0 (S^j - S).$$

In the general case we see that by the Chern-Levine-Nirenberg inequality the sequence S^j is relatively compact in the weak* topology. It therefore remains to show that if $u_0^j S^j \to \Theta$ weakly then $\Theta = u_0 S$.

First we claim that $u_0 S \ge \Theta$. For this it is enough to show that $u_0^{j_0} S \wedge \alpha \ge \Theta \wedge \alpha$ for every j_0 and positive $\alpha \in \mathbb{C}_{(n-p-q,n-p-q)}$. For any ε we have

$$u_0^j\, S^j \wedge \alpha \le u_0^{j_0}\, S^j \wedge \alpha \le u_0^{j_0} * \rho_\varepsilon\, S^j \wedge \alpha$$

and therefore $\Theta \wedge \alpha \le u_0^{j_0} * \rho_\varepsilon\, S \wedge \alpha$. From the Lebesgue monotone convergence theorem we now get $\Theta \wedge \alpha \le u_0^{j_0}\, S \wedge \alpha$.

By Proposition 1.5 to finish the proof of the theorem it remains to show that $\int_B (u_0 S - \Theta) \wedge \omega^{n-p-q} \le 0$. Integrating by parts we will get

$$\begin{aligned}\int_B u_0 dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T \wedge \omega^{n-p-q} &\le \int_B u_0^j dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T \wedge \omega^{n-p-q}\\ &= \int_B u_1 dd^c u_0^j \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_p \wedge T \wedge \omega^{n-p-q}\\ &\le \cdots \le \int_B u_0^j dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j \wedge T \wedge \omega^{n-p-q}\end{aligned}$$

and the theorem follows since the last integral in fact converges to $\int_B \Theta \wedge \omega^{n-p-q}$ (recall that $u_j^k = A\psi$ in a fixed neighborhood of ∂B). □

As proved by Demailly [25] (see also [26]), definition of the Monge-Ampère operator can be extended to psh functions that may be unbounded on a relatively compact subset. Take $u \in PSH(\Omega)$ which is locally bounded away from $\Omega' \Subset \Omega$. Withous loss of generality we may assume that u is negative. Define

$$dd^c u_j := \max\{u, -j\}.$$

Then $u_j = u$ in $\Omega \setminus \Omega'$ for j big enough and integration by parts gives

$$\int_\Omega dd^c u_j \wedge T \wedge \omega^{n-q-1} = \int_\Omega dd^c u_k \wedge T \wedge \omega^{n-q-1}$$

for j, k sufficiently large. Let $\chi \in C_0^\infty(\Omega)$ be equal to $|z|^2/4$ in Ω'. Then

$$\begin{aligned}C_1 \int_\Omega dd^c u_j \wedge T \wedge \omega^{n-q-1} &\ge \int_\Omega \chi\, dd^c u_j \wedge T \wedge \omega^{n-q-1}\\ &= \int_\Omega u_j\, dd^c \varphi \wedge T \wedge \omega^{n-q-1}\\ &\ge \int_{\Omega'} u_j\, T \wedge \omega^{n-q} + C_2.\end{aligned}$$

This shows that uT has a locally bounded mass and thus is a current. We can now define $dd^c u \wedge T$ as before. Since by the Lebesgue bounded convergence theorem $u_j T \to uT$ weakly, we see that $dd^c u \wedge T$ is a closed positive current.

Exercise 3. *Show that* $(dd^c \log|z|)^n = (2\pi)^n \delta_0$.

We also see that in the proof of the Chern-Levine-Nirenberg inequality we really get

$$||dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T||_K \le C ||u_1||_{L^\infty(\Omega\setminus K)} \dots ||u_p||_{L^\infty(\Omega\setminus K)} ||T||_{\Omega\setminus K}.$$

We have a similar result to Theorem 1.6 but we have to assume that $p + q < n$ and that functions are defined in a pseudoconvex domain:

Theorem 1.7. *For $k = 0, 1, \dots, p$, where $p + q < n$, let $\{u_k^j\}$ be a sequence of psh functions in a pseudoconvex domain Ω in $\mathbb{C}^n$ decreasing to psh u_k which is locally bounded away from a compact subset of Ω as $j \to \infty$. Then we have weak convergence*

$$u_0^j \, dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j \wedge T \longrightarrow u_0 \, dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T.$$

Proof. The proof is essentially the same as that of Theorem 1.6 with some modifications. We take B to be a strongly pseudoconvex domain (instead of a ball) and ψ its defining function. We assume that $-M \le u_k^j \le -1$ in $B \setminus B'$. We choose $A >> 1$ so that $A\psi \le u_k^j - 1$ in a neighborhood of $\partial B'$ and replace u_k^j with

$$\begin{cases} \max\{u_k^j, A\psi\} & \text{in } B \setminus B' \\ u_k^j & \text{in } B'. \end{cases}$$

The rest of the proof is the same. □

Note the pseudoconvexity assumption in Theorem 1.7 is not a real obstacle, at least when $u_0 = u_1 = \cdots = u_p$, for $\{u < const\}$ is pseudoconvex for psh u.

It is crucial in Theorems 1.6 and 1.7 that the sequences are decreasing. Cegrell [19] constructed a sequence u_j of smooth psh functions converging weakly (and thus in L^p_{loc} for every $p < \infty$) to a smooth psh u but such that $(dd^c u_j)^n$ does not converge weakly to $(dd^c u)^n$.

