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On the complex Monge–Ampère operator for quasi-plurisubharmonic functions with analytic singularities

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Abstract

We give a modified, very natural definition for the complex Monge–Ampère operator for an ω-plurisubharmonic (psh) function φ with analytic singularities on a Kähler manifold (X, ω) of dimension n which has the property $\int_X (\omega + dd^c\varphi)^n = \int_X \omega^n$ if X is compact. This means that, unlike in the previous definition, no mass is lost here. In fact, the definition works for any smooth (1,1)-form ω (we need neither closedness nor positivity) and quasi-psh φ with analytic singularities.

1. *Introduction*

A plurisubharmonic (psh) function u defined on a complex manifold X of dimension n is said to have *analytic singularities* if locally it can be written in the form

$$u = c \log |F| + v,$$

where $c \geqslant 0$ is a constant, $F = (f_1, \ldots, f_m)$ is a tuple of holomorphic functions which does not vanish everywhere, and v is bounded. By Z we will denote the singular set of u, that is the analytic variety in X where $u = -\infty$. If $m = 1$ then we say that u has *divisorial singularities*. In this case v has to be a bounded psh function.

For a psh u with analytic singularities and $k = 2, \ldots, n$ Andersson–Wulcan [2] inductively defined the complex Monge–Ampère operator as follows:

$$(dd^c u)^k := dd^c\big(u \mathbf{1}_{X\setminus Z}(dd^c u)^{k-1}\big).$$

In order for this definition to work one has to show two things: first that $T_{k-1} := \mathbf{1}_{X\setminus Z}(dd^c u)^{k-1}$ extends across Z as a closed current on X and second that uT_{k-1} has locally finite mass near Z. If $u = \log|f| + v$ has divisorial singularities then

$$(dd^c u)^k = dd^c u \wedge (dd^c v)^{k-1} = [f = 0] \wedge (dd^c v)^{k-1} + (dd^c v)^k,$$

where $[f = 0] = dd^c(\log|f|)$ is the current of integration along $\{f = 0\}$.

As long as Z is not discrete, it follows in particular that $\nabla u \notin L^2_{loc}$, and then $u \notin \mathcal{D}$ where $\mathcal{D}$ is a domain of definition of the complex Monge–Ampère equation defined in [4, 5]. It is a maximal subclass of the class of psh functions where one can define the complex Monge–Ampère operator in such a way that it is continuous (in the weak* topology of currents) for decreasing sequences. Therefore, we cannot expect that this operator will be continuous for smooth regularizations of psh functions with analytic singularities. In fact, for

$$u(z) := \log|z_1 \cdots z_n|, \quad z \in \mathbb{C}^n,$$

one has $(dd^c u)^n = 0$ but $(dd^c u_j)^n \to c_n \delta_0$ for some $c_n > 0$, where $u_j = u * \rho_{1/j}$ are the standard regularizations of u by convolution (see [6]).

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Recently in [1] it was shown however that this definition of the complex Monge–Ampère operator is continuous for special regularizations, namely if u is approximated by a sequence of the form $\chi_j \circ u$, where $\chi_j' \geqslant 0$, $\chi_j'' \geqslant 0$. Perhaps the most obvious choice would be $\chi_j(t) = \max\{t, -j\}$. This result can be treated as an alternative definition of $(dd^c u)^k$.

If ω is a Kähler form and φ is an ω-psh function with analytic singularities then $(\omega + dd^c\varphi)^k$ was defined in [1] as $(dd^c(g + \varphi))^k$, where g is a local potential for ω (that is $\omega = dd^c g$). There are two problems with this definition. First of all, if X is compact then

$$\int_X (\omega + dd^c\varphi)^n \leqslant \int_X \omega^n \tag{1}$$

but it may happen that one has a strict inequality here, that is, some mass is lost in the process. For example, if $X = \mathbb{P}^n$ is the projective space with the Fubini-Study metric ω and

$$\varphi([Z]) = \log \frac{|z_1|}{|Z|}, \quad Z = (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}. \tag{2}$$

then $(\omega + dd^c\varphi)^n = 0$ on $\mathbb{P}^n$ (provided that $n \geqslant 2$). The second problem with this definition is that it does not work if ω is not closed.

The aim of this paper is to propose a modified, probably more natural definition of the Monge–Ampère operator $(\omega + dd^c\varphi)^k$ for which we will have equality in (1). Another advantage is that it will also work in the Hermitian, not necessarily Kähler setting. The idea is to consider, instead of local approximations of the form $\chi_j \circ u$, where $u = g + \varphi$, the global ones $\chi_j \circ \varphi$. (If we assume in addition that $\chi_j' \leqslant 1$ then $\chi_j \circ \varphi$ is ω-psh.) For $\chi_j(t) = \max\{t, -j\}$ this means that instead of approximating u by $\max\{u, -j\}$ we do it by $\max\{u, g - j\}$. This also shows how, for a local psh function u with analytic singularities, we can differently define the Monge–Ampère operator $(dd^c u)^k$ relatively to a strongly psh g.

