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A note on maximal plurisubharmonic functions

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Maqolada maksimal plyurisubgarmonik funksiyaning chegaralanmaganligi lokal xossa bo‘ladimi? Biz quyidagi taxminni inkor qilamiz: agar u maksimal plyurisubgarmonik funksiya bo‘lsa u xolda j soni 1 ga intilganda (ddcmaxu, -j)n ifoda 0 ga kuchsiz yaqinlashadi.

Мы обсуждаем проблему: является ли неограниченность максимальной плюрисубгармонической функции локальным свойством. Мы опровергаем следующее предположение: если u является максимальной плюрисубгармонической функцией, то (ddcmaxu, -j)n сходится слабо к 0, когда j стремится к 1.

Introduction

The notion of maximality for plurisubharmonic functions was introduced by Sadullaev in [9]: a plurisubharmonic function u in a domain Ω in $\mathbb{C}^n$ is called *maximal* if for any other plurisubharmonic function v in Ω satisfying $v \leq u$ outside a compact subset of Ω one has $v \leq u$ in Ω. For $n = 1$ maximal functions are precisely the harmonic ones. One of the main result of the Bedford-Taylor theory of the complex Monge-Ampère operator [1]-[2] is the following characterization:

Theorem 1 A locally bounded plurisubharmonic function u is maximal if and only if $(dd^c u)^n = 0$.

The *if* part follows from the comparison principle [2], whereas the *only if* part is a consequence of the solution of the Dirichlet problem [1].

Theorem 1 immediately gives

Corollary 1 Maximality is a local notion for locally bounded plurisubharmonic functions.

The domain of definition D of the complex Monge-Ampère operator is the biggest subclass of the class of plurisubharmonic functions where the operator can be (uniquely) extended from the class of smooth plurisubharmonic functions (as a regular measure) so that it is continuous for decreasing sequences. It was characterized in [4]-[5], for example for $n = 2$ we have $D = PSH \cap W^{1,2}_{loc}$. It turns out that the class D coincides with the class E introduced by Cegrell [7].

One can generalize Theorem 1 as follows (see [4]):

Theorem 2 A function $u \in D$ is maximal if and only if $(dd^c u)^n = 0$.

Corollary 2 Maximality is a local notion for functions from the class D.

The proof of Theorem 2 is similar to that of Theorem 1, the extra result one uses is the following theorem of Sadullaev [9] (see also [3]):

Theorem 3 If u_j is a sequence of locally bounded plurisubharmonic functions decreasing to a plurisubharmonic function u such that $(dd^c u_j)^n$ tends weakly to 0, then u is maximal.

A natural question arises whether a converse is true. It turns out that the answer is *no*, as the following example of Cegrell [6] shows: $\log|zw|$ is a maximal plurisubharmonic function in $\mathbb{C}^2$ (in fact every function of the form $\log|F|$, where F is holomorphic, is maximal in dimension $n \geq 2$) but if we consider for example the sequence

$$u_j := \frac{1}{2}\log(|z|^2 + 1/j) + \frac{1}{2}\log(|w|^2 + 1/j)$$

then one can show that $(dd^c u_j)^2$ tends weakly to $2^7\pi^2\delta_0$ (δ_0 denotes the point mass at the origin).

The open problem remains whether maximality is a local notion, without any additional assumption. A positive answer to the following conjecture would solve this problem:

$$u \text{ maximal } \Rightarrow (dd^c \max\{u, -j\})^n \text{ tends weakly to 0 as } j \to \infty.$$

The main goal of this note is to give a counterexample to this conjecture.

Example

In the unit bidisk Δ^2 set

$$u(z,w) := -\sqrt{\log|z|\,\log|w|}, \quad |z| < 1, \ |w| < 1.1$$

Then u is plurisubharmonic in Δ^2. We claim that u is maximal in $\Delta^2 \setminus \{(0,0)\}$. Indeed, it follows easily from the fact that u is harmonic on the punctured disks

$$\Delta_* \ni \zeta \longmapsto (\zeta^n, \lambda\zeta^m) \in \Delta^2,$$

where $|\lambda| = 1$, $n, m = 1, 2, \ldots$ (and from the continuity of u away from the axis).

