ANNALES
POLONICI MATHEMATICI
101.3 (2011)

On the product property for the transfinite diameter

by Zbigniew Błocki, Armen Edigarian, and Józef Siciak (Kraków)

Abstract. We give a pluripotential-theoretic proof of the product property for the transfinite diameter originally shown by Bloom and Calvi. The main tool is the Rumely formula expressing the transfinite diameter in terms of the global extremal function.

1. Introduction. For a compact subset K of $\mathbb{C}^n$ the transfinite diameter $\delta(K)$ is defined as follows. Denote by $\mathcal{P}_d(\mathbb{C}^n)$ the set of complex polynomials of degree $\le d$. It is a complex vector space of dimension

$$N(d) := \binom{d+n}{d}.$$

The monomials $e_1, \dots, e_{N(d)}$ of degree $\le d$ form a basis in $\mathcal{P}_d(\mathbb{C}^n)$. We can define the Vandermonde determinant as

$$\mathrm{VDM}(\zeta^1, \dots, \zeta^{N(d)}) := \det(e_j(\zeta^k)), \qquad \zeta^1, \dots, \zeta^k \in \mathbb{C}^n.$$

It is a homogeneous polynomial in $\mathbb{C}^{nN(d)}$ of degree $dnN(d)/(n+1)$. Set

$$\delta_d(K) := \max_{\zeta^1, \dots, \zeta^{N(d)} \in K} |\mathrm{VDM}(\zeta^1, \dots, \zeta^{N(d)})|^{\frac{n+1}{dnN(d)}}.$$

Leja [L] conjectured (for $n = 2$) that the sequence $V_d(K)$ is decreasing. This problem is in fact still open. Zaharyuta [Z1] proved however that it is convergent, and we define

$$\delta(K) = \lim_{d \to \infty} \delta_d(K).$$

Methods from [Z1] were used in [J] to prove a similar result for the so-called homogeneous transfinite diameter.

Bloom and Calvi [BC1] (see also [BC2]) showed the following product property: for compact $K \subset \mathbb{C}^n$, $L \subset \mathbb{C}^m$ we have

$$\delta(K \times L) = \delta(K)^{\frac{n}{n+m}} \delta(L)^{\frac{m}{n+m}}. \tag{1}$$

2010 *Mathematics Subject Classification*: 32U15, 32W20.
Key words and phrases: transfinite diameter, complex Monge–Ampère operator.

DOI: 10.4064/ap101-3-1

The main tool to prove (1) in [BC1] was a formula for the transfinite diameter in terms of orthogonal polynomials with respect to a positive measure satisfying the Bernstein–Markov inequality.

A special case of (1) was earlier shown in [SS]:

$$\delta(K_1 \times \cdots \times K_n) = (\delta(K_1)\ldots\delta(K_n))^{1/n}, \quad K_1,\ldots,K_n \subset \mathbb{C}.$$

Other proofs of (1) were given in [CM] (in the spirit of [SS]) and [HM].

The main goal of this note is to give yet another proof of the product property. The main tool will be the formula due to Rumely [R] expressing $\delta(K)$ in terms of Monge–Ampère measures of sections of the Robin function for the global extremal function of K. We also use methods developed in [B1] where the product property for the equilibrium measure was proved.

The original proof from [R] uses nontrivial number theory (e.g. Arakelov theory). Recently, Berman and Boucksom [BB] proved the Rumely formula using essentially only pluripotential theory. Their methods (that work in a much more general setting) are presented for subsets of $\mathbb{C}^n$ in the self-contained notes of Levenberg [Le] which we have found very useful.

2. Preliminaries. For a compact $K \subset \mathbb{C}^n$ the global extremal function was originally defined in [S1] as

$$V_K := \sup\left\{\frac{1}{d}\log|P| : P \in \mathcal{P}_d(\mathbb{C}^n),\ |P| \le 1 \text{ on } K,\ d = 1,2,\ldots\right\}.$$

Zaharyuta [Z2] proved that

$$V_K = \sup\{u \in \mathcal{L}_+(\mathbb{C}^n) : u \le 0 \text{ on } K\},$$

where

$$\mathcal{L}_+(\mathbb{C}^n) = \{u \in \mathrm{PSH}(\mathbb{C}^n) : -C_1 + \log_+|z| \le u \le C_2 + \log_+|z|\}$$

(C_1, C_2 are constants depending on u, and $v_+ := \max\{v,0\}$). It was shown in [S2] that $V_K^* \in \mathcal{L}_+(\mathbb{C}^n)$ if and only if K is not pluripolar, which is equivalent to $\delta(K) > 0$. (By v^* we denote the upper regularization of v.)

Let $\mathcal{H}_+(\mathbb{C}^n)$ denote the class of homogeneous plurisubharmonic functions v in $\mathbb{C}^n$:

$$v(\lambda z) = v(z) + \log|\lambda|, \quad z \in \mathbb{C}^n,\ \lambda \in \mathbb{C},$$

such that $\max\{v,0\} \in \mathcal{L}_+(\mathbb{C}^n)$. It was shown in [S3] that the mapping

$$\mathcal{H}_+(\mathbb{C}^n) \ni v \mapsto v(\cdot,1) \in \mathcal{L}_+(\mathbb{C}^{n-1})$$

is bijective.

For $u \in \mathcal{L}_+(\mathbb{C}^n)$ the *Robin function* is defined as

$$\rho_u(z) := \limsup_{|\lambda|\to\infty}(u(\lambda z) - \log|\lambda|), \quad z \in \mathbb{C}^n.$$

Let $\widetilde{u} \in \mathcal{H}_+(\mathbb{C}^{n+1})$ be such that $u = \widetilde{u}(\cdot, 1)$. Then $\rho_u = \widetilde{u}(\cdot, 0)$ and it follows in particular that ρ_u is upper semicontinuous, in fact $\rho_u \in \mathcal{H}_+(\mathbb{C}^n)$.

From now on assume that K is not pluripolar. We consider the Robin function for K:

$$\rho_K := \rho_{V_K^*} \in \mathcal{H}_+(\mathbb{C}^n).$$

We can now recall the Rumely formula. For $p = 0, 1, \ldots, n-1$ define the following sections of ρ_K:

$$\rho_K^{(p)}(z) := \rho_K(z_1, \ldots, z_p, 1, 0, \ldots, 0), \qquad z = (z_1, \ldots, z_p) \in \mathbb{C}^p.$$

Then

$$-\log \delta(K) = \frac{1}{n}\Big(\rho_K^{(0)} + \sum_{p=1}^{n-1} \frac{1}{(2\pi)^p} \int_{\mathbb{C}^p} \rho_K^{(p)} (dd^c \rho_K^{(p)})^p\Big). \tag{2}$$

We use here Bedford–Taylor's theory of the complex Monge–Ampère operator for locally bounded plurisubharmonic functions (see [BT], and also [D], [K] or [B2]). The operator d can be written as $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$, so that $dd^c = 2i\partial\bar{\partial}$.

We will use the fact that

$$\int_{\mathbb{C}^n} (dd^c u)^n = (2\pi)^n, \qquad u \in \mathcal{L}_+(\mathbb{C}^n) \tag{3}$$

(originally proved in [T]).

3. Proof of the product property. We will need some preparatory results. The first one is from [B1], but we present a