In fact, positivity of ω is not essential. It is also convenient to assume that φ is quasi-psh, that is locally can be written as $\varphi = u + \psi$, where u is psh and ψ is smooth. We say that φ has *analytic singularities* if u does.

Our main result is the following.

THEOREM 1. *Let φ be a negative quasi-psh function with analytic singularities on a complex manifold X of dimension n and assume that η is a smooth (1,1)-form on X. Then for $k = 1, \dots, n$ the current $(\eta + dd^c\varphi)^k$ can be uniquely defined in such a way that if χ_j is a sequence of bounded nondecreasing convex functions on $(-\infty, 0]$ such that $\chi_j(t)$ decreases to t as j increases to ∞ then*

$$(\eta + dd^c(\chi_j \circ \varphi))^k \longrightarrow (\eta + dd^c\varphi)^k$$

weakly as $j \to \infty$.

This definition immediately gives

COROLLARY 2. *Assume that φ is an ω-psh function with analytic singularities on a compact Kähler manifold (X, ω). Then*

$$\int_X (\omega + dd^c\varphi)^n = \int_X \omega^n.$$

This might potentially be useful to some ω-psh functions appearing naturally in complex geometry, see, for example, [8].

The operator defined in Theorem 1 comes from the expansion

$$(\eta + dd^c\varphi)^k = \sum_{l=1}^{k} \binom{k}{l} (dd^c\varphi)^l \wedge \eta^{k-l} \tag{3}$$

which reduces the proof to the case $\eta = 0$, that is, a generalization of [1, Theorem 1.1] from psh to quasi-psh functions. In fact, then the definition from [2] works as well:

$$(dd^c\varphi)^k := dd^c\big(\varphi \mathbf{1}_{X\setminus Z}(dd^c\varphi)^{k-1}\big). \tag{4}$$

If we use (3) and (4) for the function given by (2) then for $k \geqslant 1$

$$(dd^c\varphi)^k = (-1)^k dd^c\varphi \wedge \omega^{k-1}$$

and

$$(\omega + dd^c\varphi)^k = (\omega + dd^c\varphi) \wedge \omega^{k-1} = [z_1 = 0] \wedge \omega^{k-1}.$$

The difference between these two definitions of the complex Monge–Ampère operator with respect to a Kähler form comes from the fact that if φ is quasi-psh with analytic singularities and ψ is smooth then $(dd^c(\varphi + \psi))^k$ is in general not equal to the corresponding binomial expansion. This is exploited in the next result.

THEOREM 3. *Assume that φ is a quasi-psh function with analytic singularities on a complex manifold X with singular set Z. Then for any smooth ψ and $k = 1, \dots, n$ one has*

$$\begin{aligned}(dd^c(\varphi+\psi))^k &= \sum_{l=0}^{k} \binom{k}{l} (dd^c\varphi)^l \wedge (dd^c\psi)^{k-l} - \mathbf{1}_Z \sum_{l=1}^{k-1} \binom{k-1}{l} (dd^c\varphi)^l \wedge (dd^c\psi)^{k-l} \\ &= \sum_{l=0}^{k} \binom{k}{l} (dd^c\varphi)^l \wedge (dd^c\psi)^{k-l} - \mathbf{1}_Z \sum_{l=1}^{k-1} (dd^c(\varphi+\psi))^l \wedge (dd^c\psi)^{k-l}.\end{aligned} \tag{5}$$

Note that [1, Theorem 1.2] is an immediate consequence of the second equality in (5), (3) and Corollary 2. It is also clear that this is the same measure as the one defined in [7, Remark 3.7].

One should note that the current $(\eta + dd^c\varphi)^k$ in Theorem 1 does not really depend on the (1,1)-form $\eta + dd^c\varphi$ but on both η and φ. This is clear from Theorem 3 if we take, for example, $\eta = dd^c\psi$. Therefore $(\eta + dd^c\varphi)^k$ should be viewed as the operator acting on φ and depending on η.

2. *Proofs*

Proof of Theorem 1. By (3) we may assume that $\eta = 0$. The proof will now be similar to that of [1, Theorem 1.2]. Shrinking X if necessary, we may write $\varphi = u + \psi$ where $u = c\log|F| + v$ is psh with analytic singularities and ψ is smooth. By resolution of singularities there exists a complex manifold X' and a proper holomorphic mapping $\pi : X' \to X$ such that the exceptional divisor $E := \pi^{-1}Z$ is a hypersurface in X' and $\pi|_{X'\setminus E} \to X \setminus Z$ is a biholomorphism. We then locally have $\pi^*F = f_0F'$, where f_0 is a holomorphic function such that $E = \{f_0 = 0\}$ and F' is a nonvanishing tuple of holomorphic functions. Then

$$\pi^*\varphi = \log|f_0| + \log|F'| + \pi^*v + \pi^*\psi$$

has divisorial singularities. Since

$$(dd^c(\chi_j \circ \varphi))^k = \pi_*(dd^c(\chi_j \circ \pi^*\varphi))^k$$

and since

$$\pi_*(dd^c\pi^*\varphi)^k = (dd^c\varphi)^k,$$

where we use (4) (see [**1**, **2**]), it follows that it is enough to prove the theorem when φ has divisorial singularities.