Exercise 4. *Following Cegrell* [20] *define*

$$u_j(z) := \log(|z_1 \dots z_n|^2 + 1/j)$$
$$v_j(z) = \log(|z_1|^2 + 1/j) + \cdots + \log(|z_n|^2 + 1/j),$$

so that both sequences decrease to $2 \log |z_1 \dots z_n|$. *Show that* $(dd^c u_j)^n$ *converges weakly to 0 whereas* $(dd^c v_j)^n$ *to* $\pi^n \delta_0$.

2. Domain of definition of $(dd^c)^n$

These two examples of Cegrell above suggest to introduce the domain of definition $\mathcal{D}$ of the complex Monge-Ampère operator as follows: we say that a psh u belongs to $\mathcal{D}$ if there exists a measure μ such that for every sequence u_j of smooth psh functions decreasing to u we have weak convergence $(dd^c u_j)^n \to \mu$. We then of course set $(dd^c u)^n := \mu$. Note that the definition is purely local so that the approximating sequences u_j may be defined in a smaller set than u. One can easily show that $\mathcal{D}$ is the maximal subclass of the class of psh functions where the Monge-Ampère operator can be defined so that it is continuous for decreasing sequences.

First consider the case $n=2$ studied in [14]. Note that then it is easy to define the Monge-Ampère operator for functions in $W^{1,2}_{loc}$ (see also [6]):

$$\int_\Omega \varphi(dd^cu)^2 = -\int_\Omega du\wedge d^cu\wedge dd^c\varphi, \quad \varphi\in C_0^\infty(\Omega).$$

Proposition 2.1. *If a sequence of psh functions u_j converges to a psh u in $W^{1,2}_{loc}$ then $(dd^cu_j)^2\to(dd^cu)^2$ weakly.*

Proof. For $\varphi\in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\left|\int_\Omega \varphi\big((dd^cu_j)^2-(dd^cu)^2\big)\right| &= \left|\int_\Omega \varphi\, dd^c(u_j-u)\wedge dd^c(u_j+u)\right|\\ &= \left|\int_\Omega d(u_j-u)\wedge d^c(u_j-u)\wedge dd^c\varphi\right|\\ &\le C\left(\int_\Omega |\nabla(u_j-u)|^2 d\lambda\right)^{1/2}\left(\int_\Omega |\nabla(u_j+u)|^2 d\lambda\right)^{1/2}\end{aligned}$$

and the proposition follows. □

To obtain that $PSH\cap W^{1,2}_{loc}\subset\mathcal{D}$ we need however to know that it is continuous for decreasing sequences. It was proved in [14] and slightly simplified by Cegrell [21] who showed the second part of the following result:

Theorem 2.2. *i) If $u\in SH\cap W^{1,2}_{loc}$ and $v\in SH$ are such that $u\le v$ then $v\in W^{1,2}_{loc}$.*

ii) If $u_j\in SH$ decreases to $u\in SH\cap W^{1,2}_{loc}$ then it converges in $W^{1,2}_{loc}$.

Proof. i) We will show that if u,v are subharmonic in $\Omega\subset\mathbb{R}^m$ and such that $u\le v<0$ then

$$||v||_{W^{1,2}(\Omega')}\le C(\Omega',\Omega)\,||u||_{W^{1,2}(\Omega)},\quad \Omega'\Subset\Omega. \tag{2.1}$$

Choose nonnegative $\varphi\in C_0^\infty(\Omega)$ such that $\varphi=1$ on Ω'. Then

$$\begin{aligned}\int_{\Omega'}|\nabla v|^2 d\lambda &\le \int_\Omega \varphi|\nabla v|^2 d\lambda\\ &= \int_\Omega \varphi\big(\frac12\Delta(v^2)-v\Delta v\big)d\lambda\\ &\le \frac12\int_\Omega v^2\Delta\varphi\, d\lambda - \int_\Omega \varphi u\Delta v\, d\lambda\end{aligned}$$

and it is enough to estimate the last integral. We have

$$-\int_\Omega \varphi u\Delta v\, d\lambda = -\int_\Omega v\Delta(\varphi u)\, d\lambda = -\int_\Omega v\big(\varphi\Delta u+\frac12\langle\nabla\varphi,\nabla u\rangle+u\Delta\varphi\big)d\lambda$$

and we can easily estimate every term to get (2.1).

ii) We have

$$\int_\Omega \varphi|\nabla(u_j-u)|^2 d\lambda = \int_\Omega \varphi\big[\frac{1}{2}\Delta((u_j-u)^2)-(u_j-u)\Delta(u_j-u)\big]d\lambda$$
$$\le \frac{1}{2}\int_\Omega (u_j-u)^2\Delta\varphi\, d\lambda + \int_\Omega \varphi(u_j-u)\Delta u\, d\lambda$$

and the last integral converges to 0 by the Lebesgue monotone convergence theorem. □

The second part of Theorem 2.2 and Proposition 2.1 give $PSH \cap W^{1,2}_{loc} \subset \mathcal{D}$. In fact, one can show that for $u \in PSH \setminus W^{1,2}_{loc}$ it is possible to construct an approximating sequence whose Monge-Ampère measures are not weakly bounded, see [14] for details. We thus get:

Theorem 2.3. *If* $n = 2$ *then* $\mathcal{D} = PSH \cap W^{1,2}_{loc}$.

