Chapter 5
The Calabi–Yau Theorem

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Abstract This lecture, based on a course given by the author at Toulouse in January 2005, surveys the proof of Yau's celebrated solution to the Calabi conjecture, through the solvability of inhomogeneous complex Monge–Ampère equations on compact Kähler manifolds.

5.1 Introduction

Our main goal is to present a complete proof of the Calabi–Yau theorem [Yau78] (Theorem 5.3 below). In Sect. 5.2 we collect basic notions of the Kähler geometry (proofs can be found for example in [KN69]). We then formulate the Calabi conjecture and reduce it to solving a Monge–Ampère equation. Kähler–Einstein metrics are also briefly discussed. In Sect. 5.3 we prove the uniqueness of solutions and reduce the proof of existence to a priori estimates using the continuity method and Schauder theory. Since historically the uniform estimate has caused the biggest problem, we present two different proofs of this estimate in Sect. 5.4. The first is the classical simplification of the Yau proof due to Kazdan, Aubin and Bourguignon and its main tool is the Moser iteration technique. The second is essentially due to Kolodziej and is more in the spirit of pluripotential theory. In Sect. 5.5 we show the estimate for the mixed second order complex derivatives of solutions which can also be applied in the degenerate case. The $C^{2,\alpha}$ estimate can be proved

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locally using general Evans–Krylov–Trudinger theory coming from (real) fully nonlinear elliptic equations. This is done in Sect. 5.6. Finally, in Sect. 5.7 we study a corresponding Dirichlet problem for weak (continuous) solutions.

We concentrate on the PDE aspects of the subject, whereas the geometric problems are presented only as motivation. In particular, without much more effort we could also solve the Monge–Ampère equation (5.9) below for $\lambda < 0$ and thus prove the existence of the Kähler–Einstein metric on compact complex manifolds with negative first Chern class.

We try to present as complete proofs as possible. We assume that the reader is familiar with main results from the theory of linear elliptic equations of second order with variable coefficients (as covered in [GT83, Part I]) and basic theory of functions and forms of several complex variables. Good general references are [Aub98, Dembook, GT83, KN69], whereas the lecture notes [Siu87] and [Tianbook] (as well as [Aub98]) cover the subject most closely. In Sect. 6 we assume the Bedford–Taylor theory of the complex Monge–Ampère operator in $\mathbb{C}^n$ but in fact all the results of that part are proved by means of certain stability estimates that are equally difficult to show for smooth solutions.

When proving an a priori estimate by $C_1, C_2, \dots$ we will denote constants which are as in the hypothesis of this estimate and call them *under control*.

5.2 Basic Concepts of Kähler Geometry

In this section we collect the basic notions of Kähler geometry. Let M be a complex manifold of dimension n. By TM denote the (real) tangent bundle of M - it is locally spanned over $\mathbb{R}$ by $\partial/\partial x_j, \partial/\partial y_j$, $j = 1, \dots, n$. The complex structure on M defines the endomorphism J of TM given by $J(\partial/\partial x_j) = \partial/\partial y_j$, $J(\partial/\partial y_j) = -\partial/\partial x_j$. Every hermitian form on M

$$\omega(X, Y) = \sum_{i,j=1}^{n} g_{i\bar{j}} X_i \overline{Y}_j, \quad X, Y \in TM,$$

we can associate with a real (this means that $\omega = \overline{\omega}$) (1,1)-form

$$2\sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j \tag{5.1}$$

(it is easy to check that they are transformed in the same way under a holomorphic change of coordinates). If ω is positive then $\widetilde{\omega} := \operatorname{Re}\omega$ is the Riemannian form on M. Let ∇ be the Levi–Civita connection defined by $\widetilde{\omega}$ - it is the unique torsion-free connection satisfying $\nabla\widetilde{\omega} = 0$, that is

$$(\nabla_X\widetilde{\omega})(Y,Z) = \widetilde{\omega}(\nabla_X Y, Z) + \widetilde{\omega}(Y, \nabla_X Z) - X\widetilde{\omega}(Y,Z) = 0, \quad X, Y, Z \in TM.$$

One can show that for a hermitian manifold (M,ω)

$$d\omega = 0 \;\Leftrightarrow\; \nabla\omega = 0 \;\Leftrightarrow\; \nabla J = 0. \tag{5.2}$$

Hermitian forms ω satisfying equivalent conditions (5.2) are called *Kähler*. This means that the complex structure of M is compatible with the Riemannian structure given by ω. Manifold M is called *Kähler* if there exists a Kähler form on M.

We shall use the operators $\partial, \overline{\partial}$, so that $d = \partial + \overline{\partial}$ and $2\sqrt{-1}\partial\overline{\partial} = dd^c$, where $d^c := \sqrt{-1}(\overline{\partial} - \partial)$.

Proposition 5.1 *Let ω be a closed, real (1,1) form on M. Then locally $\omega = dd^c\eta$ for some smooth η.*

Proof. Locally we can find a real 1-form γ such that $\omega = d\gamma$. We may write $\gamma = \alpha + \beta$, where α is a (1,0)-form and β a (0,1)-form. We have $\overline{\alpha} = \beta$, since γ is real. Moreover,

$$\omega = (\partial + \overline{\partial})(\alpha + \beta) = \partial\alpha + \overline{\partial}\alpha + \partial\beta + \overline{\partial}\beta,$$

and thus $\partial\alpha = 0$, $\overline{\partial}\beta = 0$, since ω is a (1,1)-form. Then locally we can find a complex-valued, smooth function f with $\beta = \overline{\partial} f$ and

$$\omega = \partial\overline{\beta} + \overline{\partial}\beta = dd^c(\operatorname{Im} f).$$

□

The condition $d\omega = 0$ reads

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \quad i, j, k = 1, \dots, n,$$

and by Proposition 5.1 this means that locally we can write $\omega = dd^c g$ for some smooth, real-valued g. We will use the notation $f_i = \partial f/\partial z_i$, $f_{\bar{j}} = \partial f/\partial \overline{z}_j$, it is then compatible with (5.1). If ω is Kähler then g is strongly plurisubharmonic (shortly psh). From now on we assume that ω is a Kähler form and g is its local potential.

By $T_{\mathbb{C}}M$ denote the complexified tangent bundle of M - it is locally spanned over $\mathbb{C}$ by $\partial_j := \partial/\partial z_j, \partial_{\bar{j}} := \partial/\partial \overline{z}_j$, $j = 1, \dots, n$. Then J, ω and ∇ can be uniquely extended to $T_{\mathbb{C}}M$ in a $\mathbb{C}$-linear way. One can check that

$$\begin{gathered}
J(\partial_j) = \sqrt{-1}\partial_j, \quad J(\partial_{\bar{j}}) = -\sqrt{-1}\partial_{\bar{j}}, \\
\omega(\partial_i, \partial_j) = \omega(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0, \quad \omega(\partial_i, \partial_{\bar{j}}) = g_{i\bar{j}}, \\
\nabla_{\partial_i}\partial_{\bar{j}} = \overline{\nabla_{\partial_{\bar{i}}}\partial_j} = 0, \quad \nabla_{\partial_i}\partial_j = \overline{\nabla_{\partial_{\bar{i}}}\partial_{\bar{j}}} = g^{k\bar{l}} g_{i\bar{l}j}\partial_k,
\end{gathered} \tag{5.3}$$

where $(g^{k\bar{l}})$ is the inverse transposed to $(g_{i\bar{j}})$, that is

$$g^{k\bar{l}}g_{j\bar{l}} = \delta_{jk}. \tag{5.4}$$

We have the following curvature tensors from Riemannian geometry

$$\begin{aligned} R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,\\ R(X,Y,W,Z) &= \omega(R(X,Y)Z,W),\\ Ric(Y,Z) &= \mathrm{tr}\{X \longmapsto R(X,Y)Z\}. \end{aligned}$$

One can then show that

$$R_{i\bar{j}k\bar{l}} = R(\partial_i, \partial_{\bar{j}}, \partial_k, \partial_{\bar{l}}) = -g_{i\bar{j}k\bar{l}} + g^{p\bar{q}}g_{p\bar{j}k}g_{i\bar{l}\bar{q}}$$

$$Ric_{k\bar{l}} = Ric(\partial_k, \partial_{\bar{l}}) = g^{i\bar{j}}R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l}\log\det(g_{i\bar{j}}). \tag{5.5}$$

Since this is the moment where the Monge–Ampère operator appears in complex geometry, let us have a look at the last equality. Let D, Q be any linear first order differential operators with constant coefficients. Then

$$Q\log\det(g_{i\bar{j}}) = \frac{a^{i\bar{j}}Qg_{i\bar{j}}}{\det(g_{i\bar{j}})} = g^{i\bar{j}}Qg_{i\bar{j}}, \tag{5.6}$$

where $(a^{i\bar{j}})$ is the (transposed) adjoint matrix of $(g_{i\bar{j}})$. Differentiating (5.4) we get

$$Dg^{i\bar{j}} = -g^{i\bar{q}}g^{p\bar{j}}Dg_{p\bar{q}},$$

thus

$$DQ\log\det(g_{i\bar{j}}) = g^{i\bar{j}}DQg_{i\bar{j}} - g^{i\bar{q}}g^{p\bar{j}}Dg_{p\bar{q}}Qg_{i\bar{j}}, \tag{5.7}$$

and (5.5) follows.

