SEVERAL COMPLEX VARIABLES
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Contents

Introduction

Our goal is two-fold: to present the most classical results of complex analysis in several variables as well as the most recent developments in the area. The central point is the Hörmander estimate for the $\bar\partial$-equation, Theorem 7.1 below. It is the principal tool for constructing holomorphic functions of several variables and it seems that essentially every problem of this kind can be solved using this estimate more or less directly.

In Sections 1-4 we present basic results on subharmonic and plurisubharmonic functions, holomorphic functions of several variables, domains of holomorphy and pseudoconvex sets in $\mathbb{C}^n$. We mostly follow Hörmander's books [36] and [37]. We assume that the reader is familiar with holomorphic and harmonic functions of one complex variable. One less classical ingredient is the pluricomplex Green function which is an important tool later on but can also be used to give a simple proof of the fact that euclidean balls are not biholomorphic to polydiscs. Section 5 is an introduction to the Bergman kernel and metric. The main result, Theorem 5.5, is a criterion due to Kobayashi [47] for completeness with respect to the Bergman metric.

The Hörmander estimate is first proved in dimension one in Section 6. The proof is simpler than in higher dimensions but gives a good idea of the general method. We follow the expositions of Hörmander [37] and Berndtsson [2, 5]. The Hörmander estimate can be quite useful already in dimension one, as an example we prove a result of Chen [21] which simplifies the Kobayashi criterion for Bergman completeness in this case. In Section 7 we prove the Hörmander estimate in arbitrary dimension following [36] and [37].

Sections 8-11 contain various more or less direct consequences of the Hörmander estimate. First, we give a solution of the classical Levi problem, originally solved independently by Oka [55], Bremermann [18] and Norguet [53]. Following Berndtsson [4] we show that other estimates for $\bar\partial$ due to Donnelly-Fefferman [28] and Berndtsson [3] are formal consequences of the Hörmander estimate. We then prove a pluripotential criterion for Bergman completeness, Theorem 8.5, due to Chen [20] and Herbort [34] (see also [15]). We also establish a lower bound for the Bergman kernel in terms of the Green function from [13] and deduce the one-dimensional Suita conjecture [60], originally shown in [12].

Another big topic is the Ohsawa-Takegoshi extension theorem. In Section 9 we present the proof of this important result recently proposed by Chen [22] (see also [11]) who was the first to notice that it can be deduced directly from the Hörmander estimate. In Section 10 we discuss applications of the Ohsawa-Takegoshi theorem for singularities of plurisubharmonic functions: the openness conjecture of Demailly-Kollár [26] recently established by Berndtsson [6] and the Demailly approximation [25] of plurisubharmonic functions. The latter can be used to give a simple proof of the Siu theorem [59] on analyticity of level sets of Lelong numbers.

Finally, in Section 11 we discuss recent approach of Nazarov [52] to the Mahler conjecture [50] and the Bourgain-Milman inequality [17] from convex analysis using several complex variables, in particular the Hörmander estimate. In this section the main tool is the Fourier-Laplace transform, in particular the Parseval formula and the Paley-Wiener theorem (they can be found for example in Chapters 7.1 and 7.3 in [38]).

1. Subharmonic Functions

Let $\Omega \subset \mathbb{C}$ be open. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called *subharmonic* if it is upper semi-continuous (usc), $u \not\equiv -\infty$ on every component of Ω and for any domain $D \Subset \Omega$ and $h \in \mathcal{H}(D) \cap C(\bar{D})$ such that $u \le h$ on ∂D we have $u \le h$ in D. The set of subharmonic functions in Ω will be denoted by $SH(\Omega)$.

Proposition 1.1. *Let u be subharmonic in a neighbourhood of $\bar{\Delta}(z_0, r)$. Then*

$$u(z) \le \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z - z_0|^2}{|z - z_0 - re^{it}|^2} \, u(z_0 + re^{it}) \, dt, \quad z \in \Delta(z_0, r).$$

Proof. Let φ_n be a sequence of continuous functions decreasing to u on $\partial\Delta(z_0, r)$. Solving the Dirchlet problem with this data we will find $h_n \in \mathcal{H}(\Delta(z_0, r)) \cap C(\bar{\Delta}(z_0, r))$ such that $h_n = \varphi_n$ on $\partial\Delta(z_0, r)$. By the definition we then have

$$u(z) \le h_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z - z_0|^2}{|z - z_0 - re^{it}|^2} \, \varphi_n(z_0 + re^{it}) \, dt, \quad z \in \Delta(z_0, r),$$

and it is enough to let $n \to \infty$. □

Theorem 1.2. *Assume that u is usc on a domain $\Omega \subset \mathbb{C}$ and $u \not\equiv -\infty$. Then u is subharmonic if and only if for every $z_0 \in \Omega$ there exists $r_0 > 0$ such that $\bar{\Delta}(z_0, r_0) \subset \Omega$ and we have the mean-value inequality*

$$u(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it}) dt, \quad 0 < r \le r_0. \tag{1.1}$$

In particular, subharmonicity is a local condition.

Proof. If u is subharmonic then (1.1) follows from Proposition 1.1. In order to show the converse take $D \Subset \Omega$ and $h \in \mathcal{H}(D) \cap C(\bar{D})$ with $u \le h$ on ∂D. If $\{u > h\} \ne \emptyset$ then $u - h$ attains maximum at some $z_0 \in D$, since u is usc. Using (1.1) one can show that the set $\{u - h = u(z_0) - h(z_0)\}$ contains all circles $\partial\Delta(z_0, r)$ such that $\bar{\Delta}(z_0, r) \subset \Omega$, and therefore is open. Since it is also closed (it is of the form $\{u - h \ge \mathit{const}\}$), it follows that $u - h = \mathit{const} > 0$ in D which contradicts the boundary condition. □

Proposition 1.1 and the proof of Theorem 1.2 immediately give the maximum principle for subharmonic functions:

Theorem 1.3. *If $u \in SH(\Omega)$ attains maximum in a domain Ω then u is constant.* □

For a real-valued function u defined on an open $\Omega \subset \mathbb{C}$ we set

$$u^*(z) := \limsup_{\zeta \to z} u(\zeta), \quad z \in \bar{\Omega}.$$

Then u^*, defined in $\bar{\Omega}$, is the smallest usc function which is $\ge u$ in Ω.

The following basic properties of subharmonic functions follow easily from the previous results:

Proposition 1.4. *(i)* $\mathcal{H}(\Omega) \subset SH(\Omega)$;

(ii) $u, v \in SH(\Omega),\ \alpha \geq 0 \ \Rightarrow\ \max\{u, v\},\ u+v,\ \alpha u \in SH(\Omega)$;

(iii) If $\bar{\Delta}(z_0, r) \subset \Omega$ *and* $u \in SH(\Omega)$ *then*

$$u(z_0) \leq \frac{1}{\pi r^2} \iint_{\Delta(z_0,r)} u\, d\lambda;$$

(iv) $SH(\Omega) \subset L^1_{loc}(\Omega)$;

(v) If $u_n \in SH(\Omega)$ *is a non-increasing sequence converging tu* u *on a domain* Ω *then either* $u \in SH(\Omega)$ *or* $u =\equiv -\infty$. □

Here are some further properties of subharmonic functions:

Theorem 1.5. *(i) If* $u \in SH(\Omega)$ *then* $\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})dt$ *is non-decreasing for* r *with* $\bar{\Delta}(z_0, r) \subset \Omega$ *and converges to* $u(z_0)$ *as* $r \to 0$;

(ii) If u *is subharmonic in the annulus* $\{r < |z - z_0| < R\}$ *then* $\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{it})dt$ *and* $\max_{|z-z_0|=\rho} u(z)$ *are logarithmically convex functions of* $\rho \in (r, R)$ *(that is convex w.r.t.* $\log\rho$*)*;

(iii) If $u \in SH(\Omega)$ *and* χ *is a convex non-decreasing function defined on an interval containing the image of* u *then* $\chi \circ u \in SH(\Omega)$;

(iv) $f \in \mathcal{O}(\Omega)$, $f \not\equiv 0$ *on every component of* Ω, $\alpha \geq 0 \ \Rightarrow\ \log|f|,\ |f|^\alpha \in SH(\Omega)$;

(v) For a non-empty family $\mathcal{F} \subset SH(\Omega)$*, locally uniformly bounded above, we have* $(\sup \mathcal{F})^* \in SH(\Omega)$.

Proof. (i) Assume that $r < R$ and let φ_n be a sequence of continuous functions on $\partial\Delta(z_0, R)$ decreasing to u there. We can find $h_n \in \mathcal{H}(\Delta(z_0, R)) \cap C(\bar{\Delta}(z_0, R))$ such that $h_n = \varphi_n$ on $\partial\Delta(z_0, R)$. Then $u \leq h_n$ in $\bar{\Delta}(z_0, R)$ and

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})dt \leq \frac{1}{2\pi}\int_0^{2\pi} h_n(z_0 + re^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} \varphi_n(z_0 + Re^{it})dt$$

and letting $n \to \infty$ we get monotonicity. The upper semi-continuity of u implies convergence to $u(z_0)$ as $r \to 0$.

(ii) Let us first prove the second statement. Set $M_\rho := \max_{|z-z_0|=\rho} u(z)$ and assume that $r < \rho_1 < \rho_2 < R$. The function

$$h(z) = M_{\rho_1} + \frac{M_{\rho_2} - M_{\rho_1}}{\log\rho_2 - \log\rho_1}(\log|z - z_0| - \log\rho_1)$$

is harmonic away from z_0 and such that $u \leq h$ on the boundary of $P := \{\rho_1 < |z - z_0| < \rho_2\}$. Since $u \leq h$ on $\bar{P}$,

$$M_\rho \leq M_{\rho_1} + \frac{M_{\rho_2} - M_{\rho_1}}{\log\rho_2 - \log\rho_1}(\log\rho - \log\rho_1)$$

if $\rho_1 \leq \rho \leq \rho_2$, that is that M_ρ is logarithmically convex.

To show the second statement let $\varphi_n \in C(\partial P)$ be a sequence decreasing to u on ∂P and $h_n \in \mathcal{H}(P) \cap C(\bar{P})$ is such that $h_n = \varphi_n$ on ∂P. It now follows easily from the fact that $u \leq h_n$ and that the corresponding mean-value for harmonic functions is logarithmically linear. (The latter follows from the fact that every harmonic function in an annulus centered at z_0 is of the form $\operatorname{Re} f + C\log|z-z_0|$ where f is holomorphic.)

(iii) The function $\chi\circ u$ is usc. For the disk $\bar\Delta(z_0,r)$ in Ω we have

$$\chi(u(z_0)) \leq \chi\left(\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})dt\right) \leq \frac{1}{2\pi}\int_0^{2\pi}\chi(u(z_0+re^{it}))dt,$$

where the first inequality follows since χ is non-decreasing and the second one since χ is convex (by the Jensen inequality).

(iv) The function $\log|f|$ is harmonic on the set $\{f\neq 0\}$ and $=-\infty$ if $f=0$, it is thus enough to use Theorem 1.2. We also have $|f|^\alpha = \chi(\log|f|)$, where $\chi(t)=e^{\alpha t}$, and we use (iii).

(v) For the disk $\bar\Delta(z_0,r)\subset\Omega$ by Proposition 1.1

$$v(z) \leq \frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-|z-z_0|^2}{|z-z_0-re^{it}|^2}\,v(z_0+re^{it})\,dt, \quad v\in\mathcal{F}.$$

Set $u:=\sup\mathcal{F}$. Then

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-|z-z_0|^2}{|z-z_0-re^{it}|^2}\,u(z_0+re^{it})\,dt.$$

By the Fatou lemma

$$u^*(z_0) = \limsup_{z\to z_0} u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} u^*(z_0+re^{it})\,dt. \qquad \square$$

Proposition 1.6. *Assume that $u\in C^2(\Omega)$. Then $u\in SH(\Omega)$ if and only if $\Delta u\geq 0$.*

Proof. Suppose u is subharmonic and $\Delta u<0$ in a disk $\Delta(z_0,r)\Subset\Omega$. Let $h\in\mathcal{H}(\Delta(z_0,r))\cap C(\bar\Delta(z_0,r))$ be such that $h=u$ on $\partial\Delta(z_0,r)$. Then $v:=u-h\in SH(\Delta(z_0,r))\cap C(\bar\Delta(z_0,r))$ has a local minimum in $\Delta(z_0,r)$ and thus $\Delta u=\Delta v\geq 0$ at this point, a contradiction.

On the other hand suppose that $\Delta u\geq 0$. Considering $u+\varepsilon|z|^2$ instead we may assume that $\Delta u>0$. Take $D\Subset\Omega$ and $h\in\mathcal{H}(D)\cap C(\bar D)$ such that $u\leq h$ on ∂D. If there exists $z\in D$ such that $u-h$ has a maximum at z then $\Delta(u-h)\leq 0$ at z, a contradiction. It follows that $u\leq h$ in D and thus u is subharmonic. $\square$

Let $\rho\in C_0^\infty(\mathbb{C})$ be radially symmetric (that is $\rho(z)$ depends only on $|z|$), non-negative and such that $\operatorname{supp}\rho=\bar\Delta$ and $\iint_{\mathbb{C}}\rho d\lambda=1$. For $\varepsilon>0$ we set $\rho_\varepsilon(z):=\varepsilon^{-2}\rho(z/\varepsilon)$, so that $\operatorname{supp}\rho_\varepsilon=\bar\Delta(0,\varepsilon)$ and $\iint_{\mathbb{C}}\rho_\varepsilon d\lambda=1$. In particular, $\rho_\varepsilon\to\delta_0$ weakly as $\varepsilon\to 0$.

Theorem 1.7. *For $u\in SH(\Omega)$ set*

$$u_\varepsilon(z) := (u*\rho_\varepsilon)(z) = \iint_{\Delta(0,\varepsilon)} u(w)\rho_\varepsilon(z-w)d\lambda(w) = \iint_{\Delta} u(z-\varepsilon w)\rho(w)d\lambda(w).$$

Then u_ε is smooth, subharmonic in

$$\Omega_\varepsilon := \{z\in\Omega\colon \Delta(z,\varepsilon)\subset\Omega\},$$

and decreases to u as ε decreases to 0.

Proof. It is clear that $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$. By the Fubini theorem u_ε satisfies the mean-value inequality and thus it is subharmonic. To show monotonicity in ε we have to use the fact that ρ is radially symmetric:

$$u_\varepsilon(z) = \iint_\Delta u(z-\varepsilon w)\rho(w)d\lambda(w) = \int_0^1 r\rho(r)\int_0^{2\pi} u(z+\varepsilon re^{it})dt\,dr$$

and use the first part of Theorem 1.5.i. By the second part u_ε converges to u. □

Theorem 1.8. *Proposition 1.6 remains true if we merely assume that u is a distribution.*

Proof. It follows easily Proposition 1.4.iii that subharmonic functions are locally integrable, they can be therefore treated as distributions. Theorem 1.7 and the previous part imply that they have a non-negative Laplacian. If u is an arbitrary distribution with $\Delta u \geq 0$ then $\Delta(u*\rho_\varepsilon) = \Delta u * \rho_\varepsilon \geq 0$ and by the previous part $u*\rho_\varepsilon$ is subharmonic. By commutativity of convolution the expression $u*\rho_\varepsilon*\rho_\delta$ is monotone both in ε and δ. It follows that it decreases to $u*\rho_\delta$ as ε decreases to 0, and $u*\rho_\delta$ decreases to some u_0 as δ decreases to 0. Since $u*\rho_\delta$ also converges weakly to u, it follows that for every test function φ the integral $\int \varphi u*\rho_\delta$ is bounded, and thus u_0 must be locally integrable, hence subharmonic. By the Lebesgue bounded convergence theorem the convergence $u*\rho_\delta \to u_0$ is also weak, and thus $u_0 = u$. □

Theorem 1.8 can be treated as an alternative definition of subharmonic functions.

Proposition 1.9. *If $f \in \mathcal{O}(\Omega_1, \Omega_2)$, $f \neq const$ on any component of Ω_1, and $u \in SH(\Omega_2)$ then $u \circ f \in SH(\Omega_1)$.*

Proof. It easily follows from Proposition 1.6 if u is smooth and from Theorem 1.7 for arbitrary u. □

We have the following versions of the Riemann removable singularity and Liouville theorems for subharmonic functions:

Proposition 1.10. *Assume that $u \in SH(\Omega \setminus \{w\}$ is bounded above near w. Then u can be uniquely extended to a subharmonic function in Ω.*

Proof. The uniqueness follows from Theorem 1.5(i). For every $n \geq 1$ the function $u_n = u + \frac{1}{n}\log|z-w|$ clearly extends to a subharmonic function in Ω and near w $(sup_n u_n)^*$ is a subharmonic extension of u. □

Proposition 1.11. *Entire subharmonic functions which are bounded above are constant.*

Proof. Follows easily from Theorem 1.5(ii) and (i). □

The following lemma due to Hartogs will be crucial in the proof that separate holomorphic functions are holomorphic.

Lemma 1.12. *Let u_k be a sequence of subharmonic functions on a domain Ω in $\mathbb{C}$ locally uniformly bounded from above. Assume that*

$$\limsup_{k\to\infty} u_k(z) \leq C, \quad z \in \Omega.$$

Then for every $\varepsilon > 0$ and K compact in Ω one has

$$u_k(z) \leq C + \varepsilon, \quad z \in K,$$

for sufficiently big k.

Proof. Without loss of generality we may assume that $u_k \leq 0$ in Ω. Choose $r > 0$ such that $\Delta(z, 2r) \subset \Omega$ for $z \in K$. For $w \in K$ by the Fatou lemma we have

$$\limsup_{k\to\infty} \frac{1}{\pi r^2} \iint_{\Delta(w,r)} u_k d\lambda \leq C$$

and therefore we can find k_0 depending on w such that

$$\frac{1}{\pi r^2} \iint_{\Delta(w,r)} u_k d\lambda \leq C + \frac{\varepsilon}{2}, \quad k \geq k_0.$$

If $|z - w| < \delta < r$ then by the mean-value inequality and since u_k is negative

$$\begin{aligned} u_k(z) &\leq \frac{1}{\pi(r+\delta)^2} \iint_{\Delta(z,r+\delta)} u_k \, d\lambda \\ &\leq \frac{1}{\pi(r+\delta)^2} \iint_{\Delta(w,r)} u_k \, d\lambda \\ &\leq \left(C + \frac{\varepsilon}{2}\right) \frac{\pi r^2}{\pi(r+\delta)^2} \\ &\leq C + \varepsilon \end{aligned}$$

if δ is sufficiently small. The lemma now follows if we cover K with finite collection of disks with radius δ. $\square$

2. Holomorphic Functions of Several Variables

Let Ω be an open set in $\mathbb{C}^n$. A function $f:\Omega\to\mathbb{C}$ is called *holomorphic* if it is continuous and holomorphic with respect to every variable. We will later see that the continuity assumption is superfluous. The class of holomorphic functions in Ω will be denoted by $\mathcal{O}(\Omega)$.

If f is holomorphic in a neighbourhood of $\bar P$, where $P=P(w,r)=\Delta(w_1,r_1)\times\dots\times\Delta(w_n,r_n)$ is a polydisk centered at w with multi-radius $r=(r_1,\dots,r_n)$, then by the one-dimensional Cauchy formula

$$f(z)=\frac{1}{(2\pi i)^n}\int_{\partial\Delta(w_1,r_1)}\dots\int_{\partial\Delta(w_n,r_n)}\frac{f(\zeta_1,\dots,\zeta_n)}{(\zeta_1-z_1)\dots(\zeta_n-z_n)}d\zeta_n\dots d\zeta_1,\quad z\in P. \tag{2.1}$$

By the continuity of f the right-hand-side can be treated as an multi-dimensional integral over $\partial_S P=\partial\Delta(w_1,r_1)\times\dots\times\partial\Delta(w_n,r_n)$ (which is called the *Shilov boundary* of P), we can write it as

$$\partial_S f(z)=\frac{1}{(2\pi i)^n}\int_{\partial_S P}\frac{f(\zeta)}{\zeta-z}d\zeta,\quad z\in P. \tag{2.2}$$

We can also differentiate under the sign of integration. We see that in fact f must be C^∞ smooth and for $\alpha\in\mathbb{N}^n$

$$\partial^\alpha f(z)=\frac{\partial^{|\alpha|}f}{\partial z_1^{\alpha_1}\dots\partial z_n^{\alpha_n}}(z)=\frac{\alpha!}{(2\pi i)^n}\int_{\partial_S P}\frac{f(\zeta)}{(\zeta-z)^{\alpha+1}}d\zeta,\quad z\in P,$$

where $|\alpha|=\alpha_1+\dots+\alpha_n$, $\alpha!=\alpha_1!\dots\alpha_n!$, $\alpha+1=(\alpha_1+1,\dots,\alpha_n+1)$ and $z^\alpha=z_1^{\alpha_1}\dots z_n^{\alpha_n}$. Applying this in slightly shrinked polydisks gives the multidimensional Cauchy inequality:

Proposition 2.1. *If f is holomorphic in a polydisk $P(w,r)$ and $|f|\le M$ there then*

$$|\partial^\alpha f(w)|\le\frac{M\alpha!}{r^\alpha}. \qquad \square$$

If f is holomorphic in a polydisk P centered at w then from the corresponding one-dimensional fact it follows that its power series converges absolutely in P:

$$f(z)=\sum_{\alpha\in\mathbb{N}^n}\frac{\partial^\alpha f(w)}{\alpha!}(z-w)^\alpha,\quad z\in P.$$

The following proposition Coupled with the Cauchy inequality implies that the convergence is also locally uniform in P:

Proposition 2.2. *If for a multi-radius r one has $|a_\alpha|r^\alpha\le M<\infty$ for α with $|\alpha|$ is sufficiently big then the power series $\sum_\alpha a_\alpha z^\alpha$ converges absolutely and locally uniformly in the polydisk $P(0,r)$.*

Proof. Fix t with $0<t<1$ and let $z\in\bar P(0,tr)$. Since $|a_\alpha z^\alpha|\le Mt^{|\alpha|}$, $|\alpha|\gg 0$, for every $m\gg 0$ if $k_1,k_2\gg 0$ we have

$$|S_{k_1}(z)-S_{k_2}(z)|\le\sum_{\alpha_j\ge m}|a_\alpha z^\alpha|\le\frac{Mt^{nm}}{(1-t)^n},$$

where $S_k(z)$ denotes the sequence of partial sums of the series $\sum_\alpha a_\alpha z^\alpha$. $\square$

The following two results can be proved using the Cauchy formula (2.2) in the same way as in dimension 1:

Theorem 2.3. *If f_j is a sequence of holomorphic functions converging locally uniformly to f then f is holomorphic and $\partial^\alpha f_j \to \partial^\alpha f$ locally uniformly for every α.* □

Theorem 2.4. *If f_j is a locally uniformly bounded sequence of holomorphic functions then it has a subsequence converging locally uniformply.* □

In what follows we will prove two theorems of Hartogs. The first says that the assumption of continuity in the definition of a holomorphic function of several variables is superfluous.