Theorems 2.3 and 2.2 give in particular (in dimension 2):

Theorem 2.4. *If* $u \in \mathcal{D}$ *and a psh* v *are such that* $u \le v$ *then* $v \in \mathcal{D}$.

In fact, the result holds in arbitrary dimension, see [16]. In this case the characterization of $\mathcal{D}$ is a bit more complicated:

Theorem 2.5 ([16])**.** *For a negative psh u the following are equivalent*

i) $u \in \mathcal{D}$;

ii) **For all** *sequences of smooth plurisubharmonic functions* u_j *decreasing to* u *the sequence* $(dd^c u_j)^n$ *is weakly bounded;*

iii) **For all** *sequences of smooth plurisubharmonic functions* u_j *decreasing to* u *the sequences*

$$|u_j|^{n-2-p} du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge \omega^{n-p-1}, \quad p = 0, 1, \dots, n-2,$$

($\omega := dd^c|z|^2$ *is the Kähler form in* $\mathbb{C}^n$) *are weakly bounded;*

iv) **There exists** *a sequence of smooth plurisubharmonic functions* u_j *decreasing to* u *such that the sequences (6) are weakly bounded.*

Note that in view of example from Exercise 4 we cannot replace the quantifier *for all* by *there exists* in ii). Theorem 2.5 also implies that if there are two approximating sequences whose Monge-Ampère measures converge weakly to two different limits (as in Exercise 4) then there exists a third approximating sequence whose Monge-Ampère measures are not weakly bounded.

3. Dirichlet problem

For a bounded domain Ω in $\mathbb{C}^n$, continuous φ on $\partial\Omega$ and regular measure μ on Ω we consider the following Dirichlet problem

$$\begin{cases} u \in PSH(\Omega) \\ (dd^c u)^n = \mu \\ \lim_{z \to w} u(z) = \varphi(w), \quad w \in \partial\Omega \end{cases} \tag{3.1}$$

(note that $(dd^c u)^n$ is well defined here since u is locally bounded near $\partial\Omega$). First we note that in such a general case we do not even have uniqueness here:

Exercise 5. *For $\alpha, \beta > 0$ consider*

$$u(z) := \max\{\alpha \log|z|, \beta \log|w|\}.$$

Then $u = 0$ on the boundary of the unit bidisc. Show that

$$(dd^c u)^2 = \pi\alpha\beta\delta_0.$$

We will get uniqueness in (3.1) if we restrict ourselves to bounded psh functions:

Theorem 3.1 (Comparison Principle [7])**.** *Let u, v be bounded psh functions in bounded domain Ω in $\mathbb{C}^n$ such that $(dd^c u)^n \le (dd^c v)^n$ and*

$$\liminf_{z \to \partial\Omega} (u(z) - v(z)) \ge 0. \tag{3.2}$$

Then $v \le u$ in Ω.

It easily follows from the following domination principle:

Theorem 3.2. *Assume that $u, v \in PSH \cap L^\infty(\Omega)$ satisfy* (3.2)*. Then*

$$\int_{\{u<v\}} (dd^c v)^n \le \int_{\{u<v\}} (dd^c u)^n.$$

Proof. We will show it for u, v continuous on $\bar{\Omega}$, for the general case see [7] or [12]. In this case we may assume that $u < v$ in Ω and $u = v$ on $\partial\Omega$. For $\varepsilon > 0$ define $v_\varepsilon := \max\{v, u + \varepsilon\}$, so that $v_\varepsilon = u + \varepsilon$ near $\partial\Omega$ and $v_\varepsilon \to v$ in Ω as $\varepsilon \to 0$. Therefore by the Stokes theorem

$$\int_\Omega (dd^c u)^n = \int_\Omega (dd^c v_\varepsilon)^n$$

and by weak convergence $(dd^c v_\varepsilon)^n \to (dd^c v)^n$

$$\liminf_{\varepsilon \to 0} \int_\Omega (dd^c v_\varepsilon)^n \ge \int_\Omega (dd^c v)^n.$$

□

Proof of Theorem 3.1. Suppose the set $\{v > u\}$ in not empty. Then for some $\varepsilon > 0$ the set $U := \{v + \psi > u\}$ is also nonempty, where $\psi := |z|^2 - M$ and M is chosen in such a way that $\psi \le 0$ in Ω. From Theorem 3.2 we now get

$$\int_U (dd^c u)^n \ge \int_U (dd^c(v+\psi))^n \ge \int_U (dd^c v)^n + \int_U (dd^c\psi)^n > \int_U (dd^c u)^n,$$

a contradiction. □

The fundamental result is due to Bedford and Taylor [5] who showed that the problem has a solution in strongly pseudoconvex Ω provided that μ has continuous density:

Theorem 3.3. *If Ω is strongly pseudoconvex, $\varphi \in C(\partial\Omega)$, $F \in C(\bar{\Omega})$, $F \ge 0$ then there exists unique solution to the following Dirichlet problem*

$$\begin{cases} u \in PSH(\Omega) \cap C(\bar{\Omega}) \\ (dd^c u)^n = F\, d\lambda \\ u|_{\partial\Omega} = \varphi \end{cases} \tag{3.3}$$

By the comparison principle a bounded solution of (3.3), if exists, has to be given by the Perron-Bremermann envelope

$$u = \big(\sup\{v \in PSH \cap L^\infty(\Omega)\colon (dd^c v)^n \ge F d\lambda,\ v^*|_{\partial\Omega} \le \varphi\}\big)^*$$

(here v^* denotes the upper regularization of v, it is defined on $\bar{\Omega}$). This was the approach in [5], see also [9] and [12] for some simplifications. Continuity of u defined this way can be proved using a method of Walsh [58].