The (real) Laplace–Beltrami operator of a smooth function u is defined as the trace of $X \longmapsto \nabla_X\nabla u$, where $\widetilde{\omega}(X,\nabla u) = Xu$, $X \in TM$. In the complex case it is convenient to define this operator as the double of the real one – then

$$\Delta u = g^{i\bar{j}}u_{i\bar{j}}$$

and

$$dd^c u \wedge \omega^{n-1} = \frac{1}{n}\Delta u\, \omega^n.$$

The form ω^n will be the volume form for us (in fact, it is $4^n n!$ times the standard volume form) and we will denote $V := \mathrm{vol}\,(M) = \int_M \omega^n$. Note that the local formulas for the quantities we have considered (the Christoffel symbols (5.3), the curvature tensors, the Laplace–Beltrami operator) are simpler in the Kähler case than in the real Riemannian case. It will also be convenient to use the notation $R_\omega = -dd^c \log\det(g_{i\bar{j}})$ (= $2Ric_\omega$ by (5.5)).

The formula (5.5) has also the following consequence: if $\widetilde{\omega}$ is another Kähler form on M then $R_\omega - R_{\widetilde{\omega}} = dd^c\eta$, where η is a globally defined function (this easily follows from Proposition 5.1), and thus R_ω, $R_{\widetilde{\omega}}$ are cohomologous (we write $R_\omega \sim R_{\widetilde{\omega}}$). The cohomology class of R_ω is precisely $c_1(M)$, the first Chern class of M, which does not depend on ω but only on the complex structure of M.

The so called *dd^c-lemma* says that in the compact case every d-exact (1,1)-form is dd^c-exact:

Lemma 5.2 *Let α be a real, d-exact (1,1)-form on a compact Kähler manifold M. Then there exists $\eta \in C^\infty(M)$ such that $\alpha = dd^c\eta$.*

Proof. Write $\alpha = d\beta$ and let ω be a Kähler form on M. Let η be the solution of the following Poisson equation

$$dd^c\eta \wedge \omega^{n-1} = \alpha \wedge \omega^{n-1}.$$

(This equation is solvable since $\int_M \alpha \wedge \omega^{n-1} = \int_M d(\beta \wedge \omega^{n-1}) = 0$.) Define $\gamma := \beta - d^c\eta$. We then have $d\gamma \wedge \omega^{n-1} = 0$ and we have to show that $d\gamma = 0$. For this we will use the Hodge theory. Note that

$$\int_M \langle d\gamma, d\gamma\rangle dV = \int_M \langle \gamma, d^*d\gamma\rangle dV,$$

it is therefore enough to show that $d^*d\gamma = 0$. From now on the argument is local: by Proposition 5.1 we may write $d\gamma = dd^c h$ and $dd^c h \wedge \omega^{n-1} = 0$ is equivalent to $d^*dh = 0$. We then have

$$d^*d\gamma = d^*dd^c h = -d^*d^c dh = d^c d^* dh = 0,$$

where we have used the equality

$$d^*d^c + d^c d^* = 0$$

(see e.g. [Dembook]). □

From now on, we always assume that M is a **compact** manifold of dimension $n \geq 2$ and ω a Kähler form with local potential g.

Calabi conjecture. [Cal56] Let $\widetilde{R}$ be a (1,1) form on M cohomologous to R_ω. Then we ask whether there exists another Kähler form $\widetilde{\omega} \sim \omega$ on M such that $\widetilde{R} = R_{\widetilde{\omega}}$. In other words, the problem is if every form representing $c_1(M)$ is the Ricci form of a certain Kähler metric on M coming from one cohomology class.

By the dd^c-lemma we have $R_\omega = \widetilde{R} + dd^c\eta$ for some $\eta \in C^\infty(M)$. We are thus looking for $\varphi \in C^\infty(M)$ such that in local coordinates $(\varphi_{i\bar{j}} + g_{i\bar{j}}) > 0$ and

$$dd^c(\log\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) - \log\det(g_{i\bar{j}}) - \eta) = 0.$$

However, $\log\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) - \log\det(g_{i\bar{j}}) - \eta$ is globally defined, and since it is pluriharmonic on a compact manifold, it must be constant. This means that

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{c+\eta}\det(g_{i\bar{j}}),$$

which is equivalent to

$$(\omega + dd^c\varphi)^n = e^{c+\eta}\omega^n.$$

Since $(\omega + dd^c\varphi)^n - \omega^n$ is exact, from the Stokes theorem we infer

$$\int_M (\omega + dd^c\varphi)^n = V,$$

and thus the constant c is uniquely determined. Therefore, solving the Calabi conjecture is equivalent to solving the following Dirichlet problem for the complex Monge–Ampère operator on M.

Theorem 5.3 [Yau78] *Let $f \in C^\infty(M)$, $f > 0$, be such that $\int_M f\omega^n = V$. Then there exists, unique up to a constant, $\varphi \in C^\infty(M)$ such that $\omega + dd^c\varphi > 0$ and*

$$(\omega + dd^c\varphi)^n = f\omega^n. \tag{5.8}$$

Kähler–Einstein metrics. A Kähler form $\widetilde{\omega}$ is called *Kähler–Einstein* if $R_{\widetilde{\omega}} = \lambda\widetilde{\omega}$ for some $\lambda \in \mathbb{R}$. Since $\lambda\widetilde{\omega} \in c_1(M)$, it follows that a necessary condition for a complex manifold M to posses a Kähler–Einstein metric is that either $c_1(M) < 0$, $c_1(M) = 0$ or $c_1(M) > 0$, that is there exists an element in $c_1(M)$ which is either negative, zero or positive. In such a case we can always find a Kähler form ω on M with $\lambda\omega \in c_1(M)$, that is $R_\omega = \lambda\omega + dd^c\eta$ for some $\eta \in C^\infty(M)$, since M is compact. We then look for $\varphi \in C^\infty(M)$ such that $\widetilde{\omega} := \omega + dd^c\varphi > 0$ (from the solution of the Calabi conjecture we know that $c_1(M) = \{R_{\widetilde{\omega}} : \widetilde{\omega} \sim \omega\}$, so we only have to look for Kähler forms that are cohomologous to the given ω) and $R_{\widetilde{\omega}} = \lambda\widetilde{\omega}$, which, similarly as before, is equivalent to

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi+\eta+c}\omega^n. \tag{5.9}$$

To find a Kähler–Einstein metric on M we thus have to find an *admissible* (that is $\omega + dd^c\varphi \geq 0$) solution to (5.9) (for some constant c).

If $c_1(M) = 0$ then $\lambda = 0$ and the solvability of (5.9) is guaranteed by Theorem 5.3. If $c_1(M) < 0$ one can solve the equation (5.9) in a similar way as (5.8). In fact, the uniform estimate for (5.9) with $\lambda < 0$ is very simple (see [Aub76], [Yau78, p. 379], and Exercise 5.9 below) and in this case, (5.9) was independently solved by Aubin [Aub76]. The case $c_1(M) > 0$ is the most difficult and it turns out that only the uniform estimate is the problem. There was a big progress in this area in the last 20 years (especially thanks to G. Tian) and, indeed, there are examples of compact manifolds with positive first Chern class not admitting a Kähler–Einstein metric. We refer to [Tianbook] for details and further references.

5.3 Reduction to A Priori Estimates

The uniqueness in Theorem 5.3 is fairly easy.

Proposition 5.4 [Cal55] *If $\varphi, \psi \in C^2(M)$ are such that $\omega + dd^c\varphi > 0$, $\omega + dd^c\psi \geq 0$ and $(\omega + dd^c\varphi)^n = (\omega + dd^c\psi)^n$ then $\varphi - \psi = const.$*

Proof. We have

$$0 = (\omega + dd^c\varphi)^n - (\omega + dd^c\psi)^n = dd^c(\varphi - \psi) \wedge T,$$

where

$$T = \sum_{j=0}^{n-1} (\omega + dd^c\varphi)^j \wedge (\omega + dd^c\psi)^{n-1-j}$$

is a positive, closed $(n-1, n-1)$-form. Integrating by parts we get

$$0 = \int_M (\psi - \varphi)((\omega + dd^c\varphi)^n - (\omega + dd^c\psi)^n) = \int_M d(\varphi - \psi) \wedge d^c(\varphi - \psi) \wedge T$$

and we conclude that $D(\varphi - \psi) = 0$. □

In subsequent sections we will show the following a priori estimate: there exists $\alpha \in (0, 1)$ and $C > 0$, depending only on M and on upper bounds for $||f||_{1,1}$ and $1/\inf_M f$, such that for any admissible solution $\varphi \in C^4(M)$ of (5.8) satisfying the normalization condition $\int_M \varphi\omega^n = 0$ we have

$$||\varphi||_{2,\alpha} \leq C, \tag{5.10}$$

where we use the following notation: in any chart $U \subset M$

$$||\varphi||_{C^{k,\alpha}(U)} := \sum_{0 \leq j \leq k} \sup_U |D^j\varphi| + \sup_{x,y \in U, x \neq y} \frac{|D^k\varphi(x) - D^k\varphi(y)|^\alpha}{|x - y|}$$

and $||\varphi||_{k,\alpha} := \sum_i ||\varphi||_{C^{k,\alpha}(U_i)}$ for a fixed finite atlas $\{U_i\}$ (for any two such atlases the obtained norms will be equivalent). In this convention

$$||f||_{k,1} = \sum_{0\le j\le k+1} \sup_M |D^j f|.$$

The aim of this section is to reduce the proof of Theorem 5.3 to showing the estimate (5.10). It will be achieved using the continuity method (which goes back to Bernstein) and the Schauder theory for linear elliptic equations of second order.