Theorem 2.5. *If a function defined on an open subset of $\mathbb{C}^n$ is holomorphic with respect to every variable separately then it is holomorphic.*

Proof. The result is of course purely local. We first claim that it is enough to show that separately holomorphic functions are locally bounded. Indeed, we claim that if f is separately holomorphic in a neighbourhood of $(\bar{\Delta}_R)^n$ (where $\Delta_R = \Delta(0,R)$) with $|f| \leq M$ then

$$|f(z) - f(w)| \leq 2M \sum_{j=1}^{n} \frac{R|z_j - w_j|}{|R^2 - z_j \bar{w}_j|}, \quad z, w \in \Delta_R^n.$$

We have

$$f(z) - f(\zeta) = \sum_{j=1}^{n} (f(\zeta_1, \dots, \zeta_{j-1}, z_j, \dots, z_n) - f(\zeta_1, \dots, \zeta_j, z_{j+1}, \dots, z_n))$$

which reduces the estimate to $n = 1$. Then it easily follows from the Schwarz lemma. The estimate clearly implies that f is continuous.

To prove that a separately holomorphic function f is locally bounded we use the induction on n. Of course the result is true for $n = 1$ and we assume that it holds for $n-1$ variables. If $f(z) = f(z', z_n)$ is defined in a neighbourhood of $(\bar{\Delta}_R)^n$, by the inductive assumption the sets

$$\{z' \in \Delta_R^{n-1} : |f(z', z_n)| \leq M \text{ for } z_n \in \Delta_R\}$$

are closed Δ_R^{n-1}. Since their union is the whole Δ_R^{n-1}, by the Baire theorem they have non-empty interiors for large M. Slightly changing the polydisk if necessary we may thus assume that f is defined in Δ_R^n, holomorphic in z' and z_n separately, and bounded by M (and in particular holomorphic) in $\Delta_r^{n-1} \times \Delta_R$, where $0 < r < R$. We can write

$$f(z', z_n) = \sum_{\alpha \in \mathbb{N}^{n-1}} f_\alpha(z_n)(z')^\alpha, \tag{2.3}$$

where

$$f_\alpha(z_n) = \frac{\partial^\alpha f(0, z_n)}{\alpha!}$$

are holomorphic in Δ_R (because f is holomorphic in $\Delta_r^{n-1} \times \Delta_R$). For $R_1 < R$ and $z_n \in \Delta_R$ we have

$$\lim_{|\alpha|\to\infty} |f_\alpha(z_n)| R_1^{|\alpha|} = 0 \tag{2.4}$$

(the series (2.3) is absolutely convergent) and by the Cauchy inequality

$$|f_\alpha(z_n)| r^{|\alpha|} \leq M.$$

Therefore the family of subharmonic functions

$$u_\alpha = \frac{1}{|\alpha|} \log |f_\alpha|$$

(for $\alpha \neq 0$) is bounded from above by $\log M - \log r$ in Δ_R. By (2.4)

$$\limsup_{|\alpha|\to\infty} u_\alpha(z_n) \leq \log \frac{1}{R_1}, \quad z_n \in \Delta_R.$$

Therefore, if $0 < R_2 < R_1$ and $|\alpha|$ is sufficiently large from Lemma 1.12 we will get

$$u_\alpha(z_n) \leq \log \frac{1}{R_2}, \quad z_n \in \Delta_{R_2},$$

that is

$$|f_\alpha(z_n)| R_2^{|\alpha|} \leq 1, \quad |z_n| < R_2.$$

By Proposition 2.2 the series (2.3) converges locally uniformly in Δ_R^n and f is holomorphic there. □

Our next result is the Hartogs extension theorem:

Theorem 2.6. *Assume that Ω is a domain in $\mathbb{C}^n$, where $n > 1$, and K is a compact subset of Ω such that $\Omega \setminus K$ is connected. Then every holomorphic function in $\Omega \setminus K$ can be extended holomorphically to Ω.*

It is clearly false in dimension one.

The main tool in the proof will be a solution of the $\bar\partial$-equation in $\mathbb{C}^n$. For a $(0,1)$-form

$$\alpha = \sum_{k=1}^n \alpha_k d\bar z_k$$

we consider the inhomogeneous Cauchy-Riemann equation (or $\bar\partial$-equation)

$$\bar\partial u = \alpha. \tag{2.5}$$

Since

$$\bar\partial u = \sum_{k=1}^n \frac{\partial u}{\partial \bar z_k} d\bar z_k,$$

(2.5) is equivalent to the system of equations

$$\frac{\partial u}{\partial \bar z_k} = \alpha_k, \quad k = 1, \dots, n.$$

It is clear that for $u \in C^1$ (and in fact for any distribution u) the condition $\bar\partial u = 0$ is equivalent to u being holomorphic.

The $(0,2)$-form $\bar\partial\alpha$ is defined as

$$\bar\partial\alpha = \sum_{k=1}^{n} \bar\partial\alpha_k \wedge d\bar z_k = \sum_{j<k} \left(\frac{\partial\alpha_k}{\partial\bar z_j} - \frac{\partial\alpha_j}{\partial\bar z_k}\right) d\bar z_j \wedge d\bar z_k.$$

The necessary condition for solvability of (2.5) is

$$\bar\partial\alpha = 0,$$

that is

$$\frac{\partial\alpha_k}{\partial\bar z_j} = \frac{\partial\alpha_j}{\partial\bar z_k}, \quad j,k = 1,\dots,n.$$

Theorem 2.6 will easily follow from the following result:

Theorem 2.7. *Assume that $n > 1$. Then for every $\alpha \in C^\infty_{0,(0,1)}(\mathbb{C}^n)$ with $\bar\partial\alpha = 0$ there exists unique $u \in C^\infty_0(\mathbb{C}^n)$ solving (2.5).*

Proof. Uniqueness follows immediately from the identity principle for holomorphic functions. Recall the Green formula from dimension 1: if $\Omega \subset \mathbb{C}$ is bounded and has C^1 boundary then for $f \in C^1(\bar\Omega)$ and $z \in \Omega$ we have

$$2\pi i f(z) = \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta + \iint_\Omega \frac{\partial f/\partial\bar z(\zeta)}{\zeta - z} d\zeta \wedge d\bar\zeta.$$

Set

$$\begin{aligned} u(z) &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\alpha_1(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar\zeta \\ &= -\frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\alpha_1(z_1 - \zeta, z_2, \dots, z_n)}{\zeta} d\zeta \wedge d\bar\zeta. \end{aligned}$$

It is clear that $u \in C^\infty(\mathbb{C}^n)$. Differentiating the second integral and using the Green formula in a big disk we will get $\partial u/\partial\bar z_1 = \alpha_1$. For $k > 1$ we have

$$\begin{aligned} \frac{\partial u}{\partial\bar z_k}(z) &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial\alpha_1/\partial\bar z_k(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar\zeta \\ &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial\alpha_k/\partial\bar z_1(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar\zeta \\ &= \alpha_k(z) \end{aligned}$$

again by the Green formula. We thus have (2.5), so in particular u is holomorphic away from the support of α. From the definition of u it follows that the support of u is contained in $\mathbb{C} \times K$ where K is compact in $\mathbb{C}^{n-1}$, so by the identity principle for holomorphic functions the support of u must in fact be compact. $\square$

This theorem is also false in dimension one: then a necessary condition for the solution of $\partial u/\partial\bar z = f$ to have a compact support is $\iint_{\mathbb{C}} f\, d\lambda = 0$.

Proof of Theorem 2.6. Let $f \in \mathcal{O}(\Omega \setminus K)$ and let $\eta \in C^\infty_0(\Omega)$ be equal to 1 in a neighbourhood of K. Then $\alpha = -f\bar\partial\eta \in C^\infty_{0,(0,1)}(\mathbb{C}^n)$, so by Theorem 2.7 there exists $u \in C^\infty_0(\mathbb{C}^n)$ with $\bar\partial u = \alpha$. We set

$$F = (1 - \eta) f - u.$$

It is clear that F is holomorphic in Ω. Since u vanishes in the unbounded component of $\mathbb{C}^n \setminus \operatorname{supp} \eta$, it follows that $F = f$ in an open subset of $\Omega \setminus \operatorname{supp} \eta$ and thus also in $\Omega \setminus K$. $\square$

A mapping $F : \Omega \to \mathbb{C}^m$ is called holomorphic if its components are holomorphic functions. By $\mathcal{O}(\Omega, \Omega')$ we will denote the set holomorphic mappings whose range is contained in Ω'. A mapping $F \in \mathcal{O}(\Omega_1, \Omega_2)$ is called *biholomorphic* if it is bijective, holomorphic and F^{-1} is also holomorphic. Biholomorphic mappings $\Omega \to \Omega$ will be called *automorphisms* of Ω, their set will be denoted by $\operatorname{Aut}(\Omega)$.

The cases $n = 1$ and $n > 1$ are quite different: for example it turns out that in the latter a ball is not biholomorphic to a polydisk. This was originally proved by Poincaré who did it comparing the automorphism groups of both domains. We will show it later using the pluricomplex Green function.

Exercise 1. *Fix r with $0 \le r < 1$. Show that the mapping*
$$F(z', z_n) = \left(\frac{\sqrt{1 - r^2}}{1 - r z_n} z', \frac{z_n - r}{1 - r z_n} \right)$$
is an automorphism of the unit ball $\mathbb{B}$. Conclude that the group $\operatorname{Aut}(\mathbb{B})$ is transitive, that is for every $z, w \in \mathbb{B}$ there exists $F \in \operatorname{Aut}(\mathbb{B})$ such that $F(z) = w$.

3. Plurisubharmonic Functions and the Pluricomplex Green Function

Let Ω be an open subset of $\mathbb{C}^n$. A function $u : \Omega \to \mathbb{R}\cup\{-\infty\}$ is called *plurisubharmonic* (psh) if u is usc, $u \not\equiv -\infty$ on any connected component of Ω, and locally u is subharmonic or $u \equiv -\infty$ on every complex line intersected with Ω, that is for every $z \in \Omega$ and $X \in \mathbb{C}^n$ the function $\zeta \mapsto u(z + \zeta X)$ is subharmonic or $\equiv -\infty$ near the origin. The set of psh functions in Ω will be denoted by $PSH(\Omega)$.

Open Problem 1. *Does upper semi-continuity in the definition of psh functions follow from the other conditions?*

It would be enough to show that u is locally bounded. Wiegerinck [62] found an example of a separately subharmonic function which is not locally bounded.

Below we list basic properties of psh functions. They follow more or less directly from the definition and corresponding one-dimensional results, the details are left to the reader.

Theorem 3.1. *(i)* $u, v \in PSH(\Omega),\ \alpha \geq 0\ \Rightarrow\ \max\{u, v\},\ u + v,\ \alpha u \in PSH(\Omega)$*;*

(ii) Psh functions are subharmonic, that is they satisfy the mean-value inequality

$$u(z) \leq \frac{1}{\sigma(S(z,r))} \int_{S(z,r)} u\, d\sigma$$

if $\bar{B}(z,r) \subset \Omega$*;*

(iii) If a psh function attains maximum in a domain then it is constant;

(iv) If $u \in PSH(\Omega)$ *then* $\dfrac{1}{\sigma(S(z,r))} \displaystyle\int_{S(z,r)} u\, d\sigma$ *is non-decreasing for* r *with* $\bar{B}(z_0, r) \subset \Omega$ *and converges to* $u(z_0)$ *as* $r \to 0$*;*

(iv) If $\bar{B}(z,r) \subset \Omega$ *then*

$$u(z) \leq \frac{1}{\lambda(B(z,r))} \int_{B(z,r)} u\, d\lambda.$$

The right-hand side is non-decreasing in r *and converges to* $u(z_0)$ *as* $r \to 0$*;*

(v) If two psh functions are equal almost everywhere then they are equal;

(vi) $PSH(\Omega) \subset L^1_{loc}(\Omega)$*;*

(vii) If u *is psh in* $\{r < |z - z_0| < R\}$ *then* $\dfrac{1}{\sigma(S(z,r))} \displaystyle\int_{S(z,r)} u\, d\sigma$ *and* $\max\limits_{|z|=\rho} u(z)$ *are logarithmically convex for* $\rho \in (r, R)$*;*

(viii) If $u \in PSH(\Omega)$ *and* χ *is a convex non-decreasing function defined on an interval containing the image of* u *then* $\chi \circ u \in PSH(\Omega)$*;*

(ix) $f \in \mathcal{O}(\Omega)$*,* $f \not\equiv 0$ *on every component of* Ω*,* $\alpha \geq 0\ \Rightarrow\ \log|f|,\ |f|^\alpha \in PSH(\Omega)$*;*

(x) If $u_n \in PSH(\Omega)$ *is a non-increasing sequence converging tu* u *on a domain* Ω *then either* $u \in PSH(\Omega)$ *or* $u \equiv -\infty$*;*

(xi) For a non-empty family $\mathcal{F} \subset PSH(\Omega)$*, locally uniformly bounded above, we have* $(\sup \mathcal{F})^* \in PSH(\Omega)$*;*

(xii) If $F \in \mathcal{O}(\Omega_1, \Omega_2)$*,* $F \neq const$ *on any component of* Ω_1*, and* $u \in PSH(\Omega_2)$ *then* $u \circ F \in PSH(\Omega_1)$*;*

(xiii) Entire psh functions bounded above are constant. □

It is clear from the definition that if $u \in C^2(\Omega)$ then it is psh if and only if

$$\frac{\partial^2 u}{\partial\zeta\partial\bar\zeta}\Big|_{\zeta=0} u(z+\zeta X) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar z_k}(z) X_j \bar X_k \geq 0, \quad z \in \Omega,\ X \in \mathbb{C}^n. \tag{3.1}$$

It is called the *Levi form* of u. We thus have

Proposition 3.2. *For $u \in C^2$*

$$u \text{ is psh} \ \Leftrightarrow\ \left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right) \geq 0. \qquad \square$$

Similarly as for $n = 1$ we can regularize psh functions using convolution. We take a radially-symmetric, non-negative $\rho \in C_0^\infty(\mathbb{C}^n)$ such that $\operatorname{supp}\rho = \bar B(0,1)$ and $\int_{\mathbb{C}^n} \rho\, d\lambda = 1$. For $\varepsilon > 0$ we set $\rho_\varepsilon(z) = \varepsilon^{-2n}\rho(z/\varepsilon)$, so that $\operatorname{supp}\rho_\varepsilon = \bar B(0,\varepsilon)$ and $\int_{\mathbb{C}^n} \rho_\varepsilon\, d\lambda = 1$. We set

$$\begin{aligned} u_\varepsilon(z) := (u * \rho_\varepsilon)(z) &= \int_{B(z,\varepsilon)} u(\zeta)\rho_\varepsilon(z-\zeta)d\lambda(\zeta) \\ &= \int_{B(0,1)} u(z-\varepsilon\zeta)\rho(\zeta)d\lambda(\zeta) \\ &= \varepsilon^{1-2n}\int_0^1 \widetilde{\rho}(r)\int_{S(z,\varepsilon r)} u d\sigma\, dr, \quad z \in \Omega_\varepsilon, \end{aligned} \tag{3.2}$$

where

$$\Omega_\varepsilon := \{z \in \Omega : B(z,\varepsilon) \subset \Omega\}$$

and $\widetilde{\rho}$ is such that $\rho(z) = \widetilde{\rho}(|z|)$. Using this and Proposition 3.2 we can prove:

Theorem 3.3. *The functions u_ε are smooth and psh in Ω_ε and decrease to u as ε decreases to 0.* $\square$

Using this regularization, similarly as in dimension 1 we can prove that Proposition 3.2 holds also for distributions:

Theorem 3.4. *Psh functions can be characterized as distributions satisfying* (3.1). $\square$

For a domain $\Omega \subset \mathbb{C}^n$ we define the *pluricomplex Green function* as

$$G_\Omega(z,w) := \sup\{u(z) \colon u \in \mathcal{B}_{\Omega,w}\}, \quad z, w \in \Omega$$

where

$$\mathcal{B}_{\Omega,w} = \{u \in PSH^-(\Omega) \colon \limsup_{z\to w}(u(z) - \log|z-w|) < \infty\}$$

(by $PSH^-(\Omega)$ we denote negative psh functions in Ω).

Proposition 3.5. *If $F \in \operatorname{Aut}(\Omega_1,\Omega_2)$ then*

$$G_{\Omega_1}(z,w) = G_{\Omega_2}(F(z),F(w)).$$

Proof. It is enough to see that the mapping

$$\mathcal{B}_{\Omega_2,F(w)} \ni u \longmapsto u \circ F \in \mathcal{B}_{\Omega_1,w}$$

is bijective. $\square$

Proposition 3.6. *(i)* $G_{B(w,r)}(z,w) = \log \dfrac{|z-w|}{r}$;

(ii) $G_{P(w,r)}(z,w) = \log \max_j \dfrac{|z_j - w_j|}{r_j}$.

Proof. (i) We may assume that $w = 0$ and $r = 1$. The inequality $\geq$ is clear. To show the reverse one take $u \in \mathcal{B}_{\mathbb{B},0}$, $X \in \mathbb{C}^n$ with $|X| = 1$ and define $v(\zeta) := u(\zeta X) - \log|\zeta|$. By Proposition 1.10 we have $v \in SH(\Delta)$ and by the maximum principle $v \leq 0$.

(ii) Similarly, we may assume that $w = 0$ and $r = (1, \dots, 1)$. Again, the inequality $\geq$ is clear. Arguing similarly for $u \in \mathcal{B}_{\Delta^n,0}$ and $X \in \mathbb{C}^n$ with $|X_1| = \cdots = |X_n| = 1$ we get $\leq$ on the set $\{z \in \Delta^n\colon |z_1| = \cdots = |z_n|\}$. Elsewhere it now follows from the maximum principle: if we fix $z_n \in \Delta$ then we will get $u(z', z_n) \leq \log \max |z_j|$ for z' with $|z_1| = \cdots = |z_{n-1}| = |z_n|$ and thus also for those with $|z_j| \leq |z_n|$, $j = 1, \dots, n-1$. $\square$

We now immediately obtain

Theorem 3.7. *For $n > 1$ the unit ball $\mathbb{B}$ and the unit polydisk Δ^n are not biholomorphic.*

Proof. If they were biholomorphic then, since $\operatorname{Aut}(\Delta^n)$ is transitive, we would find $F \in \operatorname{Aut}(\mathbb{B}, \Delta^n)$ with $F(0) = 0$. But the Green function for $\mathbb{B}$ is smooth and the one for Δ^n is not, and this contradicts Proposition 3.5. $\square$

We will now show other basic properties of the pluricomplex Green function.

Proposition 3.8. *Either $G_\Omega(\cdot, w) \in \mathcal{B}_{\Omega,w}$ or $G_\Omega(\cdot, w) \equiv -\infty$.*

Proof. If $r > 0$ is such that $B(w,r) \subset \Omega$ then $G_\Omega(z,w) \leq \log \frac{|z-w|}{r}$ and the same is valid for the usc regularization of $G_\Omega(\cdot, w)$. $\square$

Proposition 3.9. *If Ω_j is a sequence of domains increasing to Ω (that is $\Omega_j \subset \Omega_{j+1}$ and $\bigcup \Omega_j = \Omega$) then G_{Ω_j} decreases to G_Ω.*

Proof. Fix $w \in \Omega$ and $r > 0$ such that $B(w,r) \subset \Omega_j$ for j sufficiently large. Then $G_{\Omega_j}(\cdot, w)$ decreases to $u \geq G_\Omega(\cdot, w)$. If $u \equiv -\infty$ then there is nothing to prove. If $u \in PSH(\Omega)$ then, since $G_{\Omega_j}(z,w) \leq \log \frac{|z-w|}{r}$, it follows that $u \in \mathcal{B}_{\Omega,w}$ and thus $u = G_\Omega(\cdot, w)$. $\square$

The pluricomplex Green function was originally defined independently by Klimek [45] and Zakharyuta [63]. It is a classical result that G_Ω is symmetric for $n = 1$. However, this is no longer true for $n > 1$. The first example of this kind was found by Bedford and Demailly [1]. The following simple one was found by Klimek [46]:

Proposition 3.10. *Let $\Omega := \{z \in \mathbb{C}^2\colon |z_1 z_2| < 1\}$. Then*

$$G_\Omega(z,w) = \begin{cases} \log \left| \dfrac{z_1 z_2 - w_1 w_2}{1 - z_1 z_2 \overline{(w_1 w_2)}} \right|, & w \neq 0 \\ \dfrac{1}{2} \log |z_1 z_2|, & w = 0 \end{cases}.$$

In particular, $G_\Omega(0, z) = \log|z_1 z_2|$ and G_Ω is not symmetric.