Theorem 3.3 can also be easily deduced from the following deep regularity result of Cafferelli, Kohn, Nirenberg and Spruck [18] and Krylov [48]:

Theorem 3.4. *Assume that Ω is strongly pseudoconvex with C^∞ boundary, $\varphi \in C^\infty(\partial\Omega)$, $F \in C^\infty(\bar{\Omega})$, $F > 0$. Then there exists a solution of* (3.3) *in $C^\infty(\bar{\Omega})$.*

The assumption $F > 0$ in Theorem 3.4 is crucial as the following example of Gamelin-Sibony [32] shows: the function

$$u(z,w) := \big(\max\{|z|^2 - \frac{1}{2}, |w|^2 - \frac{1}{2}, 0\}\big)^2$$

is psh and $C^{1,1}$ in the unit ball $\mathbb{B}$ of $\mathbb{C}^2$, $(dd^c u)^2 = 0$ and u is C^∞ on $\partial\mathbb{B}$. But u is not C^2. Another example of this kind but of slightly different nature was constructed by Bedford and Fornæss [3].

$C^{1,1}$-regularity in the degenerate case is in fact optimal:

Theorem 3.5 (Krylov [49])**.** *Assume that Ω is strongly pseudoconvex with $C^{3,1}$ boundary, $\varphi \in C^{3,1}(\partial\Omega)$ and $F^{1/n} \in C^{1,1}(\Omega)$, $F \ge 0$. Then $u \in C^{1,1}(\bar{\Omega})$.*

$C^{3,1}$-regularity in the above theorem is also optimal: in the unit ball $\mathbb{B}$ in $\mathbb{C}^2$ set

$$u(z,w) := (1-|z|^2)^\alpha$$

where $\frac{3}{2} \le \alpha < 2$. Then $u \in C^{1,\alpha-1}(\bar{\mathbb{B}})$ but $u|_{\partial\mathbb{B}} \in C^{3,2\alpha-3}(\partial\mathbb{B})$ (and both exponents are biggest possible).

Theorem 3.3 can be easily generalized to a class of *B-regular* domains introduced by Sibony [55]. They are characterized by the following result from [55] (see also [9] and [12]):

Theorem 3.6. *For a bounded domain Ω in $\mathbb{C}^n$ the following are equivalent:*

i) For every $z_0 \in \partial\Omega$ there exists v psh in Ω such that $u^* < 0$ on $\bar{\Omega} \setminus \{z_0\}$ but $\lim_{z\to z_0} u(z) = 0$ (that is every boundary point admits a strong psh barrier);

ii) For every continous function on $\partial\Omega$ there exists a psh extension to Ω, continous on $\bar{\Omega}$;

iii) There exists a smooth psh function ψ in Ω such that $\lim_{z\to\partial\Omega} \psi(z) = 0$ and the function $\psi(z) - |z|^2$ is psh (that is ψ is uniformly strongly psh in Ω).

Another important class of domains in pluripotential theory are the ones that admit weak psh barriers. Namely, we call a bounded domain Ω in $\mathbb{C}^n$ *hyperconvex* if there exists a negative psh u in Ω which vanishes on the boundary.

Problem 2. *Assume that a bounded domain Ω has the following property: for every $z_0 \in \partial\Omega$ there exists a neighborhood U of z_0 and u a negative psh function in $U \cap \Omega$ such that $\lim_{z\to z_0} u(z) = 0$. Is Ω hyperconvex?*

Kerzman and Rosay [39] proved that hyperconvexity is a local notion of the boundary (see also [25]. Demailly [25] showed that pseudoconvex domains with Lipschitz boundary are hyperconvex. The notions of B-regular and hyperconvex domains coincide for $n = 1$ but not in higher dimensions: polydisks for example are hyperconvex but not B-regular.

Theorem 3.3 also holds for hyperconvex domains but we have to add a necessary assumption:

Theorem 3.7 ([9]). *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and assume that $\varphi \in C(\partial\Omega)$ can be extended to a psh function in Ω, continuous on $\bar{\Omega}$. Then for any nonnegative $F \in C(\bar{\Omega})$ there exists a unique solution to* (3.3).

Corollary 3.8. *For any bounded hyperconvex Ω there exists unique $u_\Omega \in PSH(\Omega) \cap C(\bar{\Omega})$ such that $u_\Omega = 0$ on $\partial\Omega$ and $(dd^c u_\Omega)^n = d\lambda$ in Ω.*

Problem 3. *Is it true that $u_\Omega \in C^\infty(\Omega)$?*

This is probably quite hard. Note that we do not assume here any regularity of the boundary. Analogous problem for the real Monge-Ampère equation and not necessarily smooth convex domains has an affirmative answer. The main ingredient is an interior C^2-estimate of Pogorelov [53]. A complex version of this estimate is not known despite some attempts (see [13]).