Continuity method. Fix arbitrary integer $k \ge 2$, $\alpha \in (0,1)$ and let f be as in Theorem 5.3. By S we denote the set of $t \in [0,1]$ such that we can find admissible $\varphi_t \in C^{k+2,\alpha}(M)$ solving

$$(\omega + dd^c\varphi_t)^n = (tf + 1 - t)\omega^n$$

and such that $\int_M \varphi_t\omega^n = 0$. It is clear that $0 \in S$ and if we show that $1 \in S$ then we will have a $C^{k+2,\alpha}$ solution of (5.8). It will be achieved if we prove that S is open and closed in $[0,1]$.

The complex Monge–Ampère operator $\mathcal{N}$, determined by

$$(\omega + dd^c\varphi)^n = \mathcal{N}(\varphi)\,\omega^n,$$

in local coordinates given by

$$\mathcal{N}(\varphi) = \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})}$$

smoothly maps the set

$$\mathcal{A} = \left\{\varphi \in C^{k+2,\alpha}(M) : \omega + dd^c\varphi > 0, \int_M \varphi\omega^n = 0\right\}$$

to

$$\mathcal{B} = \left\{\widetilde{f} \in C^{k,\alpha}(M) : \int_M \widetilde{f}\omega^n = \int_M \omega^n\right\}.$$

Then $\mathcal{A}$ is an open subset of the Banach space

$$\mathcal{E} = \left\{\eta \in C^{k+2,\alpha}(M) : \int_M \eta\omega^n = 0\right\}$$

and $\mathcal{B}$ is a hyperplane of the Banach space $C^{k+2,\alpha}(M)$ with the tangent space

$$\mathcal{F} = \left\{\widetilde{f} \in C^{k,\alpha}(M) : \int_M \widetilde{f}\omega^n = 0\right\}.$$

We want to show that for every $\varphi \in \mathcal{A}$ the differential $D\mathcal{N}(\varphi) : \mathcal{E} \to \mathcal{F}$ is an isomorphism. For $\eta \in \mathcal{E}$, denoting $\widetilde{\omega} = \omega + dd^c\varphi$, we have

$$D\mathcal{N}(\varphi).\eta = \frac{d}{dt}\mathcal{N}(\varphi + t\eta)|_{t=0} = \frac{\det(\widetilde{g}_{i\bar{j}})}{\det(g_{i\bar{j}})}\widetilde{g}^{i\bar{j}}\eta_{i\bar{j}} = \mathcal{N}(\varphi)\widetilde{\Delta}\eta.$$

It immediately follows that $D\mathcal{N}(\varphi)$ is injective. From the real theory on compact Riemannian manifolds it is known that the Laplace–Beltrami operator bijectively maps

$$\left\{\eta \in C^{k+2,\alpha}(M) : \int_M \eta = 0\right\} \longrightarrow \left\{\widetilde{f} \in C^{k,\alpha}(M) : \int_M \widetilde{f} = 0\right\}$$

(see e.g. [Aub98, Theorem 4.7]). This, applied to $(M, \widetilde{\omega})$, implies that $D\mathcal{N}(\varphi)$ is indeed surjective, and thus an isomorphism. Therefore $\mathcal{N}$ is locally invertible and in particular $\mathcal{N}(\mathcal{A})$ is open in $\mathcal{B}$, and S is open in $[0, 1]$.

If we knew that the set $\{\varphi_t : t \in S\}$ is bounded in $C^{k+2,\alpha}(M)$ then from its every sequence, by the Arzela–Ascoli theorem, we could choose a subsequence whose all partial derivatives of order $\leq k + 1$ converged uniformly. Thus, to show that S is closed, we need an a priori estimate

$$||\varphi||_{k+2,\alpha} \leq C \tag{5.11}$$

for the solutions of (5.8). We now sketch how to use (locally) the Schauder theory to show that (5.10) implies (5.11).

Schauder theory. We first analyze the complex Monge–Ampère operator

$$F(D^2u) = \det(u_{i\bar{j}})$$

for smooth psh functions u – we see that the above formula defines the real operator of second order. It is elliptic if the $2n \times 2n$ real symmetric matrix $A := (\partial F/\partial u_{pq})$ (here by u_{pq} we denote the elements of the real Hessian D^2u) is positive. Matrix A is determined by

$$\frac{d}{dt}F(D^2u + tB)|_{t=0} = \mathrm{tr}(AB^T).$$

Exercise 5.5 Show that

$$\lambda_{min}(\partial F/\partial u_{pq}) = \frac{\det(u_{i\bar{j}})}{4\lambda_{max}(u_{i\bar{j}})}, \quad \lambda_{max}(\partial F/\partial u_{pq}) = \frac{\det(u_{i\bar{j}})}{4\lambda_{min}(u_{i\bar{j}})},$$

where $\lambda_{min}A$, resp. $\lambda_{max}A$, denotes the minimal, resp. maximal, eigenvalue of A.

Thus the operator F is elliptic (in the real sense) for smooth strongly psh functions and in our case when (5.10) is satisfied (then Δu is under control and hence so are the complex mixed derivatives $u_{i\bar{j}}$) is even uniformly elliptic, that is

$$|\zeta|^2/C \le \sum_{p,q=1}^{2n} \partial F/\partial u_{pq}\zeta_p\zeta_q \le C|\zeta|^2, \quad \zeta \in \mathbb{C}^n = \mathbb{R}^{2n}$$

for some uniform constant C. We can now apply the standard elliptic theory (see [GT83, Lemma 17.16] for details) to the equation

$$F(D^2u) = f.$$

For a fixed unit vector ζ and small $h > 0$ we consider the difference quotient

$$u^h(x) = \frac{u(x+h\zeta) - u(x)}{h}$$

and

$$a_h^{pq} = \int_0^1 \frac{\partial F}{\partial u_{pq}}\big(tD^2u(x+h\zeta) + (1-t)D^2u(x)\big)dt.$$

Then

$$a_h^{pq}(x)u_{pq}^h(x) = \frac{1}{h}\int_0^1 \frac{d}{dt}F\big(tD^2u(x+h\zeta) + (1-t)D^2u(x)\big)dt = f^h(x).$$

From the Schauder theory for linear elliptic equations with variable coefficients we then infer (all corresponding estimates are uniform in h)

$$u \in C^{2,\alpha} \implies a_h^{pq} \in C^{0,\alpha} \overset{\text{Schauder}}{\implies} u^h \in C^{2,\alpha} \implies u \in C^{3,\alpha} \implies \dots$$

Coming back to our equation (5.8) for $k \ge 1$ we thus get

$$\varphi \in C^{2,\alpha},\ f \in C^{k,\alpha} \implies \varphi \in C^{k+2,\alpha}$$

and

$$||\varphi||_{k+2,\alpha} \le C,$$

where $C > 0$ depends only on M and on upper bounds for $||\varphi||_{2,\alpha}$, $||f||_{k,\alpha}$. Hence, we get (5.11), $\varphi \in C^\infty(M)$, and to prove Theorem 5.3, it is enough to establish the a priori estimate (5.10).

5.4 Uniform Estimate

The main goal of this section will be to prove the uniform estimate. We will use the notation $||\varphi||_p = ||\varphi||_{L^p(M)}$, $1 \le p \le \infty$.

Theorem 5.6 *Assume that $\varphi \in C^2(M)$ is admissible and $(\omega + dd^c\varphi)^n = f\omega^n$. Then*

$$\operatorname*{osc}_M \varphi := \sup_M \varphi - \inf_M \varphi \le C,$$

where $C > 0$ depends only on M and on an upper bound for $||f||_\infty$.

The L^p estimate for $p < \infty$ follows easily for any admissible φ (without any knowledge on f).

Proposition 5.7 *For any admissible $\varphi \in C^2(M)$ with $\max_M \varphi = 0$ one has*

$$||\varphi||_p \le C(M, p), \quad 1 \le p < \infty.$$

Proof. The case $p = 1$ follows easily from the following estimate (applied in finite number of local charts to $u = \varphi + g$): if u is a negative subharmonic function in $B(y, 3R)$ in $\mathbb{R}^m$ then for $x \in B(y, R)$ we have

$$u(x) \le \frac{1}{\operatorname{vol}(B(x, 2R))} \int_{B(x,2R)} u \le \frac{1}{\operatorname{vol}(B(y, 2R))} \int_{B(y,R)} u$$

and thus

$$||u||_{L^1(B(y,R))} \le \operatorname{vol}(B(y, 2R)) \inf_{B(y,R)} (-u).$$

For $p > 1$ we now use the following estimate: if u is a negative psh in $B(y, 2R)$ in $\mathbb{C}^n$ then

$$||u||_{L^p(B(y,R))} \le C(n, p, R)||u||_{L^1(B(y,2R))}.$$

□

We will now present two different proofs of the uniform estimate. The first one (see [Siu87, p. 92] or [Tianbook, p. 49]) is similar to the original proof of Yau, subsequently simplified by Kazdan [Kaz78] for $n = 2$ and by Aubin and Bourguignon for arbitrary n (for the detailed historical account we refer to [Yau78, p. 411] and [Siu87, p. 115]).