Proof. First note that by Propositions 1.10 and 1.11 any $u \in PSH^-(\Omega)$ must be of the form $u(z) = v(z_1 z_2)$, where $v \in SH^-(\Delta)$. It is clear that for $w = 0$ we have $|z_1 z_2| \le |z|^2/2$ and the exponent cannot be improved. Therefore $u \in \mathcal{B}_{\Omega,0}$ if and only if $v \in \frac{1}{2}\mathcal{B}_{\Delta,0}$. On the other hand, for $w \neq 0$ we have

$$z_1 z_2 - w_1 w_2 = (z_1 - w_1)(z_2 - w_2) + (z_1 - w_1)w_2 + (z_2 - w_2)w_1$$

and the extra linear term does not vanish. The best estimate for z near w we can get is $|z_1 z_2 - w_1 w_2| \le C|z - w|$ and therefore $u \in \mathcal{B}_{\Omega,w}$ if and only if $v \in \mathcal{B}_{\Delta,w_1 w_1}$. $\square$

By a deep theorem of Lempert [49] the Green function is symmetric if Ω is convex.

4. Domains of Holomorphy and Pseudoconvex Sets

Let Ω be an open set in $\mathbb{C}^n$. We say that Ω is a *domain of holomorphy* if for every open polydisk P centered at $w \in \Omega$ such that for every $f \in \mathcal{O}(\Omega)$ its Taylor series at w converges in P we have $P \subset \Omega$. It is easy to prove that for $n = 1$ all open subsets are domains of holomorphy: it is enough to consider functions of the form $1/(z - z_0)$ for $z_0 \in \partial\Omega$. On the other hand, the Hartogs extension theorem clearly shows that for $n > 1$ there are open sets which are not domains of holomorphy.

Theorem 4.1. *For an open set Ω in $\mathbb{C}^n$ the following are equivalent:*
i) Ω is a domain of holomorphy;
ii) For every compact subset K of Ω the $\mathcal{O}(\Omega)$-envelope of K

$$\widehat{K}_{\mathcal{O}(\Omega)} := \{z \in K : |f(z)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(\Omega)\}$$

is compact Ω;
iii) There exists $f \in \mathcal{O}(\Omega)$ which cannot be continued holomorphically beyond Ω, that is if P is a polydisk centered at $w \in \Omega$ such that the Taylor series of f at w converges in P then $P \subset \Omega$.

For a norm $||\cdot||$ it will be convenient to consider the distance function:

$$\delta_\Omega(z) = \inf_{w \in \mathbb{C}^n \setminus \Omega} ||z - w||, \quad z \in \Omega.$$

If not otherwise stated, δ_Ω will denote the distance with respect to the euclidean norm $|\cdot|$. We will need a lemma.

Lemma 4.2. *Assume that Ω is a domain of holomorphy, $K \subset \Omega$ and $F \in \mathcal{O}(\Omega)$ is such that $|F| \leq \delta_\Omega$ on K, where δ_Ω is take w.r.t. $||z|| = \max_j |z_j|$. Then $|F| \leq \delta_\Omega$ on $\widehat{K}_{\mathcal{O}(\Omega)}$.*

Proof. Assume that $0 < t < 1$ and $f \in \mathcal{O}(\Omega)$. The set

$$\bigcup_{w \in K} (w + t|F(w)|\bar{\Delta}^n)$$

is compact and thus $|f| \leq M < \infty$ there. The Cauchy inequality gives for $w \in K$

$$|\partial^\alpha f(w)| \leq \frac{M\alpha!}{(t|F(w)|)^{|\alpha|}},$$

that is

$$|\partial^\alpha f(w) F(w)^{|\alpha|}| \leq \frac{M\alpha!}{t^{|\alpha|}}.$$

The same inequality holds for $w \in \widehat{K}_{\mathcal{O}(\Omega)}$ and by Proposition 2.2 the Taylor series of f at w converges in $w + t|F(w)|\Delta^n$. Since Ω is a domain of holomorphy, we have $w + t|F(w)|\Delta^n \subset \Omega$ and the lemma follows. $\square$

It is easy to show that the lemma holds for arbitrary norm, it is enough to approximate it by norms whose unit ball is an arbitrary polydisk centered at the origin.

Proof of Theorem 4.1. i)$\Rightarrow$ii) follows from Lemma 4.2 applied for constant F and iii)$\Rightarrow$i) is obvious. It thus remains to prove ii)$\Rightarrow$iii). For $z \in \Omega$ by P_z denote the largest polydisk of the form $z + r\Delta^n$ contained in Ω. Let A be a countable, dense subset of Ω and let $w_j \in A$ be a sequence where every element of A is repeated infinitely many times. Let $K_1 \subset K_2 \subset \dots$ be a sequence of compact subsets of Ω whose union is Ω. Since the envelopes are compact for every j we can find $z_j \in P_{w_j} \setminus \widehat{K}_{\mathcal{O}(\Omega)}$, and therefore there exists $f_j \in \mathcal{O}(\Omega)$ such that $f(z_j) = 1$ but $|f_j| < 1$ on K_j. Replacing f_j by by a power of f_j if necessary, we may assume that $|f_j| \leq 2^{-j}$ on K_j. We may also assume that $f_j \not\equiv 1$ on any component of Ω. Define

$$f := \prod_{j=1}^{\infty} (1 - f_j)^j.$$

For a fixed l the series $\sum_j j|f_j|$ is uniformly absolutely convergent on K_l, and thus $f \in \mathcal{O}(\Omega)$ and $f \neq 0$ an any component of Ω. We have $\partial^\alpha f(z_j) = 0$ if $|\alpha| < j$. Since every element $w \in M$ is repeated infinitely many times in the sequence w_j, there exist points in P_w vanishing to arbitrary order. If the power series of f at w were convergent in a neighbourhood of $\bar{P}_w$ then we would find a point in $\bar{P}_w$ where it would vanish to infinite order and thus the function would vanish near it. This would mean that $f \equiv 0$ in a component of Ω, a contradiction. $\square$

The condition ii in Theorem 4.1 implies in particular that being a domain of holomorphy is a biholomorphically invariant notion (although it can be also deduced directly from the definition in a much more elementary way).

Exercise 2. *Let $\Omega = \{z \in \mathbb{C}^2 \colon |z_2| < |z_1| < 1\}$ be the Hartogs triangle. Show that it is a domain of holomorphy. Prove also that every holomorphic function in neighbourhood of $\bar{\Omega}$ extends holomorphically to Δ^2.*

An open set Ω in $\mathbb{C}^n$ is called *pseudoconvex* if there exists a psh exhaustion of Ω, that is $u \in PSH(\Omega)$ such that the sublevel sets $\{u < c\}$ are relatively compact for all $c \in \mathbb{R}$.

Theorem 4.3. *Domains of holomorphy are pseudoconvex.*

Proof. We may assume that $\Omega \neq \mathbb{C}^n$. Let δ_Ω be as in Lemma 4.2. The function $-\log \delta_\Omega(z) + |z|^2$ is exhaustive and it is enough to show that $-\log \delta_\Omega$ is psh in Ω. Fix $z_0 \in \Omega$ and $X \in \mathbb{C}^n$. We have to show that

$$v(\zeta) = -\log \delta_\Omega(z_0 + \zeta X)$$

is subharmonic near the origin in $\mathbb{C}$. It is enough to show that if v is defined in a neighbourhood of a closed disk, say $\bar{\Delta}$, and h is a harmonic function there with $h \leq v$ on $\partial\Delta$ then $h \leq v$ in Δ. We can find f holomorphic in a neighbourhood of $\bar{\Delta}$ such that $h = \operatorname{Re} f$. Without loss of generality we may assume that f is a polynomial. Let P be a polynomial in $\mathbb{C}^n$ such that $f(\zeta) = P(z_0 + \zeta X)$. For $K := \{z_0 + \zeta X : |\zeta| = 1\}$ by the maximum principle we have $\widehat{K}_{\mathcal{O}(\Omega)} \supset \{z_0 + \zeta X : |\zeta| \leq 1\}$. With $F := e^{-P}$ we have $|F| \leq \delta_\Omega$ on K and thus by Lemma 4.2 also on $\widehat{K}_{\mathcal{O}(\Omega)}$. This means that $h \leq v$ in Δ. $\square$

We will later show that the converse result to Theorem 4.3 also holds.

Theorem 4.4. *For an open set Ω in $\mathbb{C}^n$ the following are equivalent:*
i) Ω is pseudoconvex;
ii) $-\log \delta_\Omega$ is psh for every norm;
iii) $-\log \delta_\Omega$ is psh for some norm;
iv) If $K \Subset \Omega$ then $\widehat{K}_{PSH(\Omega)} := \{z \in \Omega : u(z) \le \sup_K u$ for all $u \in PSH(\Omega)\} \Subset \Omega$.

Proof. The implications ii)⇒iii)⇒i)⇒iv) are clear. To show iv)⇒ii) note that similarly as in the proof of Theorem 4.3 it is enough to show that if $z_0 \in \Omega$, $X \in \mathbb{C}^n$ and f is a complex polynomial such that

$$-\log \delta_\Omega(z_0 + \zeta X) \le \operatorname{Re} f(\zeta)$$

for $\zeta \in \partial\Delta$ then the inequality also holds for $\zeta \in \Delta$. This inequality is equivalent to

$$\delta_\Omega(z_0 + \zeta X) \ge |e^{-f(\zeta)}| \tag{4.1}$$

which means precisely that

$$z_0 + \zeta X + e^{-f(\zeta)} w \in \Omega, \quad \text{if } \ ||w|| < 1. \tag{4.2}$$

Fix w with $||w|| < 1$ and set

$$S := \{t \in [0,1] \colon D_t \subset \Omega\},$$

where

$$D_t = \{z_0 + \zeta X + t e^{-f(\zeta)} w \colon \zeta \in \bar{\Delta}\}.$$

We have $0 \in S$ and it is clear that S is open. It is enough to prove that it is closed.

Let K be the union of the boundaries of D_t for $t \in [0,1]$, that is

$$K = \{z_0 + \zeta X + t e^{-f(\zeta)} w \colon \zeta \in \partial\Delta,\ t \in [0,1]\}.$$

Since (4.1) holds for $\zeta \in \partial\Delta$, it follows that K is a compact subset of Ω. From the maximum principle for subharmonic functions it follows that $D_t \subset \widehat{K}_{PSH(\Omega)}$ for $t \in S$ and by (iv) also for $t \in \bar{S}$. □

It is clear that always $\widehat{K}_{PSH(\Omega)} \subset \widehat{K}_{\mathcal{O}(\Omega)}$. We will later show that in pseudoconvex domains they are actually equal.

Theorem 4.5. *Assume that $\widehat{K}_{PSH(\Omega)} \subset U \subset \Omega$, where Ω is pseudoconvex, U open and K compact. Then there exists a smooth strongly psh (that is we have strict inequality in* (3.1) *for $X \neq 0$) exhaustion u of Ω such that $u < 0$ on K and $u \ge 1$ on $\Omega \setminus U$. In particular, $\widehat{K}_{PSH(\Omega)}$ is compact.*

Proof. By Theorem 4.4 there exists a continuous psh exhaustion u_0 in Ω. We may assume that $u_0 < 0$ in K. For every $z \in L := \{u_0 \le 0\} \setminus U$ we can find $w \in PSH(\Omega)$ such that $w(z) > 0$ and $w < 0$ on K. Let Ω' be open and such that $\{u_0 \le 2\} \subset \Omega' \Subset \Omega$. Regularizing w we can find $w_1 \in PSH \cap C(\Omega')$ such that $w_1(z) > 0$ and $w_1 < 0$ on K. Since $\{w_1 > 0\}$ is an open covering of the compact set L, choosing a finite subcovering and a maximum of corresponding functions we can find $w_2 \in PSH \cap C(\Omega')$ such that $w_2 > 0$ on L and $w_2 < 0$ on K. Define

$$v(z) := \begin{cases} \max\{w_2(z), Cu_0(z)\} & \text{if } u_0(z) < 2 \\ Cu_0(z) & \text{if } u_0(z) \geq 2. \end{cases}$$

We see that $v = Cu_0$ on $\{1 \leq u_0 \leq 2\}$ for sufficiently large C and thus v is a continuous psh exhaustion of Ω. It is clear that $v < 0$ on K and $v \geq 1$ on $\Omega \setminus U$.

To construct smooth strongly psh exhaustion with the required properties set $\Omega_j := \{v < j\}$. Then, considering functions of the form $v * \rho_\varepsilon + \varepsilon|z|^2$, for every j we can find v_j such that it is smooth and strongly psh in a neighbourhood of $\bar{\Omega}_j$, $v < v_j < v + 1$ there and $v_j < 0$ on K. We may also assume that $v_j \in C^\infty(\mathbb{C}^n)$. Let $\chi \in C^\infty(\mathbb{R})$ be convex and such that $\chi(t) = 0$ for $t \leq 0$ and $\chi'(t) > 0$ for $t > 0$. Then $\chi(v_j + 1 - j)$ is strongly psh in a neighbourhood $\bar{\Omega}_j \setminus \Omega_{j-1}$. If a_j are sufficiently big then for every $m \geq 1$ the function

$$u_m = v_0 + \sum_{j=1}^{m} a_j \chi(v_j + 1 - j)$$

is strongly psh in a neighbourhood of $\bar{\Omega}_m$ and $u_m > v$ there. For $m > j$ in Ω_j we have $v_m \leq m - 1$ and thus $u_m = u_l$ there for $m, l > j$. Therefore the limit $u := \lim u_m$ exists and is a smooth strongly psh function in Ω. We also have $u = v_0 < 0$ on K and $u \geq v \geq 1$ on $\Omega \setminus U$. □

The following result is very easy for pseudoconvex sets but highly non-trivial for domains of holomorphy. For those it was called the *Levi problem.*

Theorem 4.6. *Pseudoconvexity is a local property of the boundary. More precisely: an open Ω is pseudoconvex if and only if for every $z \in \partial\Omega$ there exists a neighbourhood U of z such that $\Omega \cap U$ is pseudoconvex.*

Proof. By the condition (iii) in Theorem 4.4 an intersection of two pseudoconvex sets is pseudoconvex. It also follows that if Ω_j is a sequence of pseudoconvex increasing to Ω then Ω is pseudoconvex. Intersecting Ω with big balls we may thus assume that Ω is bounded. It follows from Theorem 4.4 that if $\partial\Omega$ has the local pseudoconvex property then $-\log \delta_\Omega \in PSH(\Omega \setminus K)$ for some $K \Subset \Omega$. Then for sufficiently big c the function $\max\{-\log \delta_\Omega + |z|^2, c\}$ is a psh exhaustion of Ω. □

To solve the Levi problem it is enough to show that pseudoconvex sets are domains of holomorphy. It was originally done independently by Oka [55], Bremermann [18] and Norguet [53]. We will later prove it using the Hörmander estimate.

We have the following characterization of pseudoconvex sets with smooth boundary:

Theorem 4.7. *Let Ω be an open set in $\mathbb{C}^n$ with C^2 boundary and let ρ be its defining function (that is ρ is C^2 in a neighbourhood of $\bar{\Omega}$, $\Omega = \{\rho < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$). Then Ω is pseudoconvex if and only if for $z \in \partial\Omega$ we have*

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) X_j \bar{X}_k \geq 0, \quad X \in T_z^{\mathbb{C}} \partial\Omega, \tag{4.3}$$

where $T_z^{\mathbb{C}}\partial\Omega$ is the complex tangent space to $\partial\Omega$ at z, that is

$$T_z^{\mathbb{C}}\partial\Omega = \big\{X \in \mathbb{C}^n : \sum_j \frac{\partial\rho}{\partial z_j}(z)X_j = 0\big\}.$$

Proof. We can choose a C^2 defining function ρ such that $\rho = -\delta_\Omega$ near $\partial\Omega$ in Ω. Then for $X \in \mathbb{C}^n$ we have near $\partial\Omega$

$$-\delta_\Omega \sum_{j,k} \frac{\partial^2(\log\delta_\Omega)}{\partial z_j \partial\bar{z}_k} X_j\bar{X}_k = -\sum_{j,k}\frac{\partial^2\delta_\Omega}{\partial z_j\partial\bar{z}_k}X_j\bar{X}_k + \delta_\Omega^{-1}\big|\sum_j \frac{\partial\delta_\Omega}{\partial z_j}X_j\big|^2.$$

If $-\log\delta_\Omega$ is psh then approaching the boundary we easily get (4.3) for this particular ρ. If $\widetilde{\rho}$ is another defining function for Ω we can find non-vanishing $h \in C^1$ in a neighbourhood of $\bar{\Omega}$ such that $\widetilde{\rho} = h\rho$ and $h > 0$. Then on $\partial\Omega$

$$\frac{\partial\widetilde{\rho}}{\partial z_j} = h\frac{\partial\rho}{\partial z_j}$$

and

$$\sum_{j,k}\frac{\partial^2\widetilde{\rho}}{\partial z_j\partial\bar{z}_k}X_j\bar{X}_k = h\sum_{j,k}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}X_j\bar{X}_k + 2\mathrm{Re}\big(\sum_j\frac{\partial\rho}{\partial z_j}X_j\sum_k\frac{\partial h}{\partial\bar{z}_k}\bar{X}_k\big).$$

It follows that the definition of $T_z^{\mathbb{C}}\partial\Omega$ and (4.3) are independent of the choice of a defining function ρ and we get (4.3) for arbitrary such a ρ.

To prove the converse assume that (4.3) holds for ρ as before and suppose that $-\log\delta_\Omega$ is not psh near $\partial\Omega$. Then we can find $z \in \Omega$ near $\partial\Omega$ (where δ_Ω is C^2) and $Y \in \mathbb{C}^n$ such that

$$c := \frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\Big|_{\zeta=0}\log\delta_\Omega(z+\zeta Y) > 0.$$

Taylor expansion gives

$$\log\delta_\Omega(z+\zeta Y) = \log\delta_\Omega(z) + \mathrm{Re}\,(a\zeta + b\zeta^2) + c|\zeta|^2 + o(|\zeta|^2) \tag{4.4}$$

for some $a, b \in \mathbb{C}$. Choose $z_0 \in \partial\Omega$ with $\delta_\Omega(z) = |z_0 - z|$ and define

$$z(\zeta) = z + \zeta Y + e^{a\zeta+b\zeta^2}(z_0 - z).$$

Then $z(0) = z_0$ and by (4.4)

$$\delta_\Omega(z+\zeta Y) = |e^{a\zeta+b\zeta^2}(z_0-z)|e^{c|\zeta|^2+o(|\zeta|^2)}.$$

Therefore, if $|\zeta|$ is sufficiently small,

$$\begin{aligned}\delta_\Omega(z+\zeta Y) - |e^{a\zeta+b\zeta^2}(z_0-z)| &= |e^{a\zeta+b\zeta^2}(z_0-z)|\left(e^{c|\zeta|^2+o(|\zeta|^2)} - 1\right)\\ &\geq |e^{a\zeta+b\zeta^2}(z_0-z)|\left(c|\zeta|^2 + o(|\zeta|^2)\right)\\ &\geq \frac{c}{2}|z_0 - z|\,|\zeta|^2.\end{aligned}$$

It follows that $z(\zeta) \in \Omega$ if $\zeta \neq 0$ and

$$\delta_\Omega(z(\zeta)) \geq \frac{c}{2}|z_0 - z|\,|\zeta|^2.$$

Therefore $\delta_\Omega(z(\zeta))$ has a minimum at 0 and for $X := z'(0)$ we have

$$\sum_j \frac{\partial\rho}{\partial z_j}(z_0)X_j = -\left.\frac{\partial}{\partial\zeta}\right|_{\zeta=0} \delta_\Omega(z(\zeta)) = 0$$

and

$$\sum_j \frac{\partial^2\rho}{\partial z_j\partial\bar z_k}(z_0)X_j\bar X_k = -\left.\frac{\partial^2}{\partial\zeta\partial\bar\zeta}\right|_{\zeta=0} \delta_\Omega(z(\zeta)) < 0$$

which contradicts (4.3). $\square$

5. Bergman Kernel and Metric

For a domain Ω in $\mathbb{C}^n$ the *Bergman space* is defined by

$$A^2(\Omega) := \mathcal{O}(\Omega) \cap L^2(\Omega).$$

By $||\cdot||$ we will denote the L^2-norm and by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\Omega)$. Since for $f \in A^2(\Omega$ the function $|f|^2$ is psh,

$$|f(w)| \leq \frac{c_n}{r^n}||f||, \quad \text{if } B(w,r) \subset \Omega, \tag{5.1}$$

and for $K \Subset \Omega$

$$\sup_K |f| \leq C||f||,$$

where C depends only on K and Ω. It follows from Theorem 2.3 that $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and thus a Hilbert space.

Open Problem 2. *Assume that Ω is pseudoconvex. Then either $A^2(\Omega) = \{0\}$ or $A^2(\Omega)$ is infinitely dimensional.*

Wiegerinck [61] showed that it is true in dimension one. He also gave examples of domains which are not pseudoconvex for which $A^2(\Omega$ has arbitrary finite dimension.

From (5.1) we also deduce that for a fixed $w \in \Omega$ the mapping

$$A^2(\Omega) \ni f \longmapsto f(w) \in \mathbb{C}$$

is a bounded linear functional on $A^2(\Omega)$. The Riesz representative of this functional defines the *Bergman kernel* K_Ω on $\Omega \times \Omega$: it is uniquely determined by the *reproducing property*

$$f(w) = \int_\Omega f\, \overline{K_\Omega(\cdot, w)} d\lambda, \quad f \in A^2(\Omega),\ w \in \Omega. \tag{5.2}$$

If we apply it for $f = K_\Omega(\cdot, z)$ we easily get that the kernel is antisymmetric:

$$K_\Omega(w,z) = \overline{K_\Omega(z,w)}.$$

In particular, $K_\Omega(z,w)$ is holomorphic in z and antiholomorphic in w. From Theorem 2.5 applied to the function $K_\Omega(\cdot, \bar{\cdot})$ it also follows that $K_\Omega \in C^\infty(\Omega \times \Omega)$.