Several important generalizations of Theorem 3.3 are due to Kołodziej. In [46] he showed that it holds for nonnegative $F \in L^p(\Omega)$ for some $p > 1$. The key is the following estimate:

Theorem 3.9 ([46])**.** *Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$. Then for smooth psh u vanishing on $\partial\mathbb{B}$ and $p > 1$ one has*

$$||u||_{L^\infty(\mathbb{B})} \leq C\big|\big| \det(u_{j\bar{k}})\big|\big|_{L^p(\mathbb{B})}^{1/n},$$

where C depends only on n and p.

For $p = 2$ it was earlier proved by Cheng and Yau (see [1], p. 75, and [22]) using the real Monge-Ampère operator. For arbitrary p Kołodziej's proof is much more complicated. It would be interesting to find a simpler PDE proof of Theorem 3.9.

Problem 4. *Is the optimal constant in Theorem 3.9 attained for radially symmetric functions?*

In fact, for such functions the estimate is rather simple, see [52].

Another interesting result of Kołodziej is the following:

Theorem 3.10 ([45])**.** (3.1) *has a bounded solution provided that it has a bounded subsolution.*

Note that this result is a generalization of Theorem 3.3.

Problem 5. *Does a continuous subsolution imply a continuous solution?*

A psh function u is called *maximal* in a domain Ω if for any other psh function v in Ω such that $v \leq u$ away from a compact subset of Ω we have $v \leq u$ in Ω. For $n = 1$ maximal psh functions are precisely harmonic ones but in higher dimensions they may be completely irregular: for example psh functions independent of one variable are maximal.

From Theorems 3.1 and 3.3 we easily infer the following:

Theorem 3.11. *A locally bounded psh function u is maximal iff $(dd^c u)^n = 0$.*

A similar characterization can be proved for functions in $\mathcal{D}$, see [14]. This implies in particular that maximality in this class is a local notion.

Problem 6. *Is maximality a local notion for arbitrary psh function?*

4. Extremal functions

The *relative* (or *Bedford-Taylor*) capacity is defined as follows:

$$c(E,\Omega) = \sup\{\int_E (dd^c u)^n \colon u \in PSH(\Omega),\ -1 \le u \le 0\}.$$

Here Ω is a bounded domain in $\mathbb{C}^n$ and E a Borel subset of Ω. One of the key results is quasicontinuity of psh functions:

Theorem 4.1 ([5])**.** *If u is psh in Ω then for every $\varepsilon > 0$ there exists open $G \subset \Omega$ such that $c(G,\Omega) < \varepsilon$ and u restricted to $\Omega \setminus G$ is continuous.*

Using this one can for example obtain a counterpart of Theorem 1.6 for sequences increasing almost everywhere.

Closely related to the relative capacity is the *relative extremal function*:

$$u_{E,\Omega} := \sup\{v \in PSH^-(\Omega) \colon v|_E \le -1\}.$$

It turns out that the supremum in the definition of capacity is essentially attained for this function:

Theorem 4.2 ([7])**.** *Assume that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ and K is compact subset of Ω. Then*

$$c(K,\Omega) = \int_K (dd^c u^*_{K,\Omega})^n.$$

Exercise 6. *Denote $B_r = B(0,r)$. Show that for $r < R$*

$$u_{\bar{B}_r,B_R} = \max\left\{\frac{\log|z| - \log R}{\log R - \log r}, -1\right\}$$

and

$$c(\bar{B}_r, B_R) = \left(\frac{2\pi}{\log R - \log r}\right)^n.$$

Theorem 4.2 was used in [7] to prove the following:

Theorem 4.3. *Assume that $P \Subset \Omega$ where Ω is a bounded hyperconvex domain in $\mathbb{C}^n$. Then the following are equivalent*

i) $P \subset \{u = -\infty\}$ for some u psh in Ω;

ii) $c(P,\Omega) = 0$.

A set $P \subset \mathbb{C}^n$ is called *locally pluripolar* if for every $z \in P$ there exists a naighborhood U and u psh in U such that $P \cap U \subset \{u = -\infty\}$ and *globally pluripolar* if $P \subset \{u = -\infty\}$ for some u psh in $\mathbb{C}^n$. For a family of psh function $\{u_\alpha\}$ in Ω locally uniformly bounded from above the sets of the form $\{u < u^*\}$, where $u = \sup_\alpha u_\alpha$, are called *negligible*.

Theorem 4.3 was used in [5] to solve two problems posed by Lelong [50]:

Theorem 4.4. *Locally pluripolar sets are globally pluripolar.*

Theorem 4.5. *Negligible sets are pluripolar.*

Theorem 4.4 is originally due to Josefson [38] who did not use the complex Monge-Ampère operator.