First proof of Theorem 5.6. Without loss of generality we may assume that $\int_M \omega^n = 1$ and $\max_M \varphi = -1$, so that $||\varphi||_p \le ||\varphi||_q$ if $p \le q < \infty$. We have

$$(f-1)\omega^n = (\omega + dd^c\varphi)^n - \omega^n = dd^c\varphi \wedge T,$$

where

$$T = \sum_{j=0}^{n-1} (\omega + dd^c\varphi)^j \wedge \omega^{n-1-j} \geq \omega^{n-1}.$$

Integrating by parts we get for $p \geq 1$

$$\begin{aligned}\int_M (-\varphi)^p (f-1)\omega^n &= \int_M (-\varphi)^p dd^c\varphi \wedge T = -\int_M d(-\varphi)^p \wedge d^c\varphi \wedge T \\ &= p\int_M (-\varphi)^{p-1} d\varphi \wedge d^c\varphi \wedge T \geq p\int_M (-\varphi)^{p-1} d\varphi \wedge d^c\varphi \wedge \omega^{n-1} \\ &= \frac{4p}{(p+1)^2}\int_M d(-\varphi)^{(p+1)/2} \wedge d^c(-\varphi)^{(p+1)/2} \wedge \omega^{n-1}\end{aligned}$$

so that

$$\int_M (-\varphi)^p (f-1)\omega^n = \frac{c_n p}{(p+1)^2} ||D(-\varphi)^{(p+1)/2}||_2^2. \tag{5.12}$$

The Sobolev inequality on compact a Riemannian manifold M with real dimension m states that

$$||v||_{mq/(m-q)} \leq C(M,q)\left(||v||_q + ||Dv||_q\right), \quad v \in W^{1,q}(M),\ q < m. \tag{5.13}$$

(it easily follows from the Sobolev inequality for $u \in W_0^{1,q}(\mathbb{R}^m)$) applied in charts forming a finite covering of M). Using (5.13) with $q = 2$ and (5.12)

$$\begin{aligned}&||(-\varphi)^{(p+1)/2}||_{2n/(n-1)} \\ &\quad \leq C_M\left(||(-\varphi)^{(p+1)/2}||_2 + \frac{p+1}{\sqrt{p}}\left(\int_M (-\varphi)^p (f-1)\omega^n\right)^{1/2}\right).\end{aligned}$$

From this (replacing $p+1$ with p) and since $|\varphi| \leq 1$ we easily get

$$||\varphi||_{np/(n-1)} \leq (Cp)^{1/p} ||\varphi||_p, \quad p \geq 2. \tag{5.14}$$

We will now apply Moser's iteration scheme (see [Mos60] or the proof of [GT83, Theorem 8.15]). Set

$$p_0 := 2, \quad p_k := \frac{np_{k-1}}{n-1}, \quad k = 1, 2, \ldots,$$

so that $p_k = 2(n/(n-1))^k$. Then by (5.14)

$$||\varphi||_\infty = \lim_{k\to\infty} ||\varphi||_{p_k} \le ||\varphi||_2 \prod_{j=0}^{\infty} (Cp_j)^{1/p_j}.$$

Taking the logarithm one can show that

$$\prod_{j=0}^{\infty} (Cp_j)^{1/p_j} = (n/(n-1))^{n(n-1)/2} (2C)^{n/2}$$

and it is enough to use Proposition 5.7 (for $p = 2$).

Exercise 5.8 Slightly modifying the above proof show that the uniform estimate follows if we assume that $||f||_q$ is under control for some $q > n$.

Exercise 5.9 Consider the equation

$$(\omega + dd^c\varphi)^n = F(\cdot, \varphi)\omega^n,$$

where $F \in C^\infty(M \times \mathbb{R})$ is positive. Show that if an admissible solution $\varphi \in C^\infty(M)$ attains maximum at $y \in M$ then $F(y, \varphi(y)) \le 1$. Deduce a uniform estimate for admissible solutions of (5.9) when $\lambda < 0$.

The second proof of the uniform estimate is essentially due to Kolodziej [Kol98] who studied pluripotential theory on compact Kähler manifolds (see also [TZ00]). The Kolodziej argument gave the uniform estimate under weaker conditions than in Theorem 5.6 – it is enough to assume that $||f||_q$ is under control for some $q > 1$. For $q = \infty$ (and even $q > 2$) this argument was simplified in [Bl05] and we will follow that proof.

The main tool in the second proof of Theorem 5.6 will the following L^2 stability for the complex Monge–Ampère equation. It was originally established by Cheng and Yau (see [B88, p. 75]). The Cheng–Yau argument was made precise by Cegrell and Persson [CP92].

Theorem 5.10 *Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that $u \in C(\overline{\Omega})$ is psh and C^2 in Ω, $u = 0$ on $\partial\Omega$. Then*

$$||u||_{L^\infty(\Omega)} \le C(n, \operatorname{diam}\Omega)\, ||f||_{L^2(\Omega)}^{1/n},$$

where $f = \det(u_{i\bar{j}})$.

Proof. We use the theory of convex functions and the real Monge–Ampère operator. From the Alexandrov–Bakelman–Pucci principle [GT83, Lemma 9.2] we get

$$||u||_{L^\infty(\Omega)} \le \frac{\operatorname{diam}\Omega}{\lambda_{2n}^{1/2n}} \left(\int_\Gamma \det D^2 u \right)^{1/2n},$$

where $\lambda_{2n} = \pi^n/n!$ is the volume of the unit ball in $\mathbb{C}^n$ and

$$\Gamma := \{x \in \Omega : u(x) + \langle Du(x), y - x\rangle \le u(y)\ \forall\, y \in \Omega\} \subset \{D^2 u \ge 0\}.$$

It will now be sufficient to prove the pointwise estimate

$$D^2 u \ge 0 \implies \det D^2 u \le c_n (\det(u_{i\bar{j}}))^2.$$

We may assume that $(u_{i\bar{j}})$ is diagonal. Then

$$\begin{aligned}\det(u_{i\bar{j}}) &= 4^{-n}(u_{x_1x_1} + u_{y_1y_1}) \dots (u_{x_nx_n} + u_{y_ny_n})\\ &\ge 2^{-n}\sqrt{u_{x_1x_1}u_{y_1y_1}\dots u_{x_nx_n}u_{y_ny_n}}\\ &\ge \sqrt{\det D^2u/c_n},\end{aligned}$$

where the last inequality follows because for real nonnegative symmetric matrices (a_{pq}) one easily gets $\det(a_{pq}) \le m! a_{11} \dots a_{mm}$ (because $|a_{pq}| \le \sqrt{a_{pp}a_{qq}}$); from Lemma 5.16 below one can deduce that in fact $\det(a_{pq}) \le a_{11} \dots a_{mm}$. □

From the comparison principle for the complex Monge–Ampère operator one can immediately obtain the estimate

$$||u||_{L^\infty(\Omega)} \le (\operatorname{diam}\Omega)^2\, ||f||^{1/n}_{L^\infty(\Omega)}$$

in Theorem 5.10. It is however not sufficient for our purposes, because it does not show that if $\operatorname{vol}(\Omega)$ is small then so is $||u||_{L^\infty(\Omega)}$.

Exercise 5.11 Using the Moser iteration technique from the first proof of Theorem 5.6 show the L^q stability for $q > n$, that is Theorem 5.10 with $||f||_{L^2(\Omega)}$ replaced with $||f||_{L^q(\Omega)}$.

The uniform estimate will easily follow from the next result.

Proposition 5.12 *Let Ω be a bounded domain in $\mathbb{C}^n$ and u is a negative C^2 psh function in Ω. Assume that $a > 0$ is such that the set $\{u < \inf_\Omega u + a\}$ is nonempty and relatively compact in Ω. Then*

$$||u||_{L^\infty(\Omega)} \le a + (C/a)^{2n} ||u||_{L^1(\Omega)} ||f||^2_{L^\infty(\Omega)},$$

where $f = \det(u_{i\bar{j}})$ and $C = C(n, \operatorname{diam}\Omega)$ is the constant from Theorem 5.10.

Proof. Set $t := \inf_\Omega u + a$, $v := u - t$ and $\Omega' := \{v < 0\}$. By Theorem 5.10

$$a = ||v||_{L^\infty(\Omega')} \le C\,(\operatorname{vol}(\Omega'))^{1/2n}\, ||f||^{1/n}_{L^\infty(\Omega')}.$$

On the other hand,

$$\mathrm{vol}\,(\Omega') \leq \frac{||u||_{L^1(\Omega)}}{|t|} = \frac{||u||_{L^1(\Omega)}}{||u||_{L^\infty(\Omega)} - a}$$

and the estimate follows. □

Second proof of Theorem 5.6 Let $y \in M$ be such that $\varphi(y) = \min_M \varphi$. The Taylor expansion of g about y gives

$$\begin{aligned} g(y+h) &= \mathrm{Re}\, P(h) + \sum_{i,j=1}^{n} g_{i\bar{j}}(y) h_i \overline{h}_j + \frac{1}{3!} D^3 g(\widetilde{y}).h^3 \\ &\geq \mathrm{Re}\, P(h) + c_1|h|^2 - c_2|h|^3, \end{aligned}$$

where

$$P(h) = g(y) + 2\sum_i g_i(y) h_i + \sum_{i,j} g_{ij}(y) h_i h_j$$

is a complex polynomial, $\widetilde{y} \in [y, y+h]$ and $c_1, c_2 > 0$ depend only on M. Modifying g by a pluriharmonic function (and thus not changing ω), we may thus assume that there exists $a, r > 0$ depending only on M such that $g < 0$ in $B(y, 2r)$, g attains minimum in $B(y, 2r)$ at y and $g \geq g(y) + a$ on $B(y, 2r) \setminus B(y, r)$. Proposition 5.12 (for $\Omega = B(y, 2r)$ and $u = g + \varphi$) combined with Proposition 5.7 (for $p = 1$) gives the required estimate.