On the other hand, note that u is not maximal in Δ^2: the function

$$v(z,w) :=$$

$$-\sqrt{-\log|z| - \log|w| + 1} \ |z| \leq 1/e, \ |w| \leq 1/e - \sqrt{\log|z|\,\log|w|} \text{otherwise}$$

is plurisubharmonic in Δ^2 but $\{u < v\} = \{|z| < 1/e, \ |w| < 1/e\}$ (note that v is maximal there).

We will need a lemma:

LemmaSet

$$L : (\mathbb{C}_*)^n \ni (z_1, \dots, z_n) \longmapsto (\log|z_1|, \dots, \log|z_n|) \in \mathbb{R}^n.$$

Assume that γ is a convex function defined on an open convex subset D of $\mathbb{R}^n$. Then for a Borel subset E of D we have

$$\int_{L^{-1}(E)} (dd^c(\gamma \circ L))^n = n!(2\pi)^n vol(N_\gamma(E)),$$

where

$$N_\gamma(E) = \bigcup_{x^0 \in E} \{y \in \mathbb{R}^n : \langle x - x^0, y\rangle + \gamma(x^0) \le \gamma(x),\ x \in D\}$$

is the gradient image of γ on E.

Proof We have

$$(dd^c(\gamma \circ L))^n = \frac{n!}{|z_1|^2 \dots |z_n|^2} L^*(M\gamma),$$

where M is the real Monge-Ampère operator ($M\gamma = \det D^2\gamma$ for smooth γ and it is a regular measure for general convex γ). Therefore

$$\int_{L^{-1}(E)} (dd^c(\gamma \circ L))^n = n! \int_{\exp E} \frac{1}{r_1^2 \dots r_n^2} \widetilde{L}^*(M\gamma),$$

where

$$\exp E = \{(e^{x_1}, \dots, e^{x_n}) : (x_1, \dots, x_n) \in E\}$$

and

$$\widetilde{L} : (\mathbb{R}_+)^n \ni (r_1, \dots, r_n) \longmapsto (\log r_1, \dots, \log r_n) \in \mathbb{R}^n.$$

The lemma now follows after a polar change of coordinates and since

$$\int_E M\gamma = vol(N_\gamma(E))$$

(see e.g. [8]).

We will now apply the lemma to the function

$$\gamma_j(x, y) = \max\{-\sqrt{xy}, -j\}, \quad x, y \in \mathbb{R}_-,$$

and the set

$$E := \{\log a \le x \le \log b\},$$

where $0 < a < b < 1$. One can then easily check that

$$N_{\gamma_j}(E) = \{(s,t) \in \mathbb{R}^2 : st \leq \frac{1}{4},\ \frac{\log^2 b}{j^2}s \leq t \leq \frac{\log^2 a}{j^2}s\}$$

and

$$vol(N_{\gamma_j}(E)) = \frac{1}{4}\log\frac{\log a}{\log b}.$$

Therefore, for u given by (1) and $u_j := \max\{u, -j\}$ we get

$$\int_{\{a\leq|z|\leq b\}}(dd^c u_j)^2 = 2\pi^2\log\frac{\log a}{\log b}.$$

Since the measures $(dd^c u_j)^2$ are supported on the set $\{u = -j\}$, it follows that on $\Delta^2 \setminus \{(0,0)\}$ they weakly tend to the measure supported on $(\Delta_* \times \{0\}) \cup (\{0\} \times \Delta_*)$. For example on $\Delta_* \times \{0\}$ it is given by

$$\frac{\pi}{-|z|^2\log|z|}d\lambda,$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}$ (and similarly on $\{0\} \times \Delta_*$).

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