With some abuse of notation we will denote the Bergman kernel on the diagonal of $\Omega \times \Omega$ also by K_Ω, that is $K_\Omega(z) = K_\Omega(z,z)$. The reproducing formula (5.2) implies that

$$K_\Omega(z) = ||K_\Omega(\cdot, z)||^2$$

and thus

$$K_\Omega(z) = \sup\{|f(z)|^2 : f \in \mathcal{O}(\Omega),\ ||f|| \leq 1\}. \tag{5.3}$$

Let be $\{\varphi_j\}$ be an orthonormal system in $A^2(\Omega)$ and write

$$K_\Omega(\cdot, w) = \sum_j a_j \varphi_j.$$

Then

$$a_j = \langle K_\Omega(\cdot, w), \varphi_j \rangle = \overline{\varphi_j(w)}$$

and it follows that

$$K_\Omega(z,w)=\sum_j \varphi_j(z)\,\overline{\varphi_j(w)} \tag{5.4}$$

and

$$K_\Omega(z)=\sum_j |\varphi_j(z)|^2. \tag{5.5}$$

Exercise 3. *Using the fact that $\{z^j\}_{j\geq 0}$ is an orthogonal system in Δ show that*

$$K_\Delta(z,w)=\frac{1}{\pi(1-z\bar{w})^2}.$$

For the annulus $P=\{z\in\mathbb{C}: r<|z|<1\}$, where $0<r<1$, prove that

$$K_P(z,w)=\frac{1}{\pi z\bar{w}}\left(\frac{1}{-2\log r}+\sum_{j\in\mathbb{Z}}\frac{j(z\bar{w})^j}{1-r^{2j}}\right).$$

Proposition 5.1. *Let Ω', Ω'' be domains in $\mathbb{C}^n$, $\mathbb{C}^m$, respectively. Then*

$$K_{\Omega'\times\Omega''}\big((z',z''),(w',w'')\big)=K_{\Omega'}(z',w')K_{\Omega''}(z'',w'').$$

Proof. By (5.4) it is enough to show that if φ_j', φ_k'' are orthonormal systems in $A^2(\Omega')$, $A^2(\Omega'')$, respectively, then $\varphi_j'(z')\varphi_k''(z'')$ is an orthonormal system in $A^2(\Omega'\times\Omega'')$. We only have to prove that it is complete. Let $f\in A^2(\Omega'\times\Omega'')$ be such that

$$\iint_{\Omega'\times\Omega''} f(z',z'')\overline{\varphi_j'(z')\varphi_k''(z'')}d\lambda(z',z'')=0$$

for all j,k. It is enough to show that for $\varphi\in A^2(\Omega')$ the function

$$g(z'')=\int_{\Omega'} f(z',z'')\overline{\varphi(z')}d\lambda(z')$$

belongs to $A^2(\Omega'')$. Let K_l be a sequence of compact subsets of Ω' increasing to Ω'. The functions

$$g_l(z'')=\int_{K_l} f(z',z'')\overline{\varphi(z')}d\lambda(z')$$

are holomorphic in Ω'' and satisfy

$$||g_l-g||_{L^2(\Omega'')}\leq ||f||_{L^2((\Omega'\setminus K_l)\times\Omega'')}||\varphi||_{L^2(\Omega')}.$$

Therefore $g_l\to g$ in $L^2(\Omega'')$ as $l\to\infty$ and it follows that $g\in A^2(\Omega)$. □

Proposition 5.2. *If Ω_j increases to Ω then $K_{\Omega_j}\to K_\Omega$ locally uniformly in $\Omega\times\Omega$.*

Proof. For $z,w\in\Omega'\Subset\Omega''\Subset\Omega$ and j big enough

$$|K_{\Omega_j}(z,w)|^2\leq K_{\Omega_j}(z)K_{\Omega_j}(w)\leq K_{\Omega''}(z)K_{\Omega''}(w),$$

hence K_{Ω_j} is locally bounded in $\Omega\times\Omega$. By Theorem 2.4 applied to the sequence $K_{\Omega_j}(\cdot,\bar{\cdot})$ it is enough to show that if $K_{\Omega_j}\to K$ locally uniformly then $K=K_\Omega$. We have

$$||K(\cdot,w)||^2_{L^2(\Omega')}=\lim_{j\to\infty}||K_{\Omega_j}(\cdot,w)||^2_{L^2(\Omega')}\leq \varliminf_{j\to\infty} K_{\Omega_j}(w)=K(w,w)$$

and therefore $||K(\cdot,w)||^2 \le K(w,w)$. In particular, $K(\cdot,w) \in A^2(\Omega)$ and it remains to show that K satisfies the reproducing property (5.2). For $f \in A^2(\Omega)$ and j sufficiently large we can write

$$\begin{aligned} f(w) - \int_\Omega f\overline{K(\cdot,w)}d\lambda &= \int_{\Omega_j} f\overline{K_{\Omega_j}(\cdot,w)}d\lambda - \int_\Omega f\overline{K(\cdot,w)}d\lambda \\ &= \int_{\Omega'} f\left(\overline{K_{\Omega_j}(\cdot,w)} - \overline{K(\cdot,w)}\right)d\lambda \\ &\quad + \int_{\Omega_j\setminus\Omega'} f\overline{K_{\Omega_j}(\cdot,w)}d\lambda - \int_{\Omega\setminus\Omega'} f\overline{K(\cdot,w)}d\lambda \end{aligned}$$

and we easily show that all three integrals are arbitrarily small as j is large and Ω' is close to Ω. □

If $F : \Omega_1 \to \Omega_2$ is a biholomorphic mapping then

$$A^2(\Omega_2) \ni f \longmapsto f \circ F\, Jac\, F \in A^2(\Omega_1)$$

is an isometry (we use the fact that $Jac_{\mathbb{R}} F = |Jac F|^2$) and

$$K_{\Omega_1}(z,w) = K_{\Omega_2}(F(z),F(w))\, Jac\, F(z)\,\overline{Jac\, F(w)}. \tag{5.6}$$

Exercise 4. *(i) Prove that* $A^2(\Delta) = A^2(\Delta_*)$.

(ii)Prove that the Hartogs triangle Ω *(defined in Exercise 2) is biholomorphic to* $\Delta_* \times \Delta$. *Use it to derive the formula*

$$K_\Omega(z,w) = \frac{z_1\bar w_1}{\pi^2(1-z_1\bar w_1)^2(z_1\bar w_1 - z_2\bar w_2)^2}. \tag{5.7}$$

It follows from (5.7) that K_Ω is exhaustive. Domains with this property are called *Bergman exhaustive*. By Exercise 4(i) and Proposition 5.2 it is clear that being Bergman exhaustive is not a biholomorphic invariant for bounded domains in $\mathbb{C}^n$ for $n \ge 2$.

Open Problem 3. *Is Bergman exhaustiveness a biholomorphic invariant for bounded domains in* $\mathbb{C}$*?*

If $z \in \Omega$ is such that $K_\Omega(z) > 0$ then $\log K_\Omega$ is a smooth psh function near z. By (5.6) we see that although K_Ω is not biholomorphically invariant, the Levi form of $\log K_\Omega$ is. For $X \in \mathbb{C}^n$ we define the *Bergman metric* on Ω by

$$B^2_\Omega(z;X) := \left.\frac{\partial^2}{\partial\zeta\partial\bar\zeta}\right|_{\zeta=0} \log K_\Omega(z+\zeta X) = \sum_{j,k} \frac{\partial^2(\log K_\Omega)}{\partial z_j \partial \bar z_k}(z) X_j \bar X_k.$$

Theorem 5.3. *Assume that* $K_\Omega(z_0) > 0$*. Then for* $X \in \mathbb{C}^n$

$$B^2_\Omega(z_0;X) = \frac{1}{K_\Omega(z_0)} \sup\{|f_X(z_0)|^2 : f \in \mathcal{O}(\Omega),\ f(z_0) = 0,\ ||f|| \le 1\}, \tag{5.8}$$

where $f_X = \sum_j \partial f/\partial z_j\, X_j$.

Proof. Define

$$\begin{aligned} H' &:= \{f \in A^2(\Omega) : f(z_0) = 0\} \\ H'' &= \{f \in H' : f_X(z_0) = 0\}. \end{aligned}$$

Then H' is a subspace of $A^2(\Omega)$ of codimension one and H'' is either a subspace of H' of codimension one or $H''=H'$. In both cases we can find an orthonormal system $\varphi_0,\varphi_1,\dots$ in $A^2(\Omega)$ such that $\varphi_1\in H'$ and $\varphi_j\in H''$ for $j\geq 2$. Write $K(z)=K_\Omega(z,z)$. Then by (5.5) at z_0 we have

$$K=|\varphi_0|^2,\quad K_X=\varphi_{0,X}\overline{\varphi_0},\quad K_{X\bar X}=|\varphi_{0,X}|^2+|\varphi_{1,X}|^2.$$

Therefore at z_0

$$B^2_\Omega(\cdot;X)=(\log K)_{X\bar X}=\frac{K\,K_{X\bar X}-|K_X|^2}{K^2}=\frac{|\varphi_{1,X}|^2}{|\varphi_0|^2}$$

and we get $\leq$ in (11.4). On the other hand for $f\in H'$ we have $\langle f,\varphi_0\rangle=0$ and thus at z_0

$$|f_X|=\big|\sum_j\langle f,\varphi_j\rangle\varphi_{j,X}\big|=\big|\langle f,\varphi_1\rangle\varphi_{1,X}\big|\leq||f||\,|\varphi_{1,X}|$$

and the result follows. □

If $K_\Omega>0$ and $\log K_\Omega$ is strongly psh then B_Ω is a Kähler metric with potential $\log K_\Omega$. We then say that Ω *admits* the Bergman metric. By Theorem 5.3 this is for example the case when Ω is bounded. The Bergman metric is in particular a Riemannian metric, for a curve $\gamma\in C^1([0,1],\Omega)$ its length is given by

$$\int_0^1 B_\Omega(\gamma(t);\gamma'(t))dt$$

and the distance between $z,w\in\Omega$ is the infimum of the lengths of curves connecting z with w. This Bergman distance will be denoted by $\operatorname{dist}^B_\Omega$. We will say that Ω is *Bergman complete* if it is complete with respect to this distance.

Proposition 5.4. *If Ω is Bergman complete then it is a domain of holomorphy.*

Proof. Suppose that Ω is not a domain of holomorphy. Then there exists an open polydisk $P\not\subset\Omega$ centered at $z_0\in\Omega$ such that for every $f\in\mathcal{O}(\Omega)$ its Taylor series at z_0 converges in P. By Theorem 2.5 there exists $K(z,w)\in C^\infty(P\times P)$, holomorphic in z, anti-holomorphic in w and such that $K=K_\Omega$ near (z_0,z_0). We can find $z'\in\partial\Omega\cap P$ which is on the boundary of the component of $\Omega\cap P$ containing z_0. Then K is an extension of K_Ω near z' and a sequence from Ω converging to z' is Cauchy with respect to $\operatorname{dist}^B_\Omega$. This means that Ω cannot be Bergman complete. □

The converse is not true: Δ_* is not Bergman complete by Exercise 4(i).

The main criterion for Bergman completeness is due to Kobayashi [47]:

Theorem 5.5. *Let Ω be a domain in $\mathbb{C}^n$ admitting the Bergman metric. Assume that for every sequence $z_j\in\Omega$ with no accumulation point in Ω one has*

$$\lim_{j\to\infty}\frac{|f(z_j)|^2}{K_\Omega(z_j)}=0,\quad f\in A^2(\Omega).\tag{5.9}$$

Then Ω is Bergman complete.

The main tool in proving Theorem 5.5 will be the mapping

$$\iota : \Omega \ni z \longmapsto \big[K_\Omega(\cdot, z)\big] \in \mathbb{P}(A^2(\Omega)).$$

It is easy to check that ι is injective if Ω is bounded.

For any Hilbert space H one can define the Fubini-Study metric $FS_{\mathbb{P}(H)}$ on $\mathbb{P}(H)$ as the push-forward $\pi_* P$, where

$$\pi : H_* \ni f \longmapsto [f] \in \mathbb{P}(H)$$

and P is the Kähler metric on H_* with the potential $\log ||f||^2$, that is for $f \in H_*$, $F \in H$ we have

$$P^2(f; F) := \frac{\partial^2}{\partial\zeta\partial\bar\zeta}\bigg|_{\zeta=0} \log ||f + \zeta F||^2 = \frac{||F||^2}{||f||^2} - \frac{|\langle F, f\rangle|^2}{||f||^4}.$$

Proposition 5.6. $B_\Omega = \iota^* FS_{\mathbb{P}(A^2(\Omega))}$.

Proof. It will be enough to show that $B_\Omega = \iota^* P$, where P is the above metric for $H = A^2(\Omega)$. For $\gamma \in C^1((-\varepsilon, \varepsilon), \Omega)$, where $\varepsilon > 0$, set $z_0 := \gamma(0)$, $X := \gamma'(0)$, $f := K_\Omega(\cdot, z_0)$ and

$$F := \frac{d}{dt}\bigg|_{t=0} K_\Omega(\cdot, \gamma(t)).$$

We have to show that

$$\frac{\partial^2}{\partial\zeta\partial\bar\zeta}\bigg|_{\zeta=0} \log K_\Omega(z_0 + \zeta X) = \frac{\partial^2}{\partial\zeta\partial\bar\zeta}\bigg|_{\zeta=0} \log ||f + \zeta F||^2. \tag{5.10}$$

If $\varphi_0, \varphi_1, \dots$ are as in the proof of Theorem 5.3 then $f = \overline{\varphi_0(z_0)}\varphi_0$ and $F = \overline{\varphi_{0,X}(z_0)}\varphi_0 + \overline{\varphi_{1,X}(z_0)}\varphi_1$. Using the fact that for any holomorphic f

$$\frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) = f_X(z_0),$$

we can show that both sides of (5.10) are equal to $|\varphi_{1,X}(z_0)|^2/|\varphi_0(z_0)|^2$. □

The distance on $\mathbb{P}(H)$ given by the Fubini-Study metric is given by

$$\arccos \frac{|\langle f, g\rangle|}{||f||\,||g||}, \quad [f],\, [g] \in \mathbb{P}(H).$$

(For details see Appendix A4 in [8].) Since ι is distance decreasing, we have obtained the following lower bound for the Bergman distance:

Corollary 5.7. *If Ω admits the Bergman metric then*

$$\operatorname{dist}^B_\Omega(z, w) \geq \arccos \frac{|K_\Omega(z, w)|}{\sqrt{K_\Omega(z) K_\Omega(w)}}.$$

□

Proof of Theorem 5.5. Let $z_j \in \Omega$ be a Cauchy sequence with respect to $\operatorname{dist}^B_\Omega$. We may assume that it has no accumulation point in Ω. Since ι is distance decreasing, $\iota(z_j)$ is a Cauchy sequence in $\mathbb{P}(A^2(\Omega))$. But $\mathbb{P}(A^2(\Omega))$ is complete and therefore we can find $f \in A^2(\Omega)$, $f \neq 0$, such that $\iota(z_j)$ converges to $[f]$. This means that there exist $\lambda_j \in \mathbb{C}$ such that $|\lambda_j| = 1$ and

$$\lambda_j \frac{K_\Omega(\cdot, z_j)}{\sqrt{K_\Omega(z_j)}} \longrightarrow \frac{f}{||f||}$$

in $A^2(\Omega)$. This implies that

$$\frac{|f(z_j)|}{\sqrt{K_\Omega(z_j)}} \longrightarrow ||f||$$

which contradicts (5.9). $\square$

Zwonek [64] showed that the converse to Theorem 5.5 does not hold: he gave an example of a bounded domain in $\mathbb{C}$ which is Bergman complete but not Bergman exhaustive. (Taking $f \equiv 1$ in (5.9) when Ω is bounded clearly shows that bounded domains satisfying (5.9) must be Bergman exhaustive.) This example was simplified by Jucha [41]: he showed that

$$\Omega := \Delta_* \setminus \left(\bigcup_{k=1}^{\infty} \bar{\Delta}(2^{-k}, r_k) \right),$$

where $r_k > 0$ are such that $\bar{\Delta}(2^{-k}, r_k) \cap \bar{\Delta}(2^{-l}, r_l) = \emptyset$ for $k \neq l$, is Bergman complete if and only if

$$\sum_{k=1}^{\infty} \frac{2^k}{\sqrt{-\log r_k}} = \infty$$

and Bergman exhaustive if and only if

$$\sum_{k=1}^{\infty} \frac{4^k}{-\log r_k} = \infty.$$

Therefore, if for example $r_k = e^{-k^2 4^k}$ then Ω is Bergman complete but not Bergman exhaustive.

The proof of Theorem 5.5 really shows something slightly stronger: instead of (5.9) it is enough to assume that

$$\varlimsup_{j\to\infty} \frac{|f(z_j)|^2}{K_\Omega(z_j)} < ||f||^2, \quad f \in A^2(\Omega),\ f \neq 0. \tag{5.11}$$

Open Problem 4. *Assume that Ω is Bergman complete. Does this imply* (5.11) *for any sequence $z_j \in \Omega$ without an accumulation point in Ω?*

6. Hörmander Estimate in Dimension One

On an open set $\Omega \subset \mathbb{C}$ we consider the inhomogeneous Cauchy Riemann equation

$$\frac{\partial u}{\partial \bar{z}} = f. \tag{6.1}$$

It makes sense for any complex-valued locally integrable u and f in the distributional sense

$$-\int_\Omega u\beta_{\bar{z}} d\lambda = \int_\Omega f\beta d\lambda, \quad \beta \in C_0^\infty(\Omega).$$

We will prove the following existence result:

Theorem 6.1. *Let Ω be an open subset of $\mathbb{C}$ and φ a C^2 strongly subharmonic function in Ω. Then for every $f \in L^2(\Omega, e^{-\varphi})$ such that*

$$\int_\Omega \frac{|f|^2}{\varphi_{z\bar{z}}} e^{-\varphi} d\lambda < \infty$$

there exists $u \in L^2(\Omega, e^{-\varphi})$ a solution of (6.1) *such that*

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq \int_\Omega \frac{|f|^2}{\varphi_{z\bar{z}}} e^{-\varphi} d\lambda. \tag{6.2}$$

In dimension one it is convenient to use the notation $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$. To prove the estimate we need to introduce the adjoint $\bar{\partial}_\varphi^*$ to the (densely defined) operator $\bar{\partial}$ in the weighted space $L^2(\Omega, e^{-\varphi})$. It is determined by the relation

$$\langle \bar{\partial} u, \beta\rangle_\varphi = \langle u, \bar{\partial}_\varphi^* \beta\rangle_\varphi, \quad \beta \in C_0^\infty(\Omega),$$

that is

$$-\int_\Omega u\bar{\partial}(\bar{\beta} e^{-\varphi}) d\lambda = \int_\Omega u\, \overline{\bar{\partial}_\varphi^* \beta}\, e^{-\varphi} d\lambda.$$

This gives

$$\bar{\partial}_\varphi^* \beta = -\partial(\beta e^{-\varphi}) e^{\varphi} = -\partial\beta + \beta\partial\varphi. \tag{6.3}$$

Theorem 6.1 will be an easy consequence of the following two propositions:

Proposition 6.2. *Assume that $f \in L^2_{loc}(\Omega, e^{-\varphi})$ where φ is continuous. Then for a given finite constant C there exists a solution u to $\bar{\partial} u = f$ satisfying*

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq C \tag{6.4}$$

if and only if

$$\left| \int_\Omega f\bar{\beta} e^{-\varphi} d\lambda \right|^2 \leq C \int_\Omega |\bar{\partial}_\varphi^* \beta|^2 e^{-\varphi} d\lambda \tag{6.5}$$

for all $\beta \in C_0^\infty(\Omega)$.