Global extremal function or *Siciak extremal function* for a bounded subset E of $\mathbb{C}^n$ is defined by

$$V_E := \sup\{u \in \mathcal{L} \colon u|_E \le 0\},$$

where

$$\mathcal{L} := \{u \in PSH(\mathbb{C}^n) \colon u \le \log_+ |z| + C \text{ for some constant } C\}$$

is the class of entire psh functions with logarithmic growth. One can show that $V_E^* \in \mathcal{L}$ iff E is not pluripolar. One of the crucial results is due to Zakharyuta [59] who proved that this definition agrees with the original one of Siciak [56]:

Theorem 4.6. *For a compact $K \subset \mathbb{C}^n$ we have*

$$V_K = \sup\{\frac{1}{d} \log |P| \colon P \text{ is a polynomial of degree } \le d \text{ such that } |P| \le 1 \text{ on } K\}.$$

Proof. We follow Demailly [26]. We clearly have $\ge$. Fix $z_0 \in \mathbb{C}^n$ and $b < a < V_K(z_0)$. We can find $v \in \mathcal{L}$ with $v \le 0$ on K and $v(z_0) > a$. Replacing v with $v * \rho_\varepsilon - \delta$ for appropriate ε and δ we may assume that $v \in \mathcal{L} \cap C^\infty$, $v < 0$ on K and $v > a$ on $\bar{B}(z_0, r)$ for some $r > 0$. We need to find $d \gg 0$ and a polynomial P of degree $\le d$ such that $|P| \le 1$ on K and $\frac{1}{d} \log |P(z_0)| \ge b$.

Take $\chi \in C_0^\infty(B(z_0, r))$ such that $\chi = 1$ in $B(z_0, r/2)$. Define a weight

$$\varphi := 2dv + 2n \log |z - z_0| + \log(1 + |z|^2)$$

so that

$$i\partial\bar{\partial}\varphi \ge \frac{1}{(1 + |z|^2)^2} i\partial\bar{\partial}|z|^2.$$

By Hörmander's theorem [37] we can find continuous u with $\bar{\partial}u = \bar{\partial}\chi$ and

$$\int_{\mathbb{C}^n} |u|^2 e^{-\varphi} d\lambda \le \int_{B(z_0,r)\setminus B(z_0,r/2)} |\bar{\partial}\chi|^2 (1 + |z|^2)^2 e^{-\varphi} d\lambda.$$

Therefore

$$\int_{\mathbb{C}^n} |u|^2 (1 + |z|^2)^{-1} |z - z_0|^{-2n} e^{-2dv} d\lambda \le C_1 e^{-2da},$$

where C_1 is independent of d. Since $|z - z_0|^{-2n}$ is not locally integrable near z_0, we see that $u(z_0) = 0$. The function $f = \chi - u$ is holomorphic in $\mathbb{C}^n$, $f(z_0) = 1$ and

$$\int_{\mathbb{C}^n} |f|^2 (1 + |z|^2)^{-n-1} e^{-2dv} d\lambda \le C_2 e^{-2da}, \tag{4.1}$$

where C_2 is also independent of d. Since $v \le \log_+ |z| + C_3$ we get in particular

$$\int_{\mathbb{C}^n} |f|^2 (1+|z|^2)^{-n-1-d} d\lambda < \infty$$

which implies that f is a polynomial of degree at most $d-1$. Using the fact that $v \le 0$ in a neighborhood of K and subharmonicity of $|f|^2$ from (4.1) we also get that $|f|^2 \le C_4 e^{-2da}$ on K, where C_4 is again independent of d (but might depend on the fixed neighborhood of K). Then $P = C_4^{-1/2} e^{da} f$ is a polynomial of degree at most $d-1$, $|P| \le 1$ on K and $\frac{1}{d} \log |P(z_0)| = a - \frac{\log C_4}{2d} \ge b$ for d sufficiently big. $\square$

Pluricomplex Green function for a domain Ω in $\mathbb{C}^n$ with pole at $w \in \Omega$ is defined by

$$G_{\Omega,w} = G_\Omega(\cdot, w) = \sup \mathcal{B}_{\Omega,w},$$

where

$$\mathcal{B}_{\Omega,w} = \{u \in PSH^-(\Omega) \colon u \le \log|z| + C \text{ for some constant } C\}.$$

This definition was originally given in [42] (and independently in a more general form in [60]).

Exercise 7. *Show that* $G_{B(w,r),w} = \log \frac{|z-w|}{r}$.

Exercise 8. *Let* $\Omega := \{z \in \mathbb{C}^2 \colon |z_1 z_2| < 1\}$. *Show that*

$$G_\Omega(z,w) = \begin{cases} \log \left| \frac{z_1 z_2 - w_1 w_2}{1 - \bar{w}_1 \bar{w}_2 z_1 z_2} \right| & w \neq 0, \\ \frac{1}{2} \log |z_1 z_2| & w = 0. \end{cases}$$

The above example is due to Klimek [43]. It shows in particular that G_Ω need not be symmetric. The first domain with this property was constructed by Bedford and Demailly [2].

One can easily show that if Ω is bounded then $G_{\Omega,w} \in \mathcal{B}_{\Omega,w}$. The basic results for pluricomplex Green function were proved by Demailly [25] who in particular essentially obtained the following (see also [12]):

Theorem 4.7. *If Ω is bounded then* $(dd^c G_{\Omega,w})^n = (2\pi)^n \delta_w$.