Slightly improving the proof of Proposition 5.12 (using the Hölder inequality) we see that the second proof of Theorem 5.6 implies that we can replace $||f||_\infty$ with $||f||_q$ for any $q > 2$. Moreover, since Kolodziej [Kol96] showed (with more complicated proof using pluripotential theory) that the (local) L^q stability for the complex Monge–Ampère equation holds for every $q > 1$ (and even for a weaker Orlicz norm), we can do this on M also for every $q > 1$. This was proved in [Kol98], where the local techniques from [Kol96] had to be repeated on M. The above argument allows to easily deduce the global uniform estimate from the local results. Exercises 5.8 and 5.11 show that both proofs of Theorem 5.6, although quite different, are related.

5.5 Second Derivative Estimate

In this section we will show the a priori estimate for the mixed complex derivatives $\varphi_{i\bar{j}}$ which is equivalent to the estimate of $\Delta\varphi$. The main idea is the same as the one in the original Yau proof [Yau78] who used the method of Pogorelov [Pog71] from the real Monge–Ampère equation. We will present an improvement of the Yau estimate that can be applied to the degenerate case (when $f \geq 0$) because it does not quantitatively depend on $\inf_M f$. It uses the idea of Guan [Gua97] (see also [GTW99]) who obtained regularity results for the degenerate real Monge–Ampère equation. It also simplifies some computations from [Yau78].

Theorem 5.13 [Bl03] *Let $\varphi \in C^4(M)$ be such that $\omega + dd^c\varphi > 0$ and $(\omega + dd^c\varphi)^n = f\omega^n$. Then*

$$\sup_M |\Delta\varphi| \leq C,$$

where C depends only on M and on an upper bound for $||f^{1/(n-1)}||_{1,1}$.

Proof. By Theorem 5.6 we may assume that

$$-C_1 \leq \varphi \leq 0. \tag{5.15}$$

Note that for any admissible φ we have $(g_{i\bar{j}} + \varphi_{i\bar{j}}) \geq 0$ and thus

$$\Delta\varphi = g^{i\bar{j}}\varphi_{i\bar{j}} \geq -n.$$

It is therefore enough to estimate $\Delta\varphi$ from above. In local coordinates the function $u = g + \varphi$ is strongly psh. It is easy to see that the expression

$$\eta := \max_{|\zeta|=1} \frac{u_{\zeta\bar{\zeta}}}{g_{\zeta\bar{\zeta}}} = \max_{\zeta\neq 0} \frac{u_{\zeta\bar{\zeta}}}{g_{\zeta\bar{\zeta}}},$$

(where $u_\zeta = \sum_i \zeta_i u_i$, $u_{\bar{\zeta}} = \sum_i \overline{\zeta}_i u_{\bar{i}}$, and $u_{\zeta\bar{\zeta}} = \sum_{i,j} \zeta_i \overline{\zeta}_j u_{i\bar{j}}$, $\zeta \in \mathbb{C}^n$) is independent of holomorphic change of coordinates, and thus η is a continuous, positive, globally defined function on M. Set

$$\alpha := \log\eta - A\varphi,$$

where $A > 0$ under control will be specified later. Since M is compact and α is continuous, we can find $y \in M$, where α attains maximum. After rotation we may assume that the matrix $(u_{i\bar{j}})$ is diagonal and $u_{1\bar{1}} \geq \cdots \geq u_{n\bar{n}}$ at y. Fix $\zeta \in \mathbb{C}^n$, $|\zeta| = 1$, such that $\eta = u_{\zeta\bar{\zeta}}/g_{\zeta\bar{\zeta}}$ at y. Then the function

$$\widetilde{\alpha} := \log\frac{u_{\zeta\bar{\zeta}}}{g_{\zeta\bar{\zeta}}} - A\varphi,$$

defined in a neighborhood of y, also has maximum at y. Moreover, $\widetilde{\alpha} \leq \alpha$ and $\widetilde{\alpha}(y) = \alpha(y)$. Since

$$u_{\zeta\bar{\zeta}}(y) \leq u_{1\bar{1}}(y) \leq C_2 u_{\zeta\bar{\zeta}}(y), \tag{5.16}$$

by (5.15) it is clear that to finish the proof it is sufficient to show the estimate

$$u_{1\bar{1}}(y) \le C_3. \tag{5.17}$$

We will use the following local estimate.

Lemma 5.14 *Let u be a C^4 psh function with $F := \det(u_{i\bar{j}}) > 0$. Then for any direction ζ*

$$u^{i\bar{j}}(\log u_{\zeta\bar{\zeta}})_{i\bar{j}} \ge \frac{(\log F)_{\zeta\bar{\zeta}}}{u_{\zeta\bar{\zeta}}}.$$

Proof. Differentiating (logarithm of) the equation $\det(u_{i\bar{j}}) = F$ twice, similarly as in (5.6), (5.7) we get

$$u^{i\bar{j}}u_{i\bar{j}\zeta} = (\log F)_\zeta,$$
$$u^{i\bar{j}}u_{i\bar{j}\zeta\bar{\zeta}} = (\log F)_{\zeta\bar{\zeta}} + u^{i\bar{l}}u^{k\bar{j}}u_{i\bar{j}\zeta}u_{k\bar{l}\bar{\zeta}}.$$

Using this we obtain

$$\begin{aligned} u_{\zeta\bar{\zeta}}\, u^{i\bar{j}}(\log u_{\zeta\bar{\zeta}})_{i\bar{j}} &= u^{i\bar{j}}u_{i\bar{j}\zeta\bar{\zeta}} - \frac{1}{u_{\zeta\bar{\zeta}}}u^{i\bar{j}}u_{\zeta\bar{\zeta}i}u_{\zeta\bar{\zeta}\bar{j}} \\ &= (\log F)_{\zeta\bar{\zeta}} + u^{i\bar{l}}u^{k\bar{j}}u_{i\bar{j}\zeta}u_{k\bar{l}\bar{\zeta}} - \frac{1}{u_{\zeta\bar{\zeta}}}u^{i\bar{j}}u_{\zeta\bar{\zeta}i}u_{\zeta\bar{\zeta}\bar{j}}. \end{aligned}$$

At a given point we may assume that the matrix $(u_{i\bar{j}})$ is diagonal. Then

$$u^{i\bar{j}}u_{\zeta\bar{\zeta}i}u_{\zeta\bar{\zeta}\bar{j}} = \sum_i \frac{|u_{\zeta\bar{\zeta}i}|^2}{u_{i\bar{i}}}$$

and

$$|u_{\zeta\bar{\zeta}i}|^2 = \left|\sum_j \bar{\zeta}_j u_{i\bar{j}\zeta}\right|^2 \le \sum_j |\zeta_j|^2 u_{j\bar{j}} \sum_j \frac{|u_{i\bar{j}\zeta}|^2}{u_{j\bar{j}}}$$

by Schwarz inequality. Therefore

$$u^{i\bar{j}}u_{\zeta\bar{\zeta}i}u_{\zeta\bar{\zeta}\bar{j}} \le u_{\zeta\bar{\zeta}} \sum_{i,j} \frac{|u_{i\bar{j}\zeta}|^2}{u_{i\bar{i}}u_{j\bar{j}}} = u_{\zeta\bar{\zeta}}\, u^{i\bar{l}}u^{k\bar{j}}u_{i\bar{j}\zeta}u_{k\bar{l}\bar{\zeta}}$$

and the lemma follows. □

As noticed by Bo Berndtsson, Lemma 4.2 has a geometric context. If ζ is a holomorphic vector field on a Kähler manifold (with potential u) then one can show that

$$\sqrt{-1}\partial\bar{\partial}\log|\zeta|^2 \geq -\frac{R(\zeta,\zeta,\cdot,\cdot)}{|\zeta|^2}.$$

Taking the trace and using that $Ric_{i\bar{j}} = -(\log F)_{i\bar{j}}$, one obtains the statement of the lemma.