Proof. It is clear that (6.4) implies (6.5), so assume (6.5) holds. Then the functional

$$F\left(\bar{\partial}_\varphi^* \beta\right) = \overline{\int_\Omega f\bar{\beta} e^{-\varphi} d\lambda}$$

is well defined on $\{\bar\partial^*_\varphi\beta : \beta \in C_0^\infty(\Omega)\}$ and its norm does not exceed $\sqrt{C}$. It can be extended to a functional defined on $L^2(\Omega, e^{-\varphi})$ with the same norm. We can find $u \in L^2(\Omega, e^{-\varphi})$ satisfying (6.4) and such that

$$F(g) = \langle g, u\rangle_\varphi, \quad g \in L^2(\Omega, e^{-\varphi}).$$

Hence u satisfies (6.4) and

$$\langle f, \beta\rangle_\varphi = \overline{F(\bar\partial^*_\varphi\beta)} = \langle u, \bar\partial^*_\varphi\beta\rangle_\varphi = \langle \bar\partial u, \beta\rangle_\varphi, \quad \beta \in C_0^\infty(\Omega).$$

It follows that $\bar\partial u = f$. □

Proposition 6.3. *For $\varphi \in C^2(\Omega)$ and $\beta \in C_0^2(\Omega)$ we have*

$$\int_\Omega |\bar\partial^*_\varphi\beta|^2 e^{-\varphi} d\lambda = \int_\Omega \left(|\bar\partial\beta|^2 + \varphi_{z\bar z}|\beta|^2\right) e^{-\varphi} d\lambda.$$

Proof. By the definition of $\bar\partial^*_\varphi$ we have

$$\int_\Omega |\bar\partial^*_\varphi\beta|^2 e^{-\varphi} d\lambda = \langle \bar\partial^*_\varphi\beta, \bar\partial^*_\varphi\beta\rangle_\varphi = \langle \bar\partial\bar\partial^*_\varphi\beta, \beta\rangle_\varphi.$$

From (6.3) we will get

$$\bar\partial\bar\partial^*_\varphi\beta = \bar\partial^*_\varphi\bar\partial\beta + \varphi_{z\bar z}\beta$$

and the proposition follows. □

Proof of Theorem 6.1. By the Cauchy-Schwarz inequality

$$\left|\int_\Omega f\bar\beta e^{-\varphi} d\lambda\right|^2 \le \int_\Omega \frac{|f|^2}{\varphi_{z\bar z}} e^{-\varphi} d\lambda \int_\Omega \varphi_{z\bar z}|\beta|^2 e^{-\varphi} d\lambda$$

and the theorem is a consequence of both propositions. □

We can now state a general existence result for the $\bar\partial$-equation:

Theorem 6.4. *For any $f \in L^2_{loc}(\Omega)$ there exists $u \in L^2_{loc}(\Omega)$ solving $\bar\partial u = f$.*

Proof. By Theorem 6.1 it is enough to find a smooth strongly subharmonic φ such that $f \in L^2(\Omega, e^{-\varphi})$ and the right-hand side of (6.2) is finite. Let ψ be a smooth strongly subharrmonic exhaustion of Ω. We may assume that $\psi \ge 0$. Considering functions of the form $\varphi = \chi \circ \psi$, where χ is smooth, increasing and convex, we can find φ satisfying the first condition. Since $\varphi_{z\bar z} \ge \chi' \circ \psi\, \psi_{z\bar z}$ we can also assume that $\varphi_{z\bar z} \ge 1$ and then the right-hand side of (6.2) is finite. □

By approximation we can show a bit more general version of Theorem 6.1, where in particular we do not have to assume that the weight is smooth:

Theorem 6.5. *Let φ be a subharmonic function in Ω and $f \in L^2_{loc}(\Omega)$. Assume that $h \in L^\infty_{loc}(\Omega)$ is such that $h \ge 0$ and $|f|^2 \le h\varphi_{z\bar z}$ (in the distributional sense). Then there exists $u \in L^2_{loc}(\Omega)$, a solution of $\bar\partial u = f$, such that*

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega h e^{-\varphi} d\lambda. \tag{6.6}$$

Proof. If the right-hand side of (6.6) is infinite then the result follows from Theorem 6.4, we may thus assume that it is finite. If φ is smooth strongly subharmonic and $f \in L^2(\Omega, e^{-\varphi})$ then it is enough to use Theorem 6.1. If f is not necessarily in $L^2(\Omega, e^{-\varphi})$ then let Ω_j be relatively compact open subsets of Ω such that $\Omega_j \uparrow \Omega$. For every j we will find a solution $u_j \in L^2(\Omega_j, e^{-\varphi})$ of $\bar\partial u = f$ satisfying

$$\int_{\Omega_j} |u_j|^2 e^{-\varphi} d\lambda \leq \int_\Omega \frac{|f|^2}{\varphi_{z\bar z}} e^{-\varphi} d\lambda.$$

By the Banach-Alaoglu theorem applied in $L^2(\Omega_k, e^{-\varphi})$ for a fixed k and using the diagonal argument one can show that u_j has a subsequence converging weakly to u which has the required properties.

We may thus assume that the right-hand side of (6.6) is finite and that the result holds for smooth strongly subharmonic φ. By the Radon-Nikodym theorem we can find $\beta \in L^1_{loc}(\Omega)$ such that $0 \leq \beta \leq \varphi_{z\bar z}$ and $|f|^2 \leq h\beta$. Let ε_j be a sequence decreasing to 0 such that $\varphi_j := \varphi * \rho_{\varepsilon_j} + \varepsilon_j |z|^2$ is defined in a neighbourhood of $\bar\Omega_j$. Set $h_j := |f|^2/\varphi_{j,z\bar z}$. By the previous part we can find $u_j \in L^2_{loc}(\Omega_j)$ such that

$$\int_{\Omega_j} |u_j|^2 e^{-\varphi_j} d\lambda \leq \int_{\Omega_j} h_j e^{-\varphi_j} d\lambda \leq \int_{\Omega_j} h_j e^{-\varphi} d\lambda.$$

We have $\beta_j := \beta * \rho_{\varepsilon_j} \leq \varphi_{j,z\bar z}$ and replecing it with a subsequence if necessary we may assume that β_j converges pointwise to β almost everywhere. Therefore

$$\limsup_{j\to\infty} h_j \leq \limsup_{j\to\infty} \frac{|f|^2}{\beta_j} \leq h$$

and by the Fatou lemma

$$\limsup_{j\to\infty} \int_{\Omega_j} |u_j|^2 e^{-\varphi_j} d\lambda \leq \int_\Omega h e^{-\varphi} d\lambda =: C.$$

For fixed $m \geq k$ we see that the $L^2(\Omega_k, e^{-\varphi_m})$-norm of u_j, $j \geq m$, is bounded and thus, replacing ε_j with a subsequence if necessary and using the diagonal argument, we can find $u \in L^2_{loc}(\Omega)$ such that u_j converges weakly to u in $L^2(\Omega_k, e^{-\varphi_m})$ for every k and m. It follows that for every $\delta > 0$ and sufficiently large m

$$\int_{\Omega_k} |u|^2 e^{-\varphi_m} d\lambda \leq C + \delta$$

and therefore u satisfies (6.6). □

The set of $u \in L^2(\Omega, e^{-\varphi})$ solving $\bar\partial u = f$ must be of the form $u_0 + \ker\bar\partial$, where u_0 is a particular solution of $\bar\partial u = f$. Since for every distribution h satisfying $\bar\partial h = 0$ we also have $\Delta h = 0$, it follows that

$$\ker\bar\partial = \mathcal{O}(\Omega) \cap L^2(\Omega, e^{-\varphi}).$$

The minimal solution to $\bar\partial u = f$ in the $L^2(\Omega, e^{-\varphi})$-norm is the only one perpendicular to $\ker\bar\partial$, that is

$$\int_\Omega u\bar h e^{-\varphi} d\lambda = 0, \quad h \in \ker\bar\partial.$$

Exercise 5. *Assume that φ is a bounded continuous radially symmetric function on the unit disk Δ. Show that $u_0 = \bar{z}$ is the minimal solution to $\bar\partial u = 1$ in the $L^2(\Delta, e^{-\varphi})$-norm.*

One inconvenience with the Hörmander estimate (6.2) is that φ appears both as a weight and in the denominator on the right-hand side of (6.2). These two roles are separated in the following estimate for the $\bar\partial$-equation due to Donnelly and Fefferman [28].

Theorem 6.6. *Let Ω be open in $\mathbb{C}$. Assume that $\psi = -\log(-v)$ where $v \in SH^-(\Omega)$. Then for any $\varphi \in SH(\Omega)$ and $f \in L^2_{loc}(\Omega)$ there exists $u \in L^2_{loc}(\Omega)$ solving $\bar\partial u = f$ such that*

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le 4 \int_\Omega h e^{-\varphi} d\lambda, \tag{6.7}$$

where $h \in L^\infty_{loc}(\Omega)$ is such that $h \ge 0$ and $|f|^2 \le h\psi_{z\bar z}$.

Proof. First assume that φ, ψ and Ω are bounded and ψ is smooth and strongly subharmonic. Then it is characterized by the condition

$$|\psi_z|^2 \le \psi_{z\bar z}. \tag{6.8}$$

Let u be the solution to $\bar\partial u = f$ which is minimal in $L^2(\Omega, e^{-\varphi-\psi/2})$. Then u is perpendicular to $\ker\bar\partial$ in $L^2(\Omega, e^{-\varphi-\psi/2})$, that is

$$\int_\Omega u\bar h\, e^{-\varphi-\psi/2} d\lambda = 0, \qquad h \in \ker\bar\partial.$$

But this means that $v := ue^{\psi/2}$ is perpendicular to $\ker\bar\partial$ in $L^2(\Omega, e^{-\varphi-\psi})$ (by our assumptions at the beginning of the proof the set $\ker\bar\partial$ is the same with respect to both weights). Therefore v is the minimal solution to $\bar\partial v = g$ where

$$g = \frac{\partial}{\partial\bar z}\big(ue^{\psi/2}\big) = e^{\psi/2}\big(f + u\psi_{\bar z}/2\big).$$

We trivially have

$$|g|^2 \le \frac{|g|^2}{\psi_{z\bar z}}\varphi_{z\bar z} + \psi_{z\bar z}$$

and by Theorem 6.5

$$\int_\Omega |v|^2 e^{-\varphi-\psi} d\lambda \le \int_\Omega \frac{|g|^2}{\psi_{z\bar z}} e^{-\varphi-\psi} d\lambda.$$

Therefore, using (6.8) for $t > 0$ we will get

$$\begin{aligned}\int_\Omega |u|^2 e^{-\varphi} d\lambda &\le \int_\Omega \frac{|f + u\,\psi_{\bar z}/2|^2}{\psi_{z\bar z}} e^{-\varphi} d\lambda \\ &\le \int_\Omega \left(\frac{|f|^2}{\psi_{z\bar z}} + \frac{|u|\,|f|}{\sqrt{\psi_{z\bar z}}} + \frac{|u|^2}{4}\right) e^{-\varphi} d\lambda \\ &\le \int_\Omega \left(\left(1 + \frac{t}{2}\right)\frac{|f|^2}{\psi_{z\bar z}} + \left(\frac14 + \frac{1}{2t}\right)|u|^2\right) e^{-\varphi} d\lambda.\end{aligned}$$

For $t = 2$ we obtain (6.7).

For arbitrary φ, ψ and Ω the result follows if we approximate similarly as before: first Ω from inside and φ from above, so that we may assume that they are bounded, and then consider $\psi_\varepsilon = -\log(-v_\varepsilon)$, where $v_\varepsilon = v * \rho_\varepsilon + \varepsilon|z|^2$. $\square$

The proof of Theorem 6.6 given here is due to Berndtsson [4] with some modifications from [9] (see also [10]) where the constant 4 was obtained. Moreover, we have the following result from [14]:

Proposition 6.7. *The constant 4 in* (6.7) *is optimal.*

Proof. Let $\Omega = \Delta$ and

$$u(z) = \frac{\eta(-\log|z|)}{z}$$

for some $\eta \in C_0^1((0,\infty))$. Then $\bar\partial u = f$ where

$$f(z) := -\frac{\eta'(-\log|z|)}{2|z|^2}.$$

We claim that it is the minimal in $L^2(\Delta)$. Indeed, for $j \geq 0$ we have

$$\int_\Delta u(z)\bar z^j d\lambda(z) = \int_0^1 \int_0^{2\pi} r^j \eta(-\log r) e^{-i(j+1)t} dt\, dr = 0$$

and since z^j is an orthogonal system in $L^2(\Delta)$ we conclude that u is perpendicular to $\ker \bar\partial$. Theorem 6.6 with $\varphi \equiv 0$ and

$$\psi(z) = -\log(-\log|z|)$$

now gives

$$\int_0^\infty \eta^2 dt \leq 4 \int_0^\infty (\eta')^2 t^2 dt \tag{6.9}$$

for all $\eta \in C_0^1((0,\infty))$ and thus for all $\eta \in W_0^{1,2}((0,\infty))$.

It is enough to show that the constant 4 is optimal in (6.9). For $\varepsilon > 0$ set

$$\eta(t) = \begin{cases} t^{-(1-\varepsilon)/2} & t \leq 1 \\ t^{-(1+\varepsilon)/2} & t > 1. \end{cases}$$

Then we can compute

$$\int_0^\infty \eta^2 dt = \frac{2}{\varepsilon}, \quad \int_0^\infty (\eta')^2 t^2 dt = \frac{1+\varepsilon^2}{2\varepsilon},$$

and the ratio tends to 4 as $\varepsilon \to 0$. □

We now illustrate the usefulness of $\bar\partial$-estimates in dimension one to show the following result proved independently in [33] and [21].

Theorem 6.8. *Let Ω be a bounded domain in $\mathbb{C}$ and let $z_0 \in \partial\Omega$. Then the subspace of $A^2(\Omega)$ of those functions from $A^2(\Omega)$ that extend holomorphically to a neighbourhood of $\Omega \cup \{z_0\}$ is dense in $A^2(\Omega)$.*

Proof. We may assume that $z_0 = 0$ and $\Omega \subset \Delta_R$. We will use Theorem 6.6 with $\varphi \equiv 0$ and

$$\psi(z) = -\log\big(-\log(|z|/R)\big).$$

For $\varepsilon > 0$ set $T := -\log(-\log(\varepsilon/R))$, so that $\Delta_\varepsilon = \{\psi < T\}$. We define $\widetilde{\varepsilon} > 0$ by the condition $\Delta_{\widetilde{\varepsilon}} = \{\psi < T+1\}$. It is easy to check that $\widetilde{\varepsilon} \to 0$ as $\varepsilon \to 0$. We also set

$$\chi(t) := \begin{cases} 0 & t \le T \\ t-T & T < t \le T+1 \\ 1 & t > T+1. \end{cases}$$

For a fixed $h \in A^2(\Omega)$ the function $h\,\chi \circ \psi$ can be trivially continuously extended to $\Omega \cup \Delta_\varepsilon$ so that it vanishes in Δ_ε. Set

$$f := \frac{\partial}{\partial \bar{z}}\big(\chi \circ \psi\, h\big) = \chi' \circ \psi\, \psi_{\bar{z}}\, h.$$

Then

$$|f|^2 = (\chi' \circ \psi)^2 |\psi_z|^2 |h|^2 = (\chi' \circ \psi)^2 |h|^2 \psi_{z\bar{z}}$$

and from Theorem 6.6 we obtain $u \in L^2_{loc}(\Omega \cup \Delta_\varepsilon)$ solving $\bar{\partial} u = f$ and such that

$$\int_{\Omega \cup \Delta_\varepsilon} |u|^2 d\lambda \le 4 \int_\Omega (\chi' \circ \psi)^2 |h|^2 d\lambda \le 4 \int_{\Omega \cap \Delta_{\widetilde{\varepsilon}}} |h|^2 d\lambda.$$

In fact, u has to be continuous, since $h\,\chi \circ \psi$ is. The function

$$h_\varepsilon := h\,\chi \circ \psi - u$$

is holomorphic in $\Omega \cup \Delta_\varepsilon$, belongs to $A^2(\Omega)$ and

$$||h_\varepsilon - h|| \le ||h\big(\chi \circ \psi - 1\big)|| + ||u|| \le 3||h||_{L^2(\Omega \cap \Delta_{\widetilde{\varepsilon}})}. \qquad \square$$

Theorem 6.8 can be used to prove the following improvement of the Kobayashi criterion for bounded domains in $\mathbb{C}$ due to Chen [21] (see also [10]).

Theorem 6.9. *Assume that Ω is a bounded domain in $\mathbb{C}$ which is Bergman exhaustive. Then it is Bergman complete.*

Proof. Take $f \in A^2(\Omega)$, $z_0 \in \partial\Omega$ and a sequence $z_j \in \Omega$ converging to z_0. By Theorem 6.8 for every $\varepsilon > 0$ we can find $f_\varepsilon \in A^2(\Omega)$ which is in particular bounded near z_0 and such that $||f - f_\varepsilon|| \le \varepsilon$. Then

$$\frac{|f(z_j)|}{\sqrt{K_\Omega(z_j)}} \le \frac{|f_\varepsilon(z_j)| + |f(z_j) - f_\varepsilon(z_j)|}{\sqrt{K_\Omega(z_j)}} \le \frac{|f_\varepsilon(z_j)|}{\sqrt{K_\Omega(z_j)}} + \varepsilon$$

and it follows that the Kobayashi criterion is satisfied. $\square$

Considering the Hartogs triangle we see that this result is no longer true in higher dimensions.

7. Hörmander Estimate in Arbitrary Dimension

We now come back to several variables and consider the $\bar\partial$-equation as in the proof of Theorem 2.6:

$$\bar\partial u = \alpha, \tag{7.1}$$

where α is a (0,1)-form satisfying $\bar\partial\alpha = 0$. The main result is the following estimate of Hörmander in its full generality for $(0,1)$-forms:

Theorem 7.1. *Let Ω be pseudoconvex in $\mathbb{C}^n$, $\varphi \in PSH(\Omega)$ and let $\alpha \in L^2_{loc,(0,1)}(\Omega)$ be $\bar\partial$-closed. Then there exists $u \in L^2_{loc}(\Omega)$ solving (7.1) and such that*

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi}\, e^{-\varphi} d\lambda. \tag{7.2}$$

Let us first explain the statement precisely. If φ is smooth and strongly psh then

$$|\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k} \varphi^{j\bar k} \alpha_j \bar\alpha_k$$

(here $(\varphi^{j\bar k}) = \overline{(\partial^2\varphi/\partial z_j\partial\bar z_k)^{-1}}$) is the square of the length of α with respect the Kähler metric $i\partial\bar\partial\varphi$. It is equal to the minimal function h satisfying

$$i\bar\alpha \wedge \alpha \le h i\partial\bar\partial\varphi, \tag{7.3}$$

that is $(\alpha_j\bar\alpha_k) \le h(\partial^2\varphi/\partial z_j\partial\bar z_k)$. If φ is arbitrary then, similarly as in Theorem 6.5, we should replace $|\alpha|^2_{i\partial\bar\partial\varphi}$ in (7.2) by any nonnegative $h \in L^\infty_{loc}(\Omega)$ satisfying (7.3). The Hörmander estimate for nonsmooth φ was first stated in [10].

Hörmander's formulation (see [35], and also [36], [37]) was also weaker in the following sense: instead of $|\alpha|^2_{i\partial\bar\partial\varphi}$ on the right-hand side of (7.2) he considered $|\alpha|^2/c$, where at every point c is the minimal eigenvalue of $(\partial^2\varphi/\partial z_j\partial\bar z_k)$. Demailly [23] was the first to note that the proof of the Hörmander estimate for $(0,1)$-forms really gives this stronger statement.

When proving Theorem 7.1 the main difficulty compared with dimension one is that we have to take into account the assumption $\bar\partial\alpha = 0$ which is not satisfied for all α when $n \ge 2$. We will consider Hilbert spaces

$$H_1 = L^2(\Omega, e^{-\varphi_1}), \quad H_2 = L^2_{(0,1)}(\Omega, e^{-\varphi_2}), \quad H_3 = L^2_{(0,2)}(\Omega, e^{-\varphi_3}),$$

where $\varphi_k \in C^2(\Omega)$, $k = 1, 2, 3$. Here, for

$$f = \sum_j f_j d\bar z_j \in H_2,$$

we have

$$|f|^2 = \sum_j |f_j|^2, \quad ||f||^2 = \int_\Omega |f|^2 e^{-\varphi_2} d\lambda,$$

and for

$$F = \sum_{j<k} F_{jk} d\bar z_j \wedge d\bar z_k \in H_3,$$

$$|F|^2 = \sum_{j<k} |F_{jk}|^2, \quad ||F||^2 = \int_\Omega |F|^2 e^{-\varphi_3} d\lambda.$$

Note that

$$|f \wedge g|^2 \leq 2|f|^2|g|^2, \quad f, g \in H_2. \tag{7.4}$$

It is also clear that the spaces of test forms $C_0^\infty(\Omega)$, $C^\infty_{0,(0,1)}(\Omega)$ and $C^\infty_{0,(0,2)}(\Omega)$ are dense in H_1, H_2 and H_3, respectively.

We will also consider linear, densely defined, closed operators given by $\bar\partial$:

$$H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3.$$

Since $\bar\partial^2 = 0$, the range of T is contained in $\ker S$. Similarly as in Proposition 6.2, our goal will be to prove

$$|\langle \alpha, f\rangle| \leq \sqrt{C}||T^* f||, \quad f \in D_{T^*}. \tag{7.5}$$

We have to take into account that $\alpha \in \ker S$. In fact, we will then prove something more:

$$|\langle \alpha, f\rangle|^2 \leq C||T^* f||^2 + \widetilde{C}||Sf||^2, \quad f \in D_{T^*} \cap D_S. \tag{7.6}$$

It is easy to see that (7.6) implies (7.5) if $\alpha \in \ker S$: for $f \in D_{T^*}$ write $f = f' + f''$ where $f' \in \ker S$ and $f'' \perp \ker S$. Then f'' is also perpendicular to the range of T and thus $T^* f'' = 0$. Since $S\alpha = 0$, we also have $\langle \alpha, f''\rangle = 0$ and thus by (7.6)

$$|\langle \alpha, f\rangle|^2 = |\langle \alpha, f'\rangle|^2 \leq C||T^* f'||^2 = C||T^* f||^2.$$

There are essentially two ways of proving the Hörmander estimate. Berndtsson [2] did it using the same weights $\varphi_k = \varphi$ on bounded Ω with smooth boundary. Then however inevitably also boundary terms have to appear, in particular one can show that $f \in C^\infty_{(0,1)}(\Omega)$ belongs to D_{T^*} if and only if $\sum_j f_j \partial\rho/\partial z_j = 0$ on $\partial\Omega$, where ρ is as in Theorem 4.7. We will follow the original Hörmander approach from [35] and [36] which is completely interior. It requires however that the weights φ_k are slightly different in order to ensure that it is enough to prove (7.6) for $f \in C^\infty_{0,(0,1)}(\Omega)$. Note that for such f we have

$$T^* f = -\sum_{j=1}^n e^{\varphi_1} \frac{\partial(e^{-\varphi_2} f_j)}{\partial z_j}. \tag{7.7}$$

This formula also holds for $f \in D_{T^*}$ and in fact can be used to define T^* and D_{T^*}.