If Ω is hyperconvex then it is easy to show that $G_{\Omega,w} = 0$ on $\partial\Omega$. Demailly [25] proved more (here we define $G_\Omega(z,w) = 0$ for $z \in \partial\Omega$, $w \in \Omega$):

Theorem 4.8. *If Ω is bounded and hyperconvex then G_Ω is continuous on $\bar{\Omega} \times \Omega \setminus \Delta$ (where Δ is the diagonal).*

Continuity on $\bar{\Omega} \times \bar{\Omega} \setminus \Delta$ is still an open problem. Equivalently, we can formulate this as follows:

Problem 7. *For bounded hyperconvex Ω, does $G_{\Omega,w}$ converge to 0 locally uniformly in Ω as $w \to \partial\Omega$?*

Herbort [36] showed that this is indeed the case if we assume in addition that $\partial\Omega$ is of class C^2 (see also [28] and [15] for a simplified proof).

In a general situation one can easily show a slightly weaker result:

Theorem 4.9 ([17]). *For bounded, hyperconvex Ω and $p < \infty$ we have*

$$\lim_{w\to\partial\Omega} ||G_{\Omega,w}||_{L^p(\Omega)} = 0.$$

Theorem 4.9 will easily follow from the following inequality which can be obtained by successive integrations by parts:

Proposition 4.10 ([8]). *Let u, v be nonpositive psh functions in bounded Ω such that $v = 0$ on $\partial\Omega$ and v is locally bounded. Then*

$$\int_\Omega |v|^n (dd^c u)^n \leq n! ||u||^{n-1}_{L^\infty(\Omega)} \int_\Omega |u| (dd^c v)^n.$$

Proof of Theorem 4.9. By Theorem 4.7 and Proposition 4.10 we get

$$||G_{\Omega,w}||^n_{L^n(\Omega)} \leq (2\pi)^n n! ||u_\Omega||^{n-1}_{L^\infty(\Omega)} |u_\Omega(w)|, \tag{4.2}$$

where u_Ω is given by Corollary 3.8. This gives Theorem 4.9 for $p = n$ and the general case is left as an exercise to the reader. □

Finally, the following regularity of the Green function is known:

Theorem 4.11 ([11], [10], [33]). *If Ω is strongly pseudoconvex with $C^{2,1}$ boundary then $G_{\Omega,w}$ is $C^{1,1}$ in $\bar{\Omega} \setminus \{w\}$.*

Bedford and Demailly [2] constructed a strongly pseudoconvex domain with C^∞ boundary whose Green function is not C^2 up to the boundary (this example heavily relies on a result from [3]).

5. Applications to the Bergman kernel

Recall that the Bergman metric on bounded Ω in $\mathbb{C}^n$ is the Kähler metric with potential $\log K_\Omega(z, z)$. We say that Ω is *Bergman complete* if it is complete w.r.t. this metric. The basic result is the following:

Theorem 5.1 ([17], [35]). *Hyperconvex domains are Bergman complete.*

The prove Theorem 5.1 one uses the following criterion of Kobayashi [44]: Ω is Bergman complete if

$$\lim_{w\to\partial\Omega} \frac{|f(w)|^2}{K_\Omega(w,w)} = 0, \quad f \in \mathcal{O} \cap L^2(\Omega). \tag{5.1}$$

To prove this he used the embedding

$$\Omega \ni w \longmapsto [K_\Omega(\cdot, w)] \in \mathbb{P}(\mathcal{O} \cap L^2(\Omega)). \tag{5.2}$$

The main observation is that the pull-back of the Fubini-Study metric in $\mathbb{P}(\mathcal{O} \cap L^2(\Omega))$ is the Bergman metric in Ω.

Zwonek [61] showed that (5.1) is not necessary for Bergman completeness - he found an example of a bounded domain in $\mathbb{C}$ which is Bergman complete but (5.1) does not hold. This condition however can be slightly relaxed: it was shown in [15] that if a bounded domain Ω in $\mathbb{C}^n$ satisfies

$$\limsup_{w\to\partial\Omega} \frac{|f(w)|^2}{K_\Omega(w,w)} < ||f||^2_{L^2(\Omega)}, \quad f \in \mathcal{O} \cap L^2(\Omega) \tag{5.3}$$

then it is Bergman complete.

Problem 8. *Does Bergman completeness imply* (5.3)*?*

The main step in the proof of Theorem 5.1 will be the following estimate of Herbort [35] (see also [23]):

Theorem 5.2. *For a pseudoconvex Ω, $w \in \Omega$ and $f \in \mathcal{O} \cap L^2(\Omega)$ one has*

$$\frac{|f(w)|^2}{K_\Omega(w,w)} \le c_n \int\limits_{\{G_{\Omega,w}<-1\}} |f|^2 d\lambda.$$

By Theorem 4.9 for hyperconvex Ω we have

$$\lim_{w\to\partial\Omega} \lambda(\{G_{\Omega,w} < -1\}) = 0.$$

Therefore Theorem 5.1 immediately follows from Theorem 5.2 and Kobayashi's criterion (5.1). Also note that setting $f \equiv 1$ in Theorem 5.2 we obtain

$$K_\Omega(w,w) \ge \frac{1}{c_n \lambda(\{G_{\Omega,w} < -1\})} \ge \frac{1}{C(n, \operatorname{diam}\Omega)|u_\Omega(w)|},$$

where the last inequality follows from (4.2). This gives a quantitative version of the following results of Ohsawa [51]:

Theorem 5.3. *For a bounded hyperconvex Ω one has*

$$\lim_{w\to\partial\Omega} K_\Omega(w,w) = \infty.$$

Proof of Theorem 5.2. Denoting $G := G_{\Omega,w}$ define

$$\alpha := \bar\partial(f\,\gamma \circ G) = f\,\gamma' \circ G\,\bar\partial G \in L^2_{loc,(0,1)}(\Omega),$$

where $\gamma \in C^\infty(\mathbb{R})$ is such that $\gamma(t) = 1$ for $t \le -2$, $\gamma(t) = 0$ for $t \ge -1$ and $|\gamma'| \le 2$. For

$$\varphi := 2nG + e^G - 1$$

we have

$$i\bar\alpha\wedge\alpha=|f|^2(\gamma'\circ G)^2 i\partial G\wedge\bar\partial G\le |f|^2(\gamma'\circ G)^2e^{-G}i\partial\bar\partial\varphi$$

which we may with some abuse of notations write

$$|\alpha|^2_{i\partial\bar\partial\varphi}\le |f|^2(\gamma'\circ G)^2e^{-G}\le 36\,\chi_{\{-2<G<-1\}}|f|^2$$

(see [15] or [12]). From Hörmander's estimate for $\bar\partial$ we obtain $u\in L^2_{loc}(\Omega)$ solving $\bar\partial u=\alpha$ and such that

$$\int_\Omega |u|^2e^{-\varphi}d\lambda\le 36\int_{\{-2<G<-1\}}|f|^2e^{-\varphi}d\lambda.$$

Since $\varphi<0$ in Ω and $\varphi\ge -4n-1$ on $\{G>-2\}$, we will get

$$||u||_{L^2(\Omega)}\le 6e^{2n+1}||f||_{L^2(\{G<-1\})}.$$

The function $\widetilde f:=f\,\gamma\circ G-u$ is holomorphic. Moreover, since $e^{-\varphi}$ is not locally integrable near w, it follows that $\widetilde f(w)=f(w)$. We also have

$$||\widetilde f||_{L^2(\Omega)}\le (1+6e^{2n+1})||f||_{L^2(\{G<-1\})}$$

and the desired estimate follows. □

Bergman completeness for bounded domains Ω is equivalent to the fact that the distance given by the Bergman metric, which we denote by $\operatorname{dist}_\Omega(z,w)$, converges to ∞ as $z\to\partial\Omega$ and w stays fixed. Theorem 5.1 implies that this is the case for hyperconvex domains but the method does not give any quantitative estimate from below for this distance. This was done in [29] and improved in [15] for sufficiently smooth domains:

Theorem 5.4. *Assume that Ω is a bounded pseudoconvex domains with C^2 boundary. Then for a fixed $w\in\Omega$ we have*

$$\operatorname{dist}_\Omega(z,w)\ge\frac{\log(1/\delta_\Omega(z))}{C\log\log(1/\delta_\Omega(z))}$$

where δ_Ω is the euclidean distance to the boundary and C is independent of $z\in\Omega$.

Since the embedding (5.2) is distance decreasing, one can show that

$$\operatorname{dist}_\Omega(z,w)\ge\arccos\frac{|K_\Omega(z,w)|}{\sqrt{K_\Omega(z,z)}\sqrt{K_\Omega(w,w)}}$$

(see [15]). This together with Hörmander's estimate for $\bar\partial$ can be used to prove the following relation between the Bergman distance and the pluricomplex Green function:

Theorem 5.5 ([15])**.** *Assume that Ω is bounded and psudoconvex. Then for $z,w\in\Omega$ with $\{G_{\Omega,z}<-1\}\cap\{G_{\Omega,w}<-1\}=\emptyset$ we have* $\operatorname{dist}_\Omega(z,w)\ge c_n>0$.

Then the proof of Theorem 5.4 boils down to uniform estimates for the Green function, see [15].

Problem 9. *Can the estimate in Theorem 5.4 be improved to*

$$\operatorname{dist}_\Omega(z,w) \geq \frac{1}{C}\log(1/\delta_\Omega(z)) \ ?$$

Such an estimate would be optimal. It is known to hold for strongly pseudoconvex domains as well as for convex ones (see [15]).

For $z \in \Omega$ and $X \in \mathbb{C}^n$ by $B_\Omega(z;X)$ denote the Levi form of $\log K_\Omega$, that is

$$B_\Omega(z,X) = \frac{\partial^2}{\partial\lambda\partial\overline{\lambda}} \log K_\Omega(z+\lambda X, z+\lambda X)\Big|_{\lambda=0}.$$

Problem 10. *Is it true that for any bounded B-regular* Ω *and fixed* $X \neq 0$ *one has*

$$\lim_{z\to\partial\Omega} B_\Omega(z;X) = \infty \ ? \tag{5.4}$$

It would be a counterpart of Theorem 5.1 which really says that for hyperconvex domains the Bergman distance goes to ∞ at the boundary. Diederich and Herbort [28] showed that (5.4) holds under additional assumption that $\partial\Omega$ is C^2 smooth. Positive answer to Problem 7 for B-regular domains would give

$$\lim_{w\to w_0} \operatorname{diam}\left(\{G_{\Omega,w} < -1\}\right) = 0, \quad w_0 \in \partial\Omega$$

and this implies a positive answer to Problem 10 (see [28]).

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