Proof of Theorem 5.13 (continued) Using the fact that $\widetilde{\alpha}$ has maximum at y, by Lemma 5.14 with $F = f\det(g_{i\bar{j}})$ we get

$$0 \geq u^{i\bar{j}}\widetilde{\alpha}_{i\bar{j}} \geq \frac{(\log f)_{\zeta\bar{\zeta}}}{u_{\zeta\bar{\zeta}}} + \frac{(\log\det(g_{p\bar{q}}))_{\zeta\bar{\zeta}}}{u_{\zeta\bar{\zeta}}} + Au^{i\bar{j}}g_{i\bar{j}} - nA.$$

By (5.16) and the elementary inequality (following from differential calculus of functions of one real variable)

$$||\sqrt{h}||_{0,1} \leq C_M(1+||h||_{1,1}), \quad h\in C^2(M),\ h\geq 0,$$

we get, denoting $\widetilde{f} := f^{1/(n-1)}$,

$$\frac{(\log f)_{\zeta\bar{\zeta}}}{u_{\zeta\bar{\zeta}}} = \frac{n-1}{u_{\zeta\bar{\zeta}}}\left(\frac{\widetilde{f}_{\zeta\bar{\zeta}}}{\widetilde{f}} - \frac{|\widetilde{f}_\zeta|^2}{\widetilde{f}^2}\right) \geq -\frac{C_4}{u_{1\bar{1}}\widetilde{f}}.$$

Therefore, using (5.16) again (recall that $(u_{i\bar{j}})$ is diagonal at y),

$$0 \geq -\frac{C_4}{u_{1\bar{1}}\widetilde{f}} - \frac{C_5}{u_{1\bar{1}}} + (-C_6 + A/C_7)\sum_i \frac{1}{u_{i\bar{i}}} - nA,$$

where $1/C_7 \leq \lambda_{min}(g_{i\bar{j}}(y))$. We choose A such that $-C_6 + A/C_7 = \max\{1, C_5\}$. The inequality between arithmetic and geometric means gives

$$\sum_{i\geq 2}\frac{1}{u_{i\bar{i}}} \geq \frac{n-1}{(u_{2\bar{2}}\dots u_{n\bar{n}})^{1/(n-1)}} = (n-1)\frac{u_{1\bar{1}}^{1/(n-1)}}{\widetilde{f}}.$$

We arrive at

$$u_{1\bar{1}}^{n/(n-1)} - C_8 u_{1\bar{1}} - C_9 \leq 0$$

(at y) from which (5.17) immediately follows. □

In the proof of Theorem 5.13, unlike in [Yau78], we used standard derivatives in local coordinates and not the covariant ones – it makes some calculations simpler.

It is rather unusual in the theory of nonlinear elliptic equations of second order that the second derivative estimate can be obtained directly from the uniform estimate, bypassing the gradient estimate. The gradient estimate follows locally (and hence globally on M) from the estimate for the Laplacian for arbitrary solutions of the Poisson equation (see e.g. [GT83, Theorem 3.9] or use the Green function and differentiate under the sign of integration).

5.6 $C^{2,\alpha}$ Estimate

Aubin [Aub70] and Yau [Yau78] proved a priori estimates for third-order derivatives of φ. The estimate from [Yau78], due to Nirenberg (see [Yau78, Appendix A]), was based on an estimate for the real Monge–Ampère equation of Calabi [Cal58]. In the meantime, a general theory of nonlinear elliptic equations of second order has been developed. It allows to obtain an interior $C^{2,\alpha}$-estimate, once an estimate for the second derivatives is known. It was done by Evans [Ev82, Ev83] (and also independently by Krylov [Kry82]) and his method was subsequently simplified by Trudinger [Trud83]. Although the complex Monge–Ampère operator is uniformly elliptic in the real sense (see Exercise 5.5), we cannot apply the estimate from the real theory directly. The reason is that Sect. 5.5 gives the control for the mixed complex derivatives $\varphi_{i\bar{j}}$ but not for $D^2\varphi$, which is required in the real estimate. We can however almost line by line repeat the real method in our case. It has been done in [Siu87], and also in [Bl00, Theorem 3.1], where an idea from [Sch86] and [WJ85] was used to write the equation in divergence form. We will get the following a priori estimate for the complex Monge–Ampère equation.

Theorem 5.15 *Let u be a C^4 psh function in an open $\Omega \subset \mathbb{C}^n$ such that $f := \det(u_{i\bar{j}}) > 0$. Then for any $\Omega' \Subset \Omega$ there exist $\alpha \in (0,1)$ depending only on n and on upper bounds for $||u||_{C^{0,1}(\Omega)}$, $sup_\Omega \Delta u$, $||f||_{C^{0,1}(\Omega)}$, $1/\inf_\Omega f$, and $C > 0$ depending in addition on a lower bound for $\operatorname{dist}(\Omega', \partial\Omega)$ such that*

$$||u||_{C^{2,\alpha}(\Omega')} \leq C.$$

A similar estimate can be proved for more general equations of the complex Hessian of the form

$$F((u_{i\bar{j}}), Du, u, z) = 0.$$

Here F is a smooth function of $\mathcal{G}\times\mathbb{R}^{2n}\times\mathbb{R}\times\Omega$, where $\mathcal{G}$ is an open subset of the set of all $n \times n$ hermitian matrices $\mathcal{H}$. In case of the complex Monge–Ampère operator we take $\mathcal{G} = \mathcal{H}_+ := \{A \in \mathcal{H} : A > 0\}$. The crucial assumption that has to be made on F in order for the Evans–Trudinger method to work is that it is concave with respect to $(u_{i\bar{j}})$. In case of the complex Monge–Ampère equation one has to use the fact that the mapping

$$\mathcal{H}_+ \ni A \longmapsto (\det A)^{1/n} \in \mathbb{R}_+ \tag{5.18}$$

is concave. This can be immediately deduced from the following very useful lemma.

Lemma 5.16 [Gav77]

$$(\det A)^{1/n} = \frac{1}{n}\inf\{\operatorname{tr}(AB) : B \in \mathcal{H}_+,\ \det B = 1\}, \quad A \in \mathcal{H}_+.$$

Proof. For every $B \in \mathcal{H}_+$ there is unique $C \in \mathcal{H}_+$ such that $C^2 = B$. We denote $C = B^{1/2}$. Then $B^{1/2}AB^{1/2} \in \mathcal{H}_+$ and after diagonalizing it, from the inequality between arithmetic and geometric means we get

$$\begin{aligned}(\det A)^{1/n}(\det B)^{1/n} &= (\det(B^{1/2}AB^{1/2}))^{1/n}\\ &\le \tfrac{1}{n}\operatorname{tr}(B^{1/2}AB^{1/2}) = \tfrac{1}{n}\operatorname{tr}(AB)\end{aligned}$$

and $\le$ follows. To show $\ge$ we may assume that A is diagonal and then we easily find B for which the infimum is attained. □

Lemma 5.16 also shows that the Monge–Ampère operator is an example of a Bellman operator.

Proof of Theorem 5.15 Fix $\zeta \in \mathbb{C}^n$, $|\zeta| = 1$. Differentiating the logarithm of both sides of the equation

$$\det(u_{i\bar{j}}) = f,$$

similarly as in (5.7) or in the proof of Lemma 5.14, we obtain

$$u^{i\bar{j}}u_{\zeta\bar{\zeta}i\bar{j}} = (\log f)_{\zeta\bar{\zeta}} + u^{i\bar{l}}u^{k\bar{j}}u_{\zeta i\bar{j}}u_{\bar{\zeta}k\bar{l}} \ge (\log f)_{\zeta\bar{\zeta}}. \tag{5.19}$$

The inequality $u^{i\bar{l}}u^{k\bar{j}}u_{\zeta i\bar{j}}u_{\bar{\zeta}k\bar{l}} \ge 0$ is equivalent to the concavity of the mapping

$$\mathcal{H}_+ \ni A \longmapsto \log\det A \in \mathbb{R}$$

which also follows from concavity of (5.18). It will be convenient to write (5.19) in divergence form. Set $a^{i\bar{j}} := fu^{i\bar{j}}$. Then for any fixed i

$$(a^{i\bar{j}})_{\bar{j}} = f(u^{i\bar{j}}u^{k\bar{l}} - u^{i\bar{l}}u^{k\bar{j}})u_{k\bar{l}\bar{j}} = 0$$

and by (5.19)

$$(a^{i\bar{j}}u_{\zeta\bar{\zeta}i})_{\bar{j}} \ge f_{\zeta\bar{\zeta}} - \frac{|f_\zeta|^2}{f} \ge -C_1 + \sum_j\left(\frac{\partial f^j}{\partial x_j} + \frac{\partial f^{j+n}}{\partial y_j}\right),$$

where $||f^l||_{L^\infty(\Omega)} \le C_2$, $l = 1, \dots, 2n$. By the assumptions on u (and Exercise 5.5) the operator $\partial_{\bar{j}}(a^{i\bar{j}}\partial_i)$ is uniformly elliptic (in the real sense) and from the weak Harnack inequality [GT83, Theorem 8.18] we now get

$$r^{-2n}\int_{B_r}\big(\sup_{B_{4r}} u_{\zeta\bar\zeta}-u_{\zeta\bar\zeta}\big)\le C_3\big(\sup_{B_{4r}} u_{\zeta\bar\zeta}-\sup_{B_r} u_{\zeta\bar\zeta}+r\big), \tag{5.20}$$

where $B_{4r}=B(z_0,4r)\subset\Omega$ and $z_0\in\Omega'$.

On the other hand, for $x,y\in\Omega$ by Lemma 5.16 we have

$$a^{i\bar j}(y)\big(u_{i\bar j}(y)-u_{i\bar j}(x)\big)\le nf(y)^{1-1/n}\big(f(y)^{1/n}-f(x)^{1/n}\big)\le C_4|x-y|. \tag{5.21}$$

We are going to combine (5.20) with (5.21). For that we will need to choose an appropriate finite set of directions ζ. The following lemma from linear algebra will be crucial.