We may then write

$$\varphi = c\log|f| + v + \psi,$$

where f is holomorphic, v is bounded psh, and ψ is smooth. First, consider the case when χ_j are smooth. Then on $\{f \neq 0\}$

$$\begin{aligned}(dd^c(\chi_j\circ\varphi))^k &= \left(\chi_j''\circ\varphi\, d\varphi\wedge d^c\varphi + \chi_j'\circ\varphi\, dd^c\varphi\right)^k\\ &= \left(k\chi_j''\circ\varphi\, d\varphi\wedge d^c\varphi + \chi_j'\circ\varphi\, dd^c\varphi\right)\wedge(\chi'\circ\varphi\, dd^c\varphi)^{k-1}\\ &= d\left((\chi_j'\circ\varphi)^k d^c\varphi\right)\wedge(dd^c\varphi)^{k-1}\\ &= dd^c(\gamma_j\circ\varphi)\wedge(dd^c(v+\psi))^{k-1}\\ &= dd^c\left(\gamma_j\circ\varphi\,(dd^c(v+\psi))^{k-1}\right),\end{aligned}$$

where γ_j is a uniquely determined convex function on $(-\infty,0]$ satisfying $\gamma_j(-1)=\chi_j(-1)$ and $\gamma_j' = (\chi_j')^k$. Then it is also bounded nondecreasing (in t) and $\gamma_j(t)$ decreases to t as $j\to\infty$. (Similar argument was used in [**3**].) Since

$$dd^c(\gamma_j\circ\varphi) \geqslant \gamma_j'\circ\varphi\, dd^c\psi$$

and $0\leqslant\chi_j'\leqslant C$ on $(-\infty,-\varepsilon]$ (where $\varphi\leqslant-\varepsilon$), it follows that locally we may write $\gamma_j\circ\varphi = u_j+\widetilde{\psi}$, where u_j is psh and $\widetilde{\psi}$ is smooth (and independent of j). Using [**1**, Theorem 2.1] we now get that

$$(dd^c(\chi_j\circ\varphi))^k \longrightarrow dd^c\varphi\wedge(dd^c(v+\psi))^{k-1}$$

weakly as $j\to\infty$ when χ_j are smooth. Approximating arbitrary functions χ_j by smooth ones we can get rid of this assumption. $\square$

Proof of Theorem 3. With the notation $\eta = dd^c\psi$ we have

$$\begin{aligned}(dd^c(\varphi+\psi))^k &= dd^c\left((\varphi+\psi)\mathbf{1}_{X\setminus Z}(dd^c(\varphi+\psi))^{k-1}\right)\\ &= dd^c\left(\varphi\mathbf{1}_{X\setminus Z}\sum_{l=0}^{k-1}\binom{k-1}{l}(dd^c\varphi)^{k-1-l}\wedge\eta^l\right) + \mathbf{1}_{X\setminus Z}(dd^c(\varphi+\psi))^{k-1}\wedge\eta\\ &= \sum_{l=0}^{k-1}\binom{k-1}{l}(dd^c\varphi)^{k-l}\wedge\eta^l + \mathbf{1}_{X\setminus Z}\sum_{l=0}^{k-1}\binom{k-1}{l}(dd^c\varphi)^l\wedge\eta^{k-l}.\end{aligned}$$

Since

$$\binom{k-1}{k-l} + \binom{k-1}{l} = \binom{k}{l},$$

the first equality follows. We have also obtained that

$$(dd^c(\varphi+\psi))^k = \sum_{l=1}^{k}\binom{k-1}{k-l}(dd^c\varphi)^l\wedge\eta^{k-l} + \mathbf{1}_{X\setminus Z}(dd^c(\varphi+\psi))^{k-1}\wedge\eta.$$

Continuing this way we will get

$$(dd^c(\varphi+\psi))^k = \sum_{l=0}^{k} a_l (dd^c\varphi)^l \wedge \eta^{k-l} - \mathbf{1}_Z \sum_{l=1}^{k-1} (dd^c(\varphi+\psi))^l \wedge (dd^c\psi)^{k-l}$$

for some a_l independent of φ and ψ. Since it also holds for nonsingular case, it is clear that $a_l = \binom{k}{l}$ and the second equality in Theorem 3 follows. □

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