Lemma 7.2. *Let $\psi \in C^2(\Omega)$ be such that there exist a sequence $\chi_\nu \in C_0^\infty(\Omega)$ such that $0 \leq \chi_\nu \leq 1$, for every $K \Subset \Omega$ one has $\chi_\nu = 1$ on K for ν big enough, and $|\bar\partial\chi_\nu|^2 \leq e^\psi$. Then for any $\varphi \in C^2(\Omega)$ and*

$$\varphi_k = \varphi + (k-3)\psi, \quad k = 1, 2, 3,$$

the space $C^\infty_{0,(0,1)}(\Omega)$ is dense in $D_{T^*} \cap D_S$ in the graph norm

$$||f|| + ||T^* f|| + ||Sf||, \quad f \in H_2.$$

Proof. Fix $f \in D_{T^*} \cap D_S$. We have $\chi_\nu T^* f \to T^* f$ in H_1, $\chi_\nu f \to f$ in H_2 and $\chi_\nu S f \to Sf$ in H_3. Since by (7.4)

$$||S(\chi_\nu f) - \chi_\nu S f||^2 \leq 2 \int_\Omega |f|^2 |\bar\partial \chi_\nu|^2 e^{-\varphi_3} d\lambda \leq 2 \int_\Omega |f|^2 e^{-\varphi_2} d\lambda,$$

it follows from the Lebesgue bounded convergence theorem that $S(\chi_\nu f) \to Sf$ in H_3. Similarly by (7.7)

$$T^*(\chi_\nu f) - \chi_\nu T^* f = -e^{\varphi_1 - \varphi_2} \sum_j f_j \frac{\partial \chi_\nu}{\partial z_j}$$

and

$$||T^*(\chi_\nu f) - \chi_\nu T^* f||^2 \leq \int_\Omega |f|^2 |\bar\partial \chi_\nu|^2 e^{-\varphi} d\lambda \leq \int_\Omega |f|^2 e^{-\varphi_2} d\lambda.$$

This implies that $T^*(\chi_\nu f) \to T^* f$ in H_1 and thus $\chi_\nu f \to f$ in the graph norm.

We may thus assume that $\operatorname{supp} f \Subset \Omega$. For sufficiently small $\varepsilon > 0$ let $f_\varepsilon := f * \rho_\varepsilon$. Since $f \in D_{T^*} \cap D_S$, we have $\sum_j \partial f_j / \partial z_j \in L^2(\Omega)$ and $Sf \in L^2_{(0,2)}(\Omega)$, and it follows that $f_\varepsilon \to f$ in the graph norm. □

We will now prove a counterpart of Proposition 6.3. This kind of results, obtained by integration by parts, are sometimes called the Bochner-Kodaira formulas.

Proposition 7.3. *For $f \in C^\infty_{0,(0,1)}(\Omega)$ and $\varphi, \psi \in C^2(\Omega)$ we have*

$$\int_\Omega \left(\Big| e^\psi T^* f + \sum_j f_j \frac{\partial \psi}{\partial z_j} \Big|^2 + |Sf|^2 \right) e^{-\varphi} d\lambda = \int_\Omega \sum_{j,k} \left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k} f_j \bar f_k + \left| \frac{\partial f_j}{\partial \bar z_k} \right|^2 \right) e^{-\varphi} d\lambda.$$

Proof. We can compute that

$$|Sf|^2 = \sum_{j<k} \left| \frac{\partial f_j}{\partial \bar z_k} - \frac{\partial f_k}{\partial \bar z_j} \right|^2 = \sum_{j,k} \left| \frac{\partial f_j}{\partial \bar z_k} \right|^2 - \sum_{j,k} \frac{\partial f_j}{\partial \bar z_k} \frac{\partial \bar f_k}{\partial z_j} \tag{7.8}$$

and

$$e^\psi T^* f = -\sum_j \frac{\partial f_j}{\partial z_j} + \sum_j f_j \frac{\partial \varphi_2}{\partial z_j} = \sum_j \delta_j f_j - \sum_j f_j \frac{\partial \psi}{\partial z_j}, \tag{7.9}$$

where

$$\delta_j \beta := -\frac{\partial \beta}{\partial z_j} + \frac{\partial \varphi}{\partial z_j} \beta = -e^\varphi \frac{\partial}{\partial z_j} (\beta e^{-\varphi}).$$

The operators $\partial / \partial \bar z_j$ and δ_j are adjoint in the sense that

$$\int_\Omega \beta_1 \overline{\frac{\partial \beta_2}{\partial \bar z_j}} e^{-\varphi} = \int_\Omega \delta_j \beta_1 \, \bar\beta_2 e^{-\varphi} d\lambda, \quad \beta_1, \beta_2 \in C^\infty_0(\Omega),$$

and they satisfy the following commutation relation:

$$\frac{\partial}{\partial \bar z_k} \delta_j - \delta_j \frac{\partial}{\partial \bar z_k} = \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k}.$$

Using that we will obtain

$$\int_\Omega \Big| \sum_j \delta_j f_j \Big|^2 e^{-\varphi} d\lambda = \int_\Omega \sum_{j,k} \left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k} f_j \bar f_k + \frac{\partial f_j}{\partial \bar z_k} \frac{\partial \bar f_k}{\partial z_j} \right) e^{-\varphi} d\lambda.$$

Combining this with (7.9) and (7.8) we get the formula. □

Assume that φ_k are as in Lemma 7.2. We want to show (7.6) for $f \in C^\infty_{0,(0,1)}(\Omega)$. By Proposition 7.3 for $t > 0$

$$\int_\Omega \sum_{j,k} \frac{\partial^2\varphi}{\partial z_j \partial \bar z_k} f_j \bar f_k e^{-\varphi} d\lambda \le (1+t^{-1})||T^* f||^2 + ||Sf||^2 + (1+t)\int_\Omega |f|^2 |\bar\partial\psi|^2 e^{-\varphi} d\lambda. \tag{7.10}$$

Proof of Theorem 7.1. Similarly as in the proof of Theorem 6.5 we reduce the proof to the case when φ is smooth strongly psh and the right-hand side of (7.2) is finite. Since Ω is pseudoconvex, there exists a smooth strongly psh exhaustion function s in Ω. Fix $t > 0$. We may assume that the cut-off functions χ_ν from Lemma 7.2 are equal to 1 on $\Omega_{t+1} := \{s < t+1\}$ and that ψ vanishes on Ω_t. Let $\gamma \in C^\infty(\mathbb{R})$ be convex and such that $\gamma = 0$ on $(-\infty, t)$, $\gamma \circ s \ge 2\psi$, and $\gamma' \circ s\, i\partial\bar\partial s \ge (1+t)|\partial\psi|^2 i\partial\bar\partial |z|^2$. Therefore for $\varphi' = \varphi + \gamma \circ s$

$$i\partial\bar\partial\varphi' \ge i\partial\bar\partial\varphi + (1+t)|\partial\psi|^2 i\partial\bar\partial|z|^2.$$

Let $\varphi_k = \varphi' + (k-3)\psi$, in particular we have $\varphi - 2\varphi_2 = -\varphi' - \gamma\circ s + 2\psi \le -\varphi'$. Therefore by (7.10) with φ replaced with φ' we get for $f \in C^\infty_{0,(0,1)}(\Omega)$

$$\int_\Omega \sum_{j,k} \frac{\partial^2\varphi}{\partial z_j \partial \bar z_k} f_j \bar f_k e^{\varphi - 2\varphi_2} d\lambda \le (1+t^{-1})||T^* f||^2 + ||Sf||^2. \tag{7.11}$$

By (7.3) for $f \in C^\infty_{0,(0,1)}(\Omega)$

$$\Big|\sum_j f_j \bar\alpha_j\Big|^2 \le h \sum_{j,k} \frac{\partial^2\varphi}{\partial z_j\partial\bar z_k} f_j \bar f_k,$$

where $h = |\alpha|^2_{i\partial\bar\partial\varphi}$. This coupled with (7.11) and the Schwarz inequality gives

$$\begin{aligned} |\langle \alpha, f\rangle|^2 &= \left| \int_\Omega \sum_j \alpha_j \bar f_j e^{-\varphi_2} d\lambda \right|^2 \\ &\le \int_\Omega h e^{-\varphi} d\lambda \int_\Omega \frac{|\sum_j \alpha_j \bar f_j|^2}{h} e^{\varphi - 2\varphi_2} d\lambda \\ &\le M\big((1+t^{-1})||T^*f||^2 + ||Sf||^2\big) \end{aligned}$$

where $M = \int_\Omega h e^{-\varphi} d\lambda < \infty$. By Lemma 7.2 it holds for all $f \in D_{T^*} \cap D_S$. As we have already seen this implies (7.5)

$$|\langle \alpha, f\rangle| \le \sqrt{M(1+t^{-1})}\,||T^* f||, \quad f \in D_{T^*}.$$

Similarly as in the proof of Proposition 6.2 we can find $u_t \in H_1$ with $||u_t|| \le \sqrt{M(1+t^{-1})}$ and

$$\langle \alpha, f\rangle = \langle u_t, T^* f\rangle, \quad f \in D_{T^*}.$$

This means that $\alpha = T^{**}u_t = Tu_t$ and, since $\varphi_1 = \varphi$ in Ω_t, we have

$$\int_{\Omega_t} |u_t|^2 e^{-\varphi} d\lambda \leq \int_{\Omega} |u_t|^2 e^{-\varphi_1} d\lambda \leq M(1 + t^{-1}).$$

We may thus find a sequence $t_j \uparrow \infty$ and $u \in L^2_{loc}(\Omega)$ such that u_{t_j} converges weakly to u in $L^2(\Omega_{t_0}, e^{-\varphi})$ for every $t_0 > 0$. $\square$

8. Some Applications of the Hörmander Estimate

Theorem 7.1 and some of its consequences are principal tools in constructing holomorphic functions. Similarly as in dimension one the condition $\bar\partial u = 0$ for functions from L^2_{loc} (and even distributions) completely characterizes the holomorphic functions.

Our first application is the solution of the Levi problem;

Theorem 8.1. *Let Ω be pseudoconvex in $\mathbb{C}^n$ and K a compact subset of Ω. Then*

$$\widehat{K}_{PSH(\Omega)} = \widehat{K}_{\mathcal{O}(\Omega)}.$$

In particular, Ω is a domain of holomorphy.

Proof. We clearly have $\subset$. To show the converse fix $z_0 \in \Omega \setminus \widehat{K}_{PSH(\Omega)}$. We may assume that $z_0 = 0$. By Theorem 4.5 there exists $v \in PSH \cap C(\Omega)$ such that $v < 0$ on K and $v(0) > 1$. Let $\chi \in C_0^\infty(\Omega)$ be such that $\chi(0) = 1$ and $\operatorname{supp}\chi \subset \{v > 1\}$. For $t \geq 1$ set $v_t := \max\{v, tv\}$, so that $v_t = v$ in $\{v < 0\}$ and $v_t > t$ on $\operatorname{supp}\chi$. By Theorem 7.1 with

$$\varphi_t = |z|^2 + 2n\log|z| + v_t$$

and $\alpha = \bar\partial\chi$ there exists $u_t \in L^2_{loc}(\Omega)$ such that $\bar\partial u_t = \bar\partial\chi$ (therefore u has to be continuous) and

$$\int_\Omega |u_t|^2 e^{-\varphi_t} d\lambda \leq \int_\Omega |\bar\partial\chi|^2 e^{-\varphi_t} d\lambda.$$

Since $e^{-\varphi_t}$ is not locally integrable near the origin, we have $u_t(0) = 0$. Therefore $f_t := \chi - u_t$ is holomorphic in Ω, $f_t(0) = 1$ and

$$\int_{\{v<0\}} |f_t|^2 d\lambda \leq C_1 e^{-t},$$

where C_1 is independent of t. Since $|f_t|^2$ is subharmonic, we have

$$\sup_K |f_t|^2 \leq C_2 e^{-t}$$

and we see that $0 \notin \widehat{K}_{\mathcal{O}(\Omega)}$.

The last statement now follows from Theorem 4.1. □

The fact that being a domain of holomorphy is equivalent to pseudoconvexity implies that the former is a local property of the boundary, see Theorem 4.6.

For our further applications we will need the following estimate for $\bar\partial$ due to Berndtsson [3] (see also [9]):

Theorem 8.2. *Assume that Ω is pseudoconvex, $\varphi \in PSH(\Omega)$ and $\psi = -\log(-v)$, where $v \in PSH^-(\Omega)$. Then for every $\alpha \in L^2_{loc,(0,1)}(\Omega)$ with $\bar\partial\alpha = 0$ and δ with $0 \leq \delta < 1$ there exists $u \in L^2_{loc}(\Omega)$, a solution of $\bar\partial u = \alpha$ such that*

$$\int_\Omega |u|^2 e^{\delta\psi - \varphi} d\lambda \leq \frac{4}{(1-\delta)^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\delta\psi-\varphi} d\lambda \tag{8.1}$$

(we use the convention explained after Theorem 7.1).

Proof. The proof will be similar to that of Theorem 6.6, the main idea is from [4]. Set $\widetilde{\varphi} := \varphi + (1-\delta)\psi/2$. As before we may assume that φ, ψ and Ω are bounded (we only need it to conclude that the spaces $L^2(\Omega, e^{-\widetilde{\varphi}})$ and $L^2(\Omega, e^{-\varphi-\psi})$ consist of the same elements) and that ψ is smooth strongly psh. The condition on ψ now means precisely that

$$|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le 1 \tag{8.2}$$

Let u be the minimal solution to $\bar\partial u = \alpha$ in $L^2(\Omega, e^{-\widetilde{\varphi}})$. Then u is perpendicular to $\ker\bar\partial$, that is

$$\int_\Omega u\bar{h}e^{-\varphi-(1-\delta)\psi/2}d\lambda = 0, \quad h \in \ker\bar\partial.$$

Then $v := e^{(1+\delta)\psi/2}u$ is perpendicular to $\ker\bar\partial$ in $L^2(\Omega, e^{-\varphi-\psi})$, and thus the minimal solution in $L^2(\Omega, e^{-\varphi-\psi})$ to $\bar\partial v = \beta$, where

$$\beta = e^{(1+\delta)\psi/2}\big(\alpha + \frac{1+\delta}{2}u\,\bar\partial\psi\big).$$

By the Hörmander estimate

$$\int_\Omega |v|^2 e^{-\varphi-\psi}d\lambda \le \int_\Omega |\beta|^2_{i\partial\bar\partial(\varphi+\psi)}e^{-\varphi-\psi}d\lambda \le \int_\Omega |\beta|^2_{i\partial\bar\partial\psi}e^{-\varphi-\psi}d\lambda$$

and thus for any $t > 0$ by (8.2)

$$\begin{aligned}\int_\Omega |u|^2 e^{\delta\psi-\varphi}d\lambda &\le \int_\Omega \big|\alpha + \frac{1+\delta}{2}u\,\bar\partial\psi\big|^2_{i\partial\bar\partial\psi}e^{\delta\psi-\varphi}d\lambda \\ &\le \int_\Omega \left((1+t)\big|\alpha\big|^2_{i\partial\bar\partial\psi} + (1+t^{-1})\frac{(1+\delta)^2}{4}|u|^2\right)e^{\delta\psi-\varphi}d\lambda.\end{aligned}$$

For $t = (1+\delta)/(1-\delta)$ we will get (8.1). □

For $\delta = 0$ we get the Donnelly-Fefferman estimate [28]. Similarly as in Proposition 6.7 one can show that the constant $4/(1-\delta)^2$ in (8.1) is optimal for any δ (see [14]). The method used to prove Theorem 8.2 is called *twisting*.

We will use Theorem 8.2 to prove the following estimate due to Herbort [34]:

Theorem 8.3. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Then for any $f \in \mathcal{O}(\Omega)$ and $w \in \Omega$ we have*

$$\frac{|f(w)|^2}{K_\Omega(w)} \le c_n \int_{\{G_\Omega(\cdot,w)<-1\}} |f|^2 d\lambda. \tag{8.3}$$

Proof. We may assume that $G = G_\Omega(\cdot, w) \in \mathcal{B}_{\Omega,w}$, otherwise the estimate follows from the trivial one $|f(w)|^2/K_\Omega(w) \le ||f||^2$. We will use Theorem 8.2 with $\delta = 0$, $\varphi = 2nG$ and $\psi = -\log(-G)$. Set

$$\alpha = \bar\partial(f\chi\circ G) = f\chi'\circ G\,\bar\partial G,$$

where χ such that $\alpha \in L^2_{loc,(0,1)}(\Omega)$ will be determined later. We have

$$i\partial\bar\partial\psi = -G^{-1}i\partial\bar\partial G + G^{-2}i\partial G \wedge \bar\partial G$$

and

$$i\bar\alpha \wedge \alpha = |f|^2(\chi'\circ G)^2 i\partial G \wedge \bar\partial G \le |f|^2(\chi'\circ G)^2 G^2 i\partial\bar\partial\psi.$$

By Theorem 8.2 there exists $u \in L^2_{loc}(\Omega)$ such that

$$\int_\Omega |u|^2 d\lambda \le \int_\Omega |u|^2 e^{-2nG} d\lambda \le 4 \int_\Omega |f|^2 (\chi' \circ G)^2 G^2 e^{-2nG} d\lambda. \tag{8.4}$$

We can now define

$$\chi(t) := \begin{cases} \displaystyle\int_1^{-t} \frac{ds}{s e^{ns}}, & t < -1 \\ 0, & t \ge -1. \end{cases}$$

The following lemma ensures that $\alpha \in L^2_{loc,(0,1)}(\Omega)$.

Lemma 8.4. *Assume that u is psh in Ω and $\chi \in C^{0,1}(\mathbb{R})$ is such that $\int_{-\infty}^0 (\chi')^2 dt < \infty$. Then $\nabla(\chi \circ u) \in L^2_{loc}(\Omega)$.*

Proof. The proof works for arbitrary subharmonic u. Without loss of generality we may assume that $u \le 0$. We will prove that for $K \Subset \Omega$ and smooth psh u in Ω one has

$$\int_K |\nabla(\chi \circ u)|^2 d\lambda \le C(K, \Omega) ||u||_{L^1(\Omega)} \int_{-\infty}^0 (\chi')^2 dt. \tag{8.5}$$

This will be sufficient because for arbitrary u one can regularize it and use the Banach-Alaoglu theorem. Set

$$f(t) := \int_{-\infty}^t (\chi'(s))^2 ds, \quad g(t) = \int_t^0 f(s) ds,$$

so that $f' = (\chi')^2$ and $g' = -f$. We have

$$g(t) = |t| \int_{-\infty}^t (\chi'(s))^2 ds + \int_t^0 |s| (\chi'(s))^2 ds \le |t| \int_{-\infty}^0 (\chi'(s))^2 ds. \tag{8.6}$$

Let $\varphi \in C_0^\infty(\Omega)$ be nonnegative and such that $\varphi = 1$ on K. Then

$$\begin{aligned} \int_K |\nabla(\chi \circ u)|^2 d\lambda &\le \int_\Omega \varphi (\chi' \circ u)^2 |\nabla u|^2 d\lambda \\ &= \int_\Omega \varphi \langle \nabla (f \circ u), \nabla u \rangle d\lambda \\ &= -\int_\Omega \varphi f \circ u \Delta u \, d\lambda - \int_\Omega f \circ u \langle \nabla \varphi, \nabla u \rangle d\lambda \\ &\le -\int_\Omega f \circ u \langle \nabla \varphi, \nabla u \rangle d\lambda \\ &= \int_\Omega \langle \nabla \varphi, \nabla (g \circ u) \rangle d\lambda \\ &= -\int_\Omega g \circ u \Delta \varphi \, d\lambda \end{aligned}$$

and (8.5) follows from (8.6). □

End of proof of Theorem 8.3. Set $\widetilde{f} := f \chi \circ G - u$, it is holomorphic in Ω. Since e^{-2nG} is not locally integrable near w, from (8.4) it follows that $u(w) = 0$ and $\widetilde{f}(w) = \chi(-\infty) f(w)$

(note that we may assume that the right-hand side of (8.3) is finite). We also have

$$||\widetilde{f}|| \le ||f\chi \circ G|| + ||u|| \le (\chi(-\infty)+2)\sqrt{\int_{\{G<-1\}} |f|^2 d\lambda}$$

and thus

$$\frac{|f(w)|^2}{K_\Omega(w)} \le \frac{||\widetilde{f}||^2}{(\chi(-\infty))^2} \le \left(1+\frac{2}{\chi(-\infty)}\right)^2 \int_{\{G<-1\}} |f|^2 d\lambda. \qquad \square$$

The proof of Theorem 8.3 presented here is from [10].

Theorem 8.3 coupled with the Kobayashi criterion Theorem 5.5 immediately gives the following class of Bergman complete domains characterized in terms of pluripotential theory. The result is due to Chen [20], see also [15] and [34].

Theorem 8.5. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ such that*

$$\lim_{j\to\infty} \lambda(\{G_\Omega(\cdot, w_j) < -1\}) = 0 \tag{8.7}$$

for every sequence $w_j \in \Omega$ without an accumulation point. Then Ω is Bergman complete. $\square$

One can show that for bounded Ω in $\mathbb{C}$ the condition (8.7) is equivalent to the regularity of Ω. In higher dimensions one can prove that (8.7) is satisfied (although not equivalent) for so called hyperconvex domains, that is domains admitting a bounded psh exhaustion, see [15].