Lemma 5.17 *Let $0<\lambda<\Lambda<\infty$ and by $S(\lambda,\Lambda)$ denote the set of hermitian matrices whose eigenvalues are in the interval $[\lambda,\Lambda]$. Then one can find unit vectors $\zeta_1,\dots,\zeta_N\in\mathbb{C}^n$ and $0<\lambda_*<\Lambda_*<\infty$, depending only on n,λ, and Λ, such that every $A\in S(\lambda,\Lambda)$ can be written as*

$$A=\sum_{k=1}^N\beta_k\zeta_k\otimes\overline{\zeta}_k,\quad \textit{i.e.}\quad a_{i\bar j}=\sum_k\beta_k\zeta_{ki}\overline{\zeta}_{kj},$$

where $\beta_k\in[\lambda_*,\Lambda_*]$, $k=1,\dots,N$. The vectors $\zeta_1,\dots,\zeta_N$ can be chosen so that they contain a given orthonormal basis of $\mathbb{C}^n$.

Proof. [Siu87, p. 103] The space $\mathcal{H}$ of all hermitian matrices is of real dimension n^2. Every $A\in\mathcal{H}$ can be written as

$$A=\sum_{k=1}^n\lambda_k w_k\otimes\overline{w}_k,$$

where $\lambda_1,\dots,\lambda_n\in\mathbb{R}$ are the eigenvalues of A and $w_1,\dots,w_n\in\mathbb{C}^n$ the corresponding unit eigenvectors. It follows that there exist unit vectors $\zeta_1,\dots,\zeta_{n^3}\in\mathbb{C}^n$ such that the matrices $\zeta_k\otimes\overline{\zeta}_k$, $k=1,\dots,n^3$, span $\mathcal{H}$ over $\mathbb{R}$. For such sets of vectors we consider the sets of matrices

$$U(\zeta_1,\dots,\zeta_{n^3})=\left\{\sum_k\beta_k\zeta_k\otimes\overline{\zeta}_k : 0<\beta_k<2\Lambda\right\}.$$

They form an open covering of $S(\lambda/2,\Lambda)$, a compact subset of $\mathcal{H}$. Choosing a finite subcovering we get unit vectors $\zeta_1,\dots,\zeta_N\in\mathbb{C}^n$ such that

$$S(\lambda/2,\Lambda)\subset\left\{\sum_{k=1}^N\beta_k\zeta_k\otimes\overline{\zeta}_k : 0<\beta_k<2\Lambda\right\}.$$

For $A \in S(\lambda, \Lambda)$ we have

$$A - \frac{\lambda}{2N} \sum_{k=1}^{N} \zeta_k \otimes \overline{\zeta}_k \in S(\lambda/2, \Lambda)$$

and the lemma follows. We see that may take arbitrary $\lambda_* < \lambda/N$ and $\Lambda_* > \Lambda$. □

Proof of Theorem 5.15. (continued) The eigenvalues of $(u_{i\bar{j}})$ are in $[\lambda, \Lambda]$, where $\lambda, \Lambda > 0$ are under control. By Lemma 5.17 we can find unit vectors $\zeta_1, \dots, \zeta_N \in \mathbb{C}^n$ such that for $x, y \in \Omega$

$$a^{i\bar{j}}(y)\big(u_{i\bar{j}}(y) - u_{i\bar{j}}(x)\big) = \sum_{k=1}^{N} \beta_k(y)\big(u_{\zeta_k\bar{\zeta}_k}(y) - u_{\zeta_k\bar{\zeta}_k}(x)\big),$$

where $\beta_k(y) \in [\lambda^*, \Lambda^*]$ and $\lambda^*, \Lambda^* > 0$ are under control. Set

$$M_{k,r} := \sup_{B_r} u_{\zeta_k\bar{\zeta}_k}, \quad m_{k,r} := \inf_{B_r} u_{\zeta_k\bar{\zeta}_k},$$

and

$$\eta(r) := \sum_{k=1}^{N} (M_{k,r} - m_{k,r}).$$

We need to show that $\eta(r) \le Cr^\alpha$. Since $\gamma_1, \dots, \gamma_N$ can be chosen so that they contain the coordinate vectors, it will then follow that $||\Delta u||_{C^\alpha(\Omega')}$ is under control and by the Schauder estimates for the Poisson equation [GT83, Theorem 4.6] also that $||D^2 u||_{C^\alpha(\Omega')}$ is under control. The condition $\eta(r) \le Cr^\alpha$ is equivalent to

$$\eta(r) \le \delta\eta(4r) + r, \quad 0 < r < r_0, \tag{5.22}$$

where $\delta \in (0, 1)$ and $r_0 > 0$ are under control (see [GT83, Lemma 8.23]).

From (5.21) we get

$$\sum_{k=1}^{N} \beta_k(y)\big(u_{\zeta_k\bar{\zeta}_k}(y) - u_{\zeta_k\bar{\zeta}_k}(x)\big) \le C_4|x - y|. \tag{5.23}$$

Summing (5.20) over $l \ne k$, where k is fixed, we obtain

$$r^{-2n} \int_{B_r} \sum_{l \ne k} \big(M_{l,4r} - u_{\zeta_l\bar{\zeta}_l}\big) \le C_3(\eta(4r) - \eta(r) + r). \tag{5.24}$$

By (5.23) for $x \in B_{4r}, y \in B_r$ we have

$$\beta_k(y)\big(u_{\zeta_k\bar{\zeta}_k}(y) - u_{\zeta_k\bar{\zeta}_k}(x)\big) \leq C_4|x-y| + \sum_{l\neq k}\beta_l(y)\big(u_{\zeta_l\bar{\zeta}_l}(x) - u_{\zeta_l\bar{\zeta}_l}(y)\big)$$

$$\leq C_5 r + \Lambda^* \sum_{l\neq k}(M_{l,4r} - u_{\zeta_l\bar{\zeta}_l}(y)).$$

Thus

$$u_{\zeta_k\bar{\zeta}_k}(y) - m_{k,4r} \leq \frac{1}{\lambda^*}\left(C_5 r + \Lambda^* \sum_{l\neq k}\big(M_{l,4r} - u_{\zeta_l\bar{\zeta}_l}(y)\big)\right)$$

and (5.24) gives

$$r^{-2n}\int_{B_r}\big(u_{\zeta_k\bar{\zeta}_k} - m_{k,4r}\big) \leq C_6(\eta(4r) - \eta(r) + r).$$

This coupled with (5.20) easily implies that

$$\eta(r) \leq C_7(\eta(4r) - \eta(r) + r),$$

and (5.22) follows.

5.7 Weak Solutions

The theory of the complex Monge–Ampère operator $(dd^c)^n$ for nonsmooth psh functions has been developed by Bedford and Taylor (see [BT76, BT82] and also general references [Dem93, Dembook, Klimbook, Bl96, Blobook, Ceg88, Kol05]). In particular, one can define $(dd^c u)^n$ as a nonnegative regular Borel measure if u is a locally bounded psh function, and this operator is continuous for monotone sequences (in the weak* topology of measures). We define the class of weakly admissible functions on M in a natural way: $\varphi : M \to \mathbb{R} \cup \{-\infty\}$ is called *admissible* (or *ω-psh*) if locally $g + \varphi$ is psh. Therefore, if φ is locally bounded and admissible then $\mathcal{M}(\varphi) := (\omega + dd^c\varphi)^n$, locally equal to $(dd^c(g+\varphi))^n$, is a measure such that $\int_M \mathcal{M}(\varphi) = V$.

We will show the following version of Theorem 5.3 for weak solutions.

Theorem 5.18 [Kol98, Kol03] *Let $f \in C(M)$, $f \geq 0$, be such that $\int_M f\omega^n = V$. Then there exists a, unique up to a constant, admissible $\varphi \in C(M)$ such that $\mathcal{M}(\varphi) = f\omega^n$.*

The existence part of Theorem 5.18 was shown in [Kol98], also for $f \in L^q(M)$, $q > 1$. As we will see, this part for $f \in C(M)$ can be proved in a simpler way. It will immediately follow from Theorem 5.3 and appropriate stability of smooth solutions (Theorem 5.21 below).

Concerning the uniqueness in Theorem 5.18 it was later shown in [Kol03] (also for more general densities f). One can however consider the uniqueness problem without any assumption on density of the Monge–Ampère measure: does $\mathcal{M}(\varphi) = \mathcal{M}(\psi)$ imply that $\varphi - \psi = const$? It was proved in [BT89] for $M = \mathbb{P}^n$ but it is true for arbitrary M and can be shown much simpler than in [BT89]. We have the following most general uniqueness result with the simplest proof.

Theorem 5.19 [Bl03b] *If $\varphi, \psi \in L^\infty(M)$ are admissible and $\mathcal{M}(\varphi) = \mathcal{M}(\psi)$ then $\varphi - \psi = const$.*

Proof. Set $\rho := \varphi - \psi$ and $\omega_\varphi := \omega + dd^c\varphi$. We start as in the proof of Proposition 5.4. We will get

$$d\rho \wedge d^c\rho \wedge \omega_\varphi^j \wedge \omega_\psi^{n-1-j} = 0, \quad j = 0, 1, \dots, n-1, \tag{5.25}$$

and we have to show that $d\rho\wedge d^c\rho\wedge\omega^{n-1} = 0$. To describe the further method we assume that $n = 2$. Using (5.25) and integrating by parts

$$\int_M d\rho \wedge d^c\rho \wedge \omega = -\int_M d\rho \wedge d^c\rho \wedge dd^c\varphi = \int_M d\varphi \wedge d^c\rho \wedge (\omega_\psi - \omega_\varphi).$$

By the Schwarz inequality

$$\left|\int_M d\varphi \wedge d^c\rho \wedge \omega_\psi\right| \le \left(\int_M d\varphi \wedge d^c\varphi \wedge \omega_\psi\right)^{1/2} \left(\int_M d\rho \wedge d^c\rho \wedge \omega_\psi\right)^{1/2} = 0$$

by (5.25) and, similarly, $\int_M d\varphi \wedge d^c\rho \wedge \omega_\varphi = 0$. Therefore $d\rho \wedge d^c\rho \wedge \omega = 0$.

For $n > 2$ the proof is similar but one has to use an appropriate inductive procedure: in the same way as before one shows for $l = 0, 1, \dots, n-1$ that

$$d\rho \wedge d^c\rho \wedge \omega_\varphi^j \wedge \omega_\psi^k \wedge \omega^l = 0$$

if $j + k + l = n - 1$ (see [Bl03b] for details). □

The Monge–Ampère measure $(dd^cu)^n$ can be defined also for some not locally bounded psh u: for example if u is bounded outside a compact set (see [Dem93]). However, there is no uniqueness in this more general class.

Exercise 5.20 Show that

$$\varphi(z) := \log|z| - g(z), \quad \psi(z) := \log||z|| - g(z), \quad z \in \mathbb{C}^n,$$

where

$$|z| = \sqrt{|z_1^2| + \dots |z_n|^2}, \quad ||z|| = \max\{|z_1|, \dots, |z_n|\}, \quad g(z) = \frac{1}{2}\log(1 + |z|^2),$$

define admissible φ, ψ on $\mathbb{P}^n$ (with the Fubini–Study metric $\omega = dd^c \log |Z|$) such that $\mathcal{M}(\varphi) = \mathcal{M}(\psi)$ but $\varphi - \psi \neq const.$

Closely analyzing the proof of Theorem 5.19 one can get the quantitative estimate

$$\int_M d(\varphi - \psi) \wedge d^c(\varphi - \psi) \wedge \omega^{n-1} \leq C \left(\int_M (\psi - \varphi)(\mathcal{M}(\varphi) - \mathcal{M}(\psi)) \right)^{2^{1-n}}, \tag{5.26}$$

where C is a constant depending only on n and upper bounds of $||\varphi||_\infty$, $||\psi||_\infty$ and V. The following Poincaré–Sobolev inequality on compact Riemannian manifolds M of real dimension m

$$||v||^2_{2m/(m-2)} \leq C_M \left(\left(\int_M v \right)^2 + ||Dv||_2^2 \right), \quad v \in W^{1,2}(M),$$

is more difficult to prove than (5.13) (see [Siu87, p.140]; the proof uses an isoperimetric inequality). This combined with (5.26) immediately gives the following stability of weak solutions whose Monge–Ampère measures have densities in L^1

$$||\varphi - \psi||_{2n/(n-1)} \leq C||f - g||_1^{2^{-n}},$$

provided that $\int_M \varphi\omega^n = \int_M \psi\omega^n$, where $\mathcal{M}(\varphi) = f\omega^n$, $\mathcal{M}(\psi) = g\omega^n$, and C depends only on M and on upper bounds for $||\varphi||_\infty$ and $||\psi||_\infty$.

For the proof of the existence part of Theorem 5.18 we will need a uniform stability.

Theorem 5.21 [Kol03] *Assume that $\varphi, \psi \in C(M)$ are admissible and that $\mathcal{M}(\varphi) = f\omega^n$, $\mathcal{M}(\psi) = g\omega^n$ for some $f, g \in C(M)$ with $||f - g||_\infty \leq 1/2$. Let φ, ψ be normalized by* $\max_M(\varphi - \psi) = \max_M(\psi - \varphi)$. *Then*

$$\operatorname*{osc}_M(\varphi - \psi) \leq C||f - g||_\infty^{1/n}, \tag{5.27}$$

where C depends only on M and on upper bounds for $||f||_\infty$, $||g||_\infty$.

Proof. First assume that we have proved the theorem for smooth, strongly admissible φ, ψ. From this and Theorem 5.3 we can easily deduce Theorem 5.18: any nonnegative $f \in C(M)$ with $\int_M f\omega^n = V$ can be uniformly approximated by positive $f_j \in C^\infty(M)$ with $\int_M f_j\omega^n = V$ and the existence part of Theorem 5.18 follows from the continuity of the Monge–Ampère operator for uniform sequences. Then obviously (5.27) will also hold for nonsmooth φ, ψ. It is thus enough to consider $\varphi, \psi \in C^\infty(M)$ with $f, g > 0$.

By Theorem 5.6 we may assume that $-C_1 \leq \varphi, \psi \leq 0$. Without loss of generality we may replace the normalizing condition $\max_M(\varphi - \psi) = \max_M(\psi - \varphi)$ with the normalizing inequalities

$$0 < \max_M(\varphi - \psi) \leq 2 \max_M(\psi - \varphi) \leq 4 \max_M(\psi - \varphi) \tag{5.28}$$

and then by the Sard theorem we may assume that 0 is the regular value $\varphi - \psi$ (we will only need that the boundaries of the sets $\{\varphi < \psi\}$ and $\{\psi < \varphi\}$ have volume zero). We will need the following comparison principle.

Proposition 5.22 *If $\varphi, \psi \in C(M)$ are admissible then*

$$\int_{\{\psi<\varphi\}} \mathcal{M}(\varphi) \leq \int_{\{\psi<\varphi\}} \mathcal{M}(\psi).$$

Proof. It is a repetition of the proof for psh functions in domains in $\mathbb{C}^n$ (see [Ceg88, p. 43]). For $\varepsilon > 0$ let $\varphi_\varepsilon := \max\{\varphi, \psi + \varepsilon\}$. Then $\varphi_\varepsilon = \psi + \varepsilon$ in a neighborhood of the boundary of $\{\psi < \varphi\}$ and by the Stokes theorem

$$\int_{\{\psi<\varphi\}} \mathcal{M}(\varphi_\varepsilon) = \int_{\{\psi<\varphi\}} \mathcal{M}(\psi).$$

But φ_ε decreases to φ in $\{\psi < \varphi\}$ as ε decreases to 0 and we get the result from the weak convergence $\mathcal{M}(\varphi_\varepsilon) \to \mathcal{M}(\varphi)$. □

Proof of Theorem 5.21, (continued) Set $\delta := ||f - g||_\infty$. We may assume that $\int_{\{\psi<\varphi\}} (f + g)\omega^n \leq V$ (otherwise replace φ with ψ). Then

$$\int_{\{\psi<\varphi\}} f\omega^n \leq \frac{1+\delta}{2} V \leq \frac{3}{4} V.$$

We can find $h \in C^\infty(M)$ such that $0 < h \leq C_2$, $\int_M h\omega^n = V$ and $h \geq f + 1/C_3$ in $\{\psi < \varphi\}$ (here we use the fact that the boundary of $\{\psi < \varphi\}$ has volume zero, and thus $\int_{\mathrm{int}\{\psi \geq \varphi\}} f\omega^n \geq V/4$). Since $||f||_\infty$ is under control, we will get

$$h^{1/n} \geq f^{1/n} + 1/C_4 \quad \text{in } \{\psi < \varphi\}.$$

By Theorem 5.3 there is an admissible $\rho \in C^\infty(M)$ such that $(\omega + dd^c\rho)^n = h\omega^n$ and $-C_5 \leq \rho \leq -C_1$.

Let a be such that $0 < a < \max_M(\varphi - \psi)$. Then

$$\emptyset \neq \{\psi < \varphi - a\} \subset E := \{\psi < (1 - t)\varphi + t\rho\} \subset \{\psi < \varphi\},$$

where $t = a/C_5 \leq 1$. Using Proposition 5.22 and the concavity of (5.18) we get

$$\begin{aligned}\int_E g\omega^n \geq \int_E \left(\omega + (1-t)dd^c\varphi + tdd^c\rho\right)^n &\geq \int_E \left((1-t)f^{1/n} + th^{1/n}\right)^n \omega^n \\ &\geq \int_E \left(f^{1/n} + t/C_4\right)^n \omega^n \\ &\geq \int_E f\omega^n + \frac{t^n}{C_4^n}\mathrm{vol}\,(E).\end{aligned}$$

On the other hand, we have $g \leq f + \delta$ and therefore

$$\int_E g\omega^n \leq \int_E f\omega^n + \delta \mathrm{vol}\,(E).$$

Hence $a \leq C_4 C_5 \delta^{1/n}$ and the estimate follows, since by (5.28)

$$\operatorname*{osc}_M(\varphi - \psi) \leq 3 \max_M(\varphi - \psi).$$

□

Note that in the proof of Theorem 5.21, contrary to Theorem 5.19, we have heavily relied on Theorem 5.3 (in the construction of ρ).

From Theorems 5.3 and 5.13 we get the following regularity in the nondegenerate ($f > 0$) and degenerate ($f \geq 0$) case.

Theorem 5.23 *Let $\varphi \in C(M)$ be admissible and assume that $\mathcal{M}(\varphi) = f\omega^n$. Then*
i) $f \in C^\infty$, $f > 0 \Longrightarrow \varphi \in C^\infty$;
ii) $f^{1/(n-1)} \in C^{1,1} \Longrightarrow \Delta\varphi \in L^\infty \Longrightarrow \varphi \in C^{1,\alpha}$, $\alpha < 1$.