Note that for $f \equiv 1$ the estimate (8.3) gives the following lower bound for the Bergman kernel

$$K_\Omega(w) \ge \frac{c_n}{\lambda(\{G_\Omega(\cdot, w) < -1\})}.$$

It turns out that establishing this estimate for all sublevel sets and finding optimal constants leads to very interesting consequences. The following estimate was obtained in [13]:

Theorem 8.6. *Let Ω be pseudoconvex in $\mathbb{C}^n$. Then for $w \in \Omega$ and $t \le 0$*

$$K_\Omega(w) \ge \frac{e^{2nt}}{\lambda(\{G_\Omega(\cdot, w) < t\})}. \tag{8.8}$$

Proof. Repeating the proof of Theorem 8.3 with $f \equiv 1$ for arbitrary sublevel set we will get

$$K_\Omega(w) \ge \frac{c(n,t)}{\lambda(\{G_\Omega(\cdot, w) < t\})}, \tag{8.9}$$

where

$$c(n,t) = \left(1+\frac{2}{\mathrm{Ei}\,(nt)}\right)^2$$

and

$$\mathrm{Ei}\,(a) = \int_a^\infty \frac{ds}{se^s}.$$

To improve the constant in (8.9) we can now use the tensor power trick: for $m \gg 0$ take $\widetilde{\Omega} = \Omega^m \subset \mathbb{C}^{nm}$ and $\widetilde{w} = (w, \dots, w) \in \widetilde{\Omega}$. We have $K_{\widetilde{\Omega}}(\widetilde{w}) = (K_\Omega(w))^m$ (by Proposition 5.1) and for $\widetilde{z} = (z^1, \dots, z^m) \in \widetilde{\Omega}$

$$\max_j G_\Omega(z^j, w) \leq G_{\widetilde{\Omega}}(\widetilde{z}, \widetilde{w}),$$

since the left-hand side belongs to $\mathcal{B}_{\widetilde{\Omega}, \widetilde{w}}$. (In fact, one always has equality here, see [39] and [29].) It follows that

$$\{G_{\widetilde{\Omega}}(\cdot, \widetilde{w}) < t\} \subset \{G_\Omega(\cdot, w) < t\}^m$$

and (8.9) gives

$$(K_\Omega(w))^m = K_{\widetilde{\Omega}}(\widetilde{w}) \geq \frac{c(nm, t)}{\lambda(\{G_{\widetilde{\Omega}}(\cdot, \widetilde{w}) < t\})} \geq \frac{c(nm, t)}{(\lambda(\{G_\Omega(\cdot, w) < t\}))^m}.$$

It is now enough to check that

$$\lim_{m \to \infty} c(nm, t)^{1/m} = e^{2nt}. \qquad \square$$

It is easy to see that if Ω is a ball centered at w then we have equality in (8.8). It is especially interesting to see what happens with the right-hand side of (8.8) when $t \to -\infty$. For $n = 1$ we can write

$$G_\Omega(z, w) = \log |z - w| + \varphi(z),$$

where φ is harmonic in Ω. Then

$$\Delta(w, e^{t - M_t}) \subset \{G_\Omega(\cdot, w) < t\} \subset \Delta(w, e^{t - m_t}),$$

where

$$m_t := \inf_{\{G_\Omega(\cdot, w) < t\}} \varphi, \qquad M_t := \sup_{\{G_\Omega(\cdot, w) < t\}} \varphi.$$

It follows that the right-hand side of (8.8) converges to $(c_\Omega(w))^2/\pi$, where

$$c_\Omega(w) = \exp\left(\lim_{z \to w} (G_\Omega(z, w) - \log |z - w|)\right)$$

is the logarithmic capacity of the complement of Ω with respect to w. We have thus proved the Suita conjecture from [60], originally shown in [12]:

Theorem 8.7. *For $\Omega \subset \mathbb{C}$ one has $c_\Omega^2 \leq \pi K_\Omega$.* $\square$

Carleson [19] proved that

$$K_\Omega(w) = 0 \;\Leftrightarrow\; c_\Omega(w) = 0. \tag{8.10}$$

Theorem 8.7 gives a quantitative version of $\Rightarrow$ in (8.10). On the other hand, it is known, see [16], that the reverse inequality $K_\Omega \leq C c_\Omega^2$ in general does not hold for any constant C.

Open Problem 5. *Find a quantitative version of $\Leftarrow$ in* (8.10).

Another interesting case are convex domains in $\mathbb{C}^n$. Letting $t \to -\infty$ in (8.8) and using Lempert's theory [49] one can then show that

$$K_\Omega(w) \le \frac{1}{\lambda(I_\Omega(w))},$$

where

$$I_\Omega(w) = \{\varphi'(0)\colon \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$$

is the Kobayashi indicatrix, see [13].

9. Ohsawa-Takegoshi Extension Theorem

The following extension result is due to Ohsawa and Takegoshi [54].

Theorem 9.1. *Let Ω be a bounded pseudoconvex open set and H a complex affine subspace of $\mathbb{C}^n$. Assume that φ is psh in Ω and f is holomorphic in $\Omega' := \Omega \cap H$. Then there exists $F \in \mathcal{O}(\Omega)$ such that $F = f$ on Ω' and*

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \le C \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda',$$

where C depends on n and the diameter of Ω.

An important recent contribution is due to Chen [22] who showed that the Ohsawa-Takegoshi extension theorem can be deduced directly from the Hörmander estimate. We will essentially follow his proof here, with some modifications from [11].

The Berndtsson estimate, Theorem 8.2, is closely related to Theorem 9.1. If it were true for $\delta = 1$ (with some finite constant) then it would be sufficient to prove the extension theorem. The following estimate from [11], motivated by the method from [22], can be treated as a counterpart of the Berndtsson estimate for $\delta = 1$.

Theorem 9.2. *Let Ω, φ, ψ and α be as in Theorem 8.2. Assume in addition that $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le a < 1$ on $\operatorname{supp}\alpha$. Then there exists $u \in L^2_{loc}(\Omega)$ such that $\bar\partial u = \alpha$ and*

$$\int_\Omega (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\psi}) |u|^2 e^{\psi-\varphi} d\lambda \le \frac{1+\sqrt{a}}{1-\sqrt{a}} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\psi-\varphi} d\lambda. \tag{9.1}$$

Proof. We essentially repeat the proof of Theorem 8.2. Assuming that the data is sufficiently regular let u be the minimal solution to $\bar\partial u = \alpha$ in $L^2(\Omega, e^{-\varphi})$. We will obtain that $v := ue^\psi$ is the minimal solution to $\bar\partial v = \beta$, where

$$\beta := (\alpha + u\bar\partial\psi)e^\psi,$$

in $L^2(\Omega, e^{-\varphi-\psi})$. From Theorem 7.1 we will get

$$\begin{aligned}
\int_\Omega |u|^2 e^{\psi-\varphi} d\lambda &= \int_\Omega |v|^2 e^{-\varphi-\psi} d\lambda \\
&\le \int_\Omega |\beta|^2_{i\partial\bar\partial(\varphi+\psi)} e^{-\varphi-\psi} d\lambda \\
&\le \int_\Omega |\alpha + u\bar\partial\psi|^2_{i\partial\bar\partial\psi} e^{\psi-\varphi} d\lambda \\
&\le \int_\Omega \left(|\alpha|^2_{i\partial\bar\partial\varphi} + 2|u|\sqrt{h}|\alpha|_{i\partial\bar\partial\varphi} + |u|^2 h \right) e^{\psi-\varphi} d\lambda,
\end{aligned}$$

where $h := |\bar\partial\psi|^2_{i\partial\bar\partial\psi}$. Thus for $t > 0$

$$\begin{aligned}
&\int_\Omega |u|^2(1-h) e^{\psi-\varphi} d\lambda \\
&\qquad \le \int_\Omega \left[\left(1 + \frac{th}{1-h}\right) |\alpha|^2_{i\partial\bar\partial\varphi} + t^{-1}|u|^2(1-h) \right] e^{\psi-\varphi} d\lambda \\
&\qquad \le \int_\Omega \left[\left(1 + \frac{ta}{1-a}\right) |\alpha|^2_{i\partial\bar\partial\varphi} + t^{-1}|u|^2(1-h) \right] e^{\psi-\varphi} d\lambda.
\end{aligned}$$

We obtain (9.1) if we take $t := 1 + a^{-1/2}$. $\square$

To prove Theorem 9.1 we may assume that H is a hyperplane and then obtain the general result by iteration. As noticed by Siu [59] and Berndtsson [3] it is enough to assume that Ω is bounded in the direction orthogonal to H. We can formulate it is as follows:

Theorem 9.3. *Assume that $\Omega \subset \mathbb{C}^{n-1} \times \Delta$ is pseudoconvex and set $\Omega' := \Omega \cap \{z_n = 0\}$. Then for any $\varphi \in PSH(\Omega)$ and $f \in \mathcal{O}(\Omega')$ there exists $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega'} = f$ and*

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \le C \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda', \tag{9.2}$$

where C is an absolute constant.

Proof. We may assume that the right-hand side of (9.2) is finite, otherwise it is enough to construct φ growing sufficiently quickly to ∞ at the boundary. Approximating Ω from inside and regularizing φ we may assume that Ω is bounded, f is defined in a neighbourhood of $\bar\Omega \cap \{z_n = 0\}$ in $\{z_n = 0\}$ and φ is smooth and defined in a neighbourhood of $\bar\Omega$.

Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) = 1$ for $t \le -2$ and $\chi(t) = 0$ for $t \ge 0$. For $\varepsilon > 0$ sufficiently small the function $f(z')v_\varepsilon(z_n)$, where

$$v_\varepsilon(\zeta) = v_\varepsilon(\zeta) := \chi\big(2\log(|\zeta|/\varepsilon)\big),$$

is defined in Ω. We will use Theorem 9.2 with

$$\alpha_\varepsilon := \bar\partial(f v_\varepsilon) = f(z')\, \chi'\big(2\log(|z_n|/\varepsilon)\big) \frac{d\bar z_n}{\bar z_n},$$

$\widetilde\varphi := \varphi + 2\log|z_n|$, and psh ψ_ε depending only on $|z_n|$. We will find $u_\varepsilon \in L^2_{loc}(\Omega)$ such that $\bar\partial u_\varepsilon = \alpha_\varepsilon$ (in fact u_ε has to be continuous, since $f v_\varepsilon$ is) and

$$\int_\Omega |u_\varepsilon|^2 (1 - |\bar\partial\psi_\varepsilon|^2_{i\partial\bar\partial\psi_\varepsilon}) e^{\psi_\varepsilon - \widetilde\varphi} d\lambda \le \frac{1 + \sqrt{a(\varepsilon)}}{1 - \sqrt{a(\varepsilon)}} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi_\varepsilon} e^{\psi_\varepsilon - \widetilde\varphi} d\lambda, \tag{9.3}$$

provided that

$$\frac{|\psi_{\varepsilon,\bar z_n}|^2}{\psi_{\varepsilon, z_n \bar z_n}} \begin{cases} < 1, & |z_n| < 1 \\ \le a(\varepsilon) < 1, & |z_n| \le \varepsilon \end{cases}. \tag{9.4}$$

We will need the following completely elementary lemma:

Lemma 9.4. *For $\zeta \in \mathbb{C}$ with $|\zeta| \le (2e)^{-1/2}$ and $\varepsilon > 0$ sufficiently small set*

$$\psi(\zeta) = \psi_\varepsilon(\zeta) := -\log\big[-\log(|\zeta|^2 + \varepsilon^2) + \log\big(-\log(|\zeta|^2 + \varepsilon^2)\big)\big].$$

Then ψ is subharmonic in $\{|\zeta| < (2e)^{-1/2}\}$ and there exist constants C_1, C_2, C_3 such that

(i) $\left(1 - \dfrac{|\psi_\zeta|^2}{\psi_{\zeta\bar\zeta}}\right) e^\psi \ge \dfrac{1}{C_1 \log^2(|\zeta|^2 + \varepsilon^2)}$ *on* $\{|\zeta| \le (2e)^{-1/2}\}$;

(ii) $\dfrac{|\psi_\zeta|^2}{\psi_{\zeta\bar\zeta}} \le \dfrac{C_2}{-\log\varepsilon}$ *on* $\{|\zeta| \le \varepsilon\}$;

(iii) $\dfrac{e^\psi}{|\zeta|^2 \psi_{\zeta\bar\zeta}} \le C_3$ *on* $\{\varepsilon/2 \le |\zeta| \le \varepsilon\}$.

Proof. Write $t = 2\log|\zeta|$ and let γ be such that $\psi = \gamma(t)$. That is

$$\gamma = -\log(-\delta + \log(-\delta)),$$

where $\delta = -\log(e^t + \varepsilon^2)$. We have $\psi_\zeta = \gamma'/\zeta$, $\psi_{\zeta\bar\zeta} = \gamma''/|\zeta|^2$ and thus

$$\frac{|\psi_\zeta|^2}{\psi_{\zeta\bar\zeta}} = \frac{(\gamma')^2}{\gamma''}.$$

We have to prove that

$$\left(1 - \frac{(\gamma')^2}{\gamma''}\right) \geq \frac{-\delta + \log(-\delta)}{C_1\delta^2} \qquad \text{if } \; t \leq -\log(2e) \tag{9.5}$$

$$\frac{(\gamma')^2}{\gamma''} \leq \frac{C_2}{-\log\varepsilon} \qquad \text{if } \; t \leq 2\log\varepsilon \tag{9.6}$$

$$(-\delta + \log(-\delta))\gamma'' \geq \frac{1}{C_3} \qquad \text{if } \; 2\log(\varepsilon/2) \leq t \leq 2\log\varepsilon. \tag{9.7}$$

We can compute that

$$\gamma' = \frac{1 - \delta^{-1}}{-\delta + \log(-\delta)}\delta'$$

and

$$\gamma'' \geq \frac{1 - \delta^{-1}}{-\delta + \log(-\delta)}\delta''.$$

Therefore we get (9.7) and since

$$\frac{(\gamma')^2}{\gamma''} \leq \frac{1 - \delta^{-1}}{-\delta + \log(-\delta)}\frac{(\delta')^2}{\delta''} = \frac{1 - \delta^{-1}}{-\delta + \log(-\delta)}e^t,$$

we also obtain (9.5) and (9.6). □

End of proof of Theorem 9.3. It is no loss of generality to assume that $\Omega \subset \mathbb{C}^{n-1} \times \Delta_r$ where $r = (2e)^{-1/2}$. Defining ψ_ε as in Lemma 9.4 we see by (ii) that (9.4) is satisfied with $a(\varepsilon) = -C_2/\log\varepsilon \to 0$ as $\varepsilon \to \infty$. For a fixed $\varepsilon > 0$ the function $(1 - |\bar\partial\psi_\varepsilon|^2_{i\partial\bar\partial\psi_\varepsilon})e^{\psi_\varepsilon - \widetilde\varphi}$ is not integrable near $\{z_n = 0\}$ and therefore by (9.3) u_ε vanishes there. It follows that $F_\varepsilon := f v_\varepsilon - u_\varepsilon$ is a holomorphic extension of f to Ω. Combining (9.3) with (i) and (iii) we obtain

$$\int_\Omega \frac{|u|^2}{|z_n|^2\log^2(|z_n|^2 + \varepsilon^2)}e^{-\varphi}d\lambda \leq C\int_{\Omega'}|f|^2e^{-\varphi}d\lambda',$$

where C is independent of ε. We immediately obtain (9.2), even for a small fixed $\varepsilon > 0$. □

In fact, as in [22] we have obtained a slightly better estimate than (9.2):

$$\int_\Omega \frac{|F|^2}{|z_n|^2\log^2(2|z_n|)}e^{-\varphi}d\lambda \leq C\int_{\Omega'}|f|^2e^{-\varphi}d\lambda'.$$

It was proved earlier by McNeal-Varolin [51]. We also see that the $\bar\partial$-estimate from Theorem 9.2 essentially reduced the proof of the Ohsawa-Takegoshi theorem to an elementary ODE problem. Pushing these ideas further, it was shown in [12] that the optimal constant in (9.2) is $C = \pi$. This method also gave the original proof of the Suita conjecture, Theorem 8.7.

In the next section we will present various applications of the Ohsawa-Takegoshi theorem to singularities of psh functions. In fact, the original motivation behind this theorem was the following lower bound for the Bergman kernel:

Theorem 9.5. *If Ω is a bounded pseudoconvex domain with C^2 boundary then*

$$K_\Omega \geq \frac{1}{C\delta_\Omega^2} \tag{9.8}$$

for some positive constant C.

Proof. We can find $r > 0$ such that for every $w \in \partial\Omega$ there exists $w^* \notin \Omega$ such that $\bar{\Omega} \cap \bar{B}(w^*, r) = \{w\}$. For $z \in \Omega$ choose $w \in \partial\Omega$ such that $|z - w| = \delta_\Omega(z)$. Then z, w and w^* lie on one complex line H. By the Ohsawa-Takegoshi theorem we have $K_{\Omega'} \leq CK_\Omega$, where $\Omega' = \Omega \cap H$, and the problem is reduced to dimension one. Then

$$K_{\Omega'}(z) \geq K_{\mathbb{C}\setminus\bar{\Delta}_r}(r + |z - w|) = \frac{r^2}{\pi|z - w|^2(2r + |z - w|)^2}$$

and the estimate follows. $\square$

One can easily check that the exponent 2 in (9.8) is optimal: consider for example pseudoconvex Ω with smooth boundary such that $B'_r \times \Delta \subset \Omega \subset B'_R \times \Delta$, where B'_r denotes the ball in $\mathbb{C}^{n-1}$ centered at the origin with radius r. Previously, Pflug [56], using the Hörmander estimate directly, proved such an estimate with exponent arbitrarily smaller than 2.

10. Singularities of Plurisubharmonic Functions

The following recent result of Berndtsson [6] solved the so-called *openness conjecture* of Demailly-Kollàr [26]. Its proof presented here, taken from [7] based on ideas of Guan-Zhou [31], is a remarkable application of the Ohsawa-Takegoshi theorem. The result for $n = 2$ was proved earlier by Favre-Jonsson [30].

Theorem 10.1. *Let φ be a psh function defined in a neighbourhood of $z_0 \in \mathbb{C}^n$. Then the set of $p \in \mathbb{R}$ such that $e^{-p\varphi}$ is integrable near z_0 is an open integral of the form $(-\infty, p_0)$.*

Proof. It will be an induction on n. For $n = 1$ it follows from the following:

Exercise 6. *For a subharmonic φ the function $e^{-\varphi}$ is integrable near z_0 if and only if $\Delta\varphi(\{z_0\}) < 4\pi$.*

Note that if μ is a positive measure with compact support in $\mathbb{C}$ such that $\mu = \Delta\varphi/2\pi$ near z_0 then we can write $\varphi = U^\mu + h$, where

$$U^\mu(z) = \log|z| * \mu = \int_{\mathbb{C}} \log|\zeta - z| d\mu(\zeta)$$

and h is harmonic near z_0. It is therefore enough to prove Exercise 6 for U^μ.

We may assume that z_0 is the origin, φ is defined in a neighbourhood of $\bar{\Delta}^n$ and $\varphi \le 0$. We first claim that if φ is not locally integrable near the origin then

$$\int_{\Delta^{n-1}} e^{-\varphi(\cdot, z_n)} d\lambda' \ge \frac{c_n}{|z_n|^2}, \quad |z_n| \le 1/2, \tag{10.1}$$

where c_n is a positive constant depending only on n. For a fixed z_n we may assume that the left-hand side of (10.1) is finite. By the Ohsawa-Takegoshi theorem there exists a holomorphic F in Δ^n such that $F(\cdot, z_n) = 1$ in Δ^{n-1} and

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda \le C_1 \int_{\Delta^{n-1}} e^{-\varphi(\cdot, z_n)} d\lambda' < \infty. \tag{10.2}$$

It is elementary that

$$|F(0, \zeta)|^2 \le C_2 \int_{\Delta^n} |F|^2 d\lambda \le C_2 \int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda, \quad |\zeta| \le 1/2. \tag{10.3}$$

Since $e^{-\varphi}$ is not locally integrable near the origin, by (10.2) we have $F(0, 0) = 0$, and thus by (10.3) and the Schwarz lemma

$$|F(0, \zeta)|^2 \le C_3 |\zeta|^2 \int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda, \quad |\zeta| \le 1/2.$$

For $\zeta = z_n$ using (10.2) and the fact that $F(0, z_n) = 1$ we get (10.1).

Now assume that the result is true for functions of $n - 1$ variables and suppose that

$$\int_{\Delta^n} e^{-p_0\varphi} d\lambda < \infty. \tag{10.4}$$

Since for $p > p_0$ we know that $e^{-p\varphi}$ is not locally integrable near the origin, by (10.1)

$$\int_{\Delta^{n-1}} e^{-p\varphi(\cdot, z_n)} d\lambda' \ge \frac{c_n}{|z_n|^2}, \quad |z_n| \le 1/2. \tag{10.5}$$

From (10.4) it follows that for almost all $z_n \in \Delta$

$$\int_{\Delta^{n-1}} e^{-p_0\varphi(\cdot,z_n)}d\lambda' < \infty$$

and thus by the inductive assumption for p sufficiently close to p_0

$$\int_{\Delta^{n-1}} e^{-p\varphi(\cdot,z_n)}d\lambda' < \infty.$$

The Lebesgue dominated convergence theorem now implies that (10.5) also holds for $p = p_0$ which contradics (10.4). $\square$

We had seen that psh functions are useful when proving various results on holomorphic functions. The proof of Theorem 10.1 shows that a reverse situation is also possible.

If a psh φ is defined near z_0 then its *Lelong number* is defined by

$$\nu_\varphi(z_0) = \liminf_{z\to z_0} \frac{\varphi(z)}{\log|z-z_0|} = \lim_{r\to 0^+} \frac{\varphi^r(z_0)}{\log r},$$

where

$$\varphi^r(z) := \max_{\bar B(z,r)} \varphi, \quad z \in \Omega_r.$$

Proposition 10.2. *For $\varphi \in PSH(\Omega)$ and $r > 0$ we have $\varphi^r \in PSH \cap C(\Omega_r)$. Also, $\varphi^r(z)$ is logarithmically convex in r and and decreases to $\varphi(z)$ as r decreases to 0.*

Proof. From Theorem 3.1(vii) it follows that $\varphi^r(z)$ is logarithmically convex in r and decreases to $\varphi(z)$ as r decreases to 0. It remains to prove that φ^r is continuous for a fixed r. If $z_j \to z$ and $\lambda > 1$ then for sufficiently large j we have $\bar B(z_j, r) \subset \bar B(z, \lambda r)$ and

$$\varphi^r(z_j) \le \varphi^{\lambda r}(z) \le \varphi^r(z) + \frac{\varphi^{r_0}(z) - \varphi^r(z)}{\log r_0 - \log r}\log\lambda,$$

where r_0 is such that $r < r_0 < \delta_\Omega(z)$. Similarly we can show the other bound. $\square$

Since $\varphi^r(z_0)$ is logarithmically convex in r, it follows that if $\varphi \le 0$ then $\varphi^r(z_0)/\log r$ is increasing in r. Therefore $\nu_\varphi(z_0)$ is the maximal number $c \ge 0$ such that

$$\varphi(z) \le c\log|z-z_0| + A$$

for some constant A and z in a neighbourhood of z_0. The Lelong number measures the singularity of a psh function at a point. For $n = 1$ one has $\nu_\varphi(z_0) = \Delta\varphi(\{z_0\})/2\pi$ - it is essentially equivalent to Exercise 6.

The classical result on Lelong numbers is due to Siu [58]:

Theorem 10.3. *For any psh φ defined in a bounded pseudoconvex Ω and $c \in \mathbb{R}$ the superlevel set $\{\nu_\varphi \ge c\}$ is globally analytic in Ω, that is it can be written as $\bigcap_{f\in\mathcal{F}}\{f = 0\}$ for some $\mathcal{F} \subset \mathcal{O}(\Omega)$.*

Note that if $\varphi = \log|f|$ where f is holomorphic then $\nu_\varphi(z_0) = |\beta|$ where $\beta \in \mathbb{N}^n$ is such that $f(z) = (z-z_0)^\beta h(z)$ and $h(z_0) \ne 0$. Since $\partial^\beta f(z_0) \ne 0$ and $\partial^\alpha f(z_0) = 0$ if $\alpha_j < \beta_j$ for some $j = 1, \dots, n$, it follows that

$$\{\nu_\varphi \ge c\} = \bigcap_{|\alpha|<c} \{\partial^\alpha f = 0\}.$$

Similarly, if

$$\varphi = \frac{1}{2}\log\sum_l |f_l|^2, \tag{10.6}$$

where f_l is a sequence of holomorphic functions, then

$$\{\nu_\varphi \geq c\} = \bigcap_{l,\ |\alpha|<c} \{\partial^\alpha f_l = 0\}. \tag{10.7}$$

The original proof of Theorem 10.3 in [58] was very complicated. It was later simplified and generalized by Kiselman [43], [44] (see also [36]) and Demailly [24]. It was Demailly [25] who found a surprisingly simple proof of the Siu theorem using the Ohsawa-Takegoshi theorem. It was done using the following approximation of psh functions by functions of the form (10.6):

Theorem 10.4. *Let φ be psh in a bounded pseudoconvex Ω in $\mathbb{C}^n$. For $m = 1, 2, \dots$ define*

$$\varphi_m := \frac{1}{2m}\log\sup\left\{|f|^2\colon f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 e^{-2m\varphi} d\lambda \leq 1\right\}.$$

Then there exist positive constants C_1 depending only on n and the diameter of Ω and C_2 depending only on n such that

$$\varphi - \frac{C_1}{m} \leq \varphi_m \leq \varphi^r + \frac{1}{m}\log\frac{C_2}{r^n} \quad \textit{in } \Omega_r \tag{10.8}$$

and

$$\nu_\varphi - \frac{n}{m} \leq \nu_{\varphi_m} \leq \nu_\varphi. \tag{10.9}$$

In particular, $\varphi_m \to \varphi$ pointwise and in L^1_{loc}.

Proof. By the Ohsawa-Takegoshi theorem for every $z \in \Omega$ we can find $f \in \mathcal{O}(\Omega)$ such that

$$\int_\Omega |f|^2 e^{-2m\varphi} d\lambda \leq C|f(z)|^2 e^{-2m\varphi(z)} = 1.$$

This implies that

$$\varphi_m(z) \geq \frac{1}{2m}\log|f(z)|^2 = \varphi(z) - \frac{\log C}{2m}$$

and we obtain the first inequality in (10.8). The proof of the second one is completely elementary: $|f|^2$ is in particular subharmonic and thus for $r < \operatorname{dist}(z, \partial\Omega)$

$$|f(z)|^2 \leq \frac{1}{\lambda(B(z,r))}\int_{B(z,r)} |f|^2 d\lambda \leq \frac{n!}{\pi^n r^{2n}} e^{2m\varphi^r(z)} \int_\Omega |f|^2 e^{-2m\varphi} d\lambda$$

which gives the second inequality in (10.8).

Now (10.9) easily follows from (10.8): the first inequality in (10.8) implies that $\nu_{\varphi_m} \leq \nu_{\varphi - C_1/m} = \nu_\varphi$ and the second one gives

$$\varphi_m^r \leq \varphi^{2r} + \frac{1}{m}\log\frac{C_2}{r^n},$$

hence $\nu_\varphi - n/m \leq \varphi_{n/m}$. $\square$

Proof of Theorem 10.3. By (10.9)

$$\{\nu_\varphi \geq c\} = \bigcap_m \{\nu_{\varphi_m} \geq c - \frac{n}{m}\},$$

it thus remains to prove the result for φ_m. If $\{\sigma_l\}$ is an orthonormal basis of $\mathcal{O}(\Omega) \cap L^2(\Omega, e^{-2m\varphi})$ then

$$\varphi_m = \frac{1}{2m} \log \sum_l |\sigma_l|^2 \tag{10.10}$$

and by (10.7)

$$\{\nu_{\varphi_m} \geq c - \frac{n}{m}\} = \bigcap_{\substack{l \\ |\alpha| < mc-n}} \{\partial^\alpha \sigma_l = 0\}$$

which finishes the proof. $\square$

The Ohsawa-Takegoshi theorem also gives the following subadditivity of the Demailly approximation from [27]:

Theorem 10.5. *Under the assumptions of Theorem 10.4 there exists a positive constant C depending only on n and the diameter of Ω such that*

$$(m_1 + m_2)\varphi_{m_1+m_2} \leq m_1\varphi_{m_1} + m_2\varphi_{m_2} + C. \tag{10.11}$$

In particular, the sequence $\varphi_{2^k} + C/2^{k+1}$ is decreasing.

Proof. By the Ohsawa-Takegoshi theorem for every $f \in \mathcal{O}(\Omega)$ with

$$\int_\Omega |f|^2 e^{-2(m_1+m_2)\varphi} d\lambda \leq 1$$

there exists $F \in \mathcal{O}(\Omega \times \Omega)$ such that $F(z, z) = f(z)$ for $z \in \Omega$ and

$$\iint_{\Omega\times\Omega} |F(z,w)|^2 e^{-2m_1\varphi(z) - m_2\varphi(w)} d\lambda(z) d\lambda(w) \leq \widetilde{C}. \tag{10.12}$$

Let $\{\sigma_l\}$ be an orthonormal basis in $\mathcal{O}(\Omega) \cap L^2(\Omega, e^{-2m_1\varphi})$ and $\{\sigma'_k\}$ an orthonormal basis in $\mathcal{O}(\Omega) \cap L^2(\Omega, e^{-2m_2\varphi})$. Similarly as in the proof of Proposition 5.1 we can prove that $\{\sigma_l(z)\sigma'_k(w)\}$ is an orthonormal basis in $\mathcal{O}(\Omega \times \Omega) \cap L^2(\Omega \times \Omega, e^{-2m_1\varphi(z) - 2m_2\varphi(w)})$. If

$$F(z, w) = \sum_{l,k} c_{lk}\sigma_l(z)\sigma'_k(w)$$

then $\sum_{l,k} |c_{lk}|^2 \leq \widetilde{C}$ by (10.12) and thus by the Schwarz inequality and (10.10)

$$|f(z)|^2 = |F(z, z)|^2 \leq \widetilde{C} \sum_l |\sigma_l(z)|^2 \sum_k |\sigma'_k(z)|^2 = \widetilde{C} e^{2m_1\varphi_{m_1}(z)} e^{2m_2\varphi_{m_2}(z)}.$$

This gives (10.11) with $C = \log \widetilde{C}/2$. $\square$

It was recently showed by D. Kim [42] that in general one cannot expect monotonicity of the entire sequence φ_m, even after adding a sequence of constants converging to 0.

11. Mahler Conjecture and Bourgain-Milman Inequality

Let K be a convex symmetric body in $\mathbb{R}^n$. This means that K is compact, convex, $-K = K$ and K has a non-empty interior. Note that there is a one-to-one correspondence between such objects and norms in $\mathbb{R}^n$: every such K is the unit ball of the norm given by its Minkowski functional:

$$q_K = \inf\{t > 0\colon t^{-1}x \in K\} = \sup\{x \cdot y\colon y \in K'\},$$

where

$$K' = \{y \in \mathbb{R}^n\colon x \cdot y \le 1 \text{ for all } x \in K\}$$

is the dual of K. The *Mahler volume* of K is defined by $\lambda(K)\lambda(K')$. One can easily show that it is independent of linear transformations in $\mathbb{R}^n$ and thus also on the inner product in $\mathbb{R}^n$.

The Blaschke-Santaló inequality says that the Mahler volume is maximal for balls. Still open Mahler conjecture [50] predicts that the Mahler volume is minimal for cubes. Since for $K = [-1, 1]^n$ we have

$$K' = \{x \in \mathbb{R}^n\colon |x_1| + \cdots + |x_n| \le 1\}$$

and $\lambda(K') = 2^n/n!$, it follows that the Mahler conjecture is equivalent to the following lower bound for the Mahler volume:

$$\lambda(K)\lambda(K') \ge \frac{4^n}{n!}.$$

For $n = 2$ the Mahler conjecture can be shown by approximating a convex symmetric body in $\mathbb{R}^2$ by polygons and showing that a proper modification of a polygon which reduces the number of vertices keeping the area unchanged decreases the area of the dual polygon.

It is known that the equality in the Blasche-Santaló inequality is attained only for balls (up to linear transformations). As for the Mahler conjecture, for $n = 2$ one can show that the square is the only minimizer. However, the cube $[-1, 1]^3$ cannot be the only minimizer for $n = 3$ because its dual, the octahedron, is not linearly equivalent to the cube. In general, it is conjectured that all minimizers of the Mahler volume are the so-called *Hansen-Lima bodies* [32]: for $n = 1$ these are precisely symmetric intervals and in higher dimensions they are obtained either by taking products of lower-dimensional Hansen-Lima bodies or by taking their duals.

Nazarov [52] has recently proposed a complex analytic approach to the Mahler conjecture. The first step is to express $\lambda(K')$ in terms of entire holomorphic functions using the Fourier-Laplace transform. For $u \in L^2(K')$ we consider

$$\widehat{u}(z) = \int_{K'} u(y) e^{-iz\cdot y} d\lambda(y) \in \mathcal{O}(\mathbb{C}^n).$$

By the Schwarz inequality

$$|\widehat{u}(0)|^2 = \left|\int_{K'} u\, d\lambda\right|^2 \le \lambda(K')||u||^2_{L^2(K')} = (2\pi)^{-n}\lambda(K')||\widehat{u}||^2_{L^2(\mathbb{R}^n)},$$

where the last equality follows from the Parseval formula. Since we have equality for $u \equiv 1$ on K', we get

$$\lambda(K') = (2\pi)^n \sup_{f\in\mathcal{P},\ f\not\equiv 0} \frac{|f(0)|^2}{||f||^2_{L^2(\mathbb{R}^n)}}, \tag{11.1}$$

where $\mathcal{P} := \{\widehat{u} \colon u \in L^2(K')\}$.

Proposition 11.1. *The class $\mathcal{P}$ consists precisely of those $f \in \mathcal{O}(\mathbb{C}^n)$ satisfying*

$$|f(z)| \le C e^{q_K(\operatorname{Im} z)}, \quad z \in \mathbb{C}^n, \tag{11.2}$$

for some constant C, and

$$\int_{\mathbb{R}^n} |f(x)|^2 d\lambda(x) < \infty. \tag{11.3}$$

Proof. We have $|\widehat{u}(z)| \le ||u||_{L^1(K')} e^{q_K(\operatorname{Im} z)}$ and thus every element of $\mathcal{P}$ satisfies (11.2). (11.3) follows from the Parseval formula. On the other hand, if $f \in \mathcal{O}(\mathbb{C}^n)$ satisfies (11.3) then $f = \widehat{u}$ for some $u \in L^2(\mathbb{R}^n)$. By (11.2) and the Paley-Wiener theorem we also have $\operatorname{supp} u \subset K'$. $\square$

Combining this with (11.1) we have thus obtained the following equivalent formulation of the Mahler conjecture: there exists $f \in \mathcal{O}(\mathbb{C}^n)$ satisfying (11.2) such that $f(0) = 1$ and

$$\int_{\mathbb{R}^n} |f(x)|^2 d\lambda(x) \le n!(\pi/2)^n \lambda(K).$$

The Mahler conjecture remains open. The most important lower bound for the Mahler volume is the following inequality of Bourgain-Milman [17]:

Theorem 11.2. *There exists $c > 0$ such that for any convex symmetric body K in $\mathbb{R}^n$ one has*

$$\lambda(K)\lambda(K') \ge c^n \frac{4^n}{n!}. \tag{11.4}$$

Of course the Mahler conjecture is equivalent to the Bourgain-Milman inequality with $c = 1$. The best known constant so far, $c = \pi/4$, was obtained by G. Kuperberg [48]. Following Nazarov [52] we will show the Bourgain-Milman inequality with $c = (\pi/4)^3$ using several complex variables.

Theorem 11.3. *For a convex symmetric body K by Ω denote the tube domain*

$$T_K := \operatorname{int} K + i\mathbb{R}^n.$$

Then

$$K_\Omega(0) \le \frac{n!}{\pi^n} \frac{\lambda(K')}{\lambda(K)} \tag{11.5}$$

and

$$K_\Omega(0) \ge \left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda(K))^2}. \tag{11.6}$$

In particular, (11.4) *holds with* $(\pi/4)^3$.

The upper bound (11.5) will be proved using the following general integral formula for the Bergman kernel in arbitrary convex tube domains due to Rothaus [57]:

Theorem 11.4. *For a domain D in $\mathbb{R}^n$*

$$K_{T_D}(z,w) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{(z+\bar{w})\cdot y}}{J_D(y)} d\lambda(y), \quad z,w\in T_D, \tag{11.7}$$

where

$$J_D(y) = \int_D e^{2x\cdot y} d\lambda(x), \quad y\in\mathbb{R}^n.$$

The result will easily follow from the following two lemmas:

Lemma 11.5. *Assume that $r>0$ and $x\in D$, where D is a domain in $\mathbb{R}^n$, are such that $x+r(-1,1)^n\subset D$. Then*

$$\int_{\mathbb{R}^n} \frac{e^{2x\cdot y}}{J_D(y)} d\lambda(y) \le \left(\frac{\pi}{\sqrt{8}r}\right)^{2n}. \tag{11.8}$$

Proof. With $C := x+r(-1,1)^n$ we have

$$J_D(y)\ge J_C(y) = e^{2x\cdot y}\frac{\sinh(2ry_1)}{y_1}\dots\frac{\sinh(2ry_n)}{y_n}$$

and the lemma follows since

$$\int_{-\infty}^{\infty} \frac{t}{\sinh t}\, dt = \frac{\pi^2}{4}. \qquad \square$$

Lemma 11.6. *For a domain D in $\mathbb{R}^n$ the mapping*

$$L^2(\mathbb{R}^n, J_D)\ni u\longmapsto \widetilde{u}\in A^2(T_D), \tag{11.9}$$

where

$$\widetilde{u}(z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} u(y)e^{z\cdot y} d\lambda(y),$$

is an isomorphism of the Hilbert spaces.

Proof. For $u\in L^2(\mathbb{R}^n, J_D)$ by Lemma 11.5 the integral is convergent and thus $\widetilde{u}$ is holomorphic in T_D. It also follows that $h(y) := (2\pi)^{-n/2}u(y)e^{\operatorname{Re} z\cdot y}\in L^2(\mathbb{R}^n)$ and we can write $\widetilde{u}(z) = \widehat{h}(-\operatorname{Im} z)$. By the Parseval formula and the Fubini theorem

$$||\widetilde{u}||^2_{L^2(T_D)} = \int_K\int_{\mathbb{R}^n} |u(y)|^2 e^{2x\cdot y} d\lambda(y)d\lambda(x) = ||u||^2_{L^2(\mathbb{R}^n, J_D)}.$$

It remains to prove that the mapping (11.9) is onto. For $f\in A^2(T_D)$ approximating D by relatively compact subsets from inside and using the fact that $|f|^2$ is subharmonic we may assume that f is bounded in T_D. Multiplying f by functions of the form $e^{\varepsilon z\cdot z}$ we may also assume that f satisfies the estimate

$$|f(z)|\le Me^{-\varepsilon|\operatorname{Im} z|^2} \tag{11.10}$$

for some positive constants M and ε. For a fixed $x\in D$ with the notation $f_x(\eta) = f(x+i\eta)$, using the fact that $g(\eta) = (2\pi)^{-n}\widehat{\widehat{g}}(-\eta)$, we have $f(x+i\eta) = \widetilde{u_x}(x+i\eta)$ where $u_x(\xi) =$

$(2\pi)^{-n/2}\widehat{f}_x(\xi)e^{-x\cdot\xi}$. We have to prove that $u_x(\xi)$ is independent of x. From (11.10) it follows that we can differentiate under the sign of integration

$$\frac{\partial}{\partial x_j}\int_{\mathbb{R}^n} f(x+iy)e^{-(x+iy)\cdot\xi}d\lambda(y) = \int_{\mathbb{R}^n}\left(\frac{\partial f}{\partial x_j}(x+iy) - \xi_j f(x+iy)\right)e^{-(x+iy)\cdot\xi}d\lambda(y).$$

We have $\partial f/\partial x_j = -i\partial f/\partial y_j$ and by (11.10) we can integrate by parts. Therefore

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_j}(x+iy)e^{-(x+iy)\cdot\xi}d\lambda(y) &= -i\int_{\mathbb{R}^n}\frac{\partial f}{\partial y_j}(x+iy)e^{-(x+iy)\cdot\xi}d\lambda(y)\\ &= \int_{\mathbb{R}^n}\xi_j f(x+iy)e^{-(x+iy)\cdot\xi}d\lambda(y)\end{aligned},$$

hence $u_x(\xi)$ is independent of x and the mapping (11.9) is onto. □

Proof of Theorem 11.4. By $K(z,w)$ denote the right-hand side of (11.7) and fix $w\in T_D$. Then, with the notation of Lemma 11.6, we have $K(\cdot,w) = (2\pi)^{-n/2}\widetilde{v}$, where

$$v(y) = \frac{e^{\bar{w}\cdot y}}{J_D(y)} \in L^2(\mathbb{R}^n, J_D)$$

by Lemma 11.5. It follows from Lemma 11.6 that $K(\cdot,w)\in A^2(T_D)$ and to finish the proof we have to show that it has the reproducing property. For $f=\widetilde{u}\in A^2(T_D)$ where $u\in L^2(\mathbb{R}^n,J_D)$ by Lemma 11.6

$$\langle f, K(\cdot,w)\rangle_{A^2(T_D)} = (2\pi)^{-n/2}\langle\widetilde{u},\widetilde{v}\rangle_{A^2(T_D)} = (2\pi)^{-n/2}\langle u,v\rangle_{L^2(\mathbb{R}^n,J_D)} = f(w).$$ □

Proof of Theorem 11.3. We first show the upper bound (11.5). For $y\in\mathbb{R}^n$ and $\widetilde{x}\in K$, since $K\supset(\widetilde{x}+K)/2$,

$$J_K(y) \geq \frac{1}{2^n}\int_K e^{(\widetilde{x}+x)\cdot y}d\lambda(x) \geq \frac{\lambda(K)}{2^n}e^{\widetilde{x}\cdot y}$$

and therefore

$$J_K(y) \geq \frac{\lambda(K)}{2^n}e^{q_{K'}(y)}.$$

Since by the Fubini theorem

$$\int_{\mathbb{R}^n} e^{-q_{K'}}d\lambda = \int_{\mathbb{R}^n}\int_{q_{K'}(y)}^{\infty} e^{-t}dt\,d\lambda(y) = \int_0^\infty e^{-t}\lambda(\{q_{K'}<t\})dt = n!\lambda(K'),$$

from (11.7) with $D=\operatorname{int}K$ and $z=w=0$ we get (11.5).

To prove the lower bound (11.6) we will use Theorem 8.6. Let Φ be a conformal mapping from the strip $\{|\operatorname{Re}\zeta|<1\}$ to Δ such that $\Phi(0)=0$, so that in particular $|\Phi'(0)|=4/\pi$. For a fixed $y\in K'$ set $u(z):=\log|\Phi(z\cdot y)|$. Then $u\in\mathcal{B}_{\Omega,0}$ and thus $u\leq G:=G_\Omega(\cdot,0)$. Therefore for $t<0$

$$\{G<t\}\subset\{z\in\Omega\colon |\Phi(z\cdot y)|<e^t\}\subset\{z\in\Omega\colon |z\cdot y|\leq\big(4/\pi+\varepsilon(t)\big)e^t\},$$

where ε is such that $\varepsilon(t)\to 0$ as $t\to-\infty$. Since $K''=K$, we conclude that

$$\{G<t\}\subset\big(4/\pi+\varepsilon(t)\big)e^t(K+iK)$$

and (11.6) follows from (8.8) as $t\to-\infty$. □

Open Problem 6. *If* $\Omega = T_K$, *where* K *is a convex symmetric body in* $\mathbb{R}^n$, *then*

$$K_\Omega(0) \geq \left(\frac{\pi}{4}\right)^n \frac{1}{(\lambda(K))^2}.$$

Note that this would be optimal since for $K = [-1,1]^n$ one has

$$K_\Omega(0) = \big(K_{\{|\mathrm{Re}\,\zeta|<1\}}(0)\big)^n = \big(|(\Phi^{-1})'(0)|^2 K_\Delta(0)\big)^n = (\pi/16)^n,$$

where Φ is as in the proof of Theorem 11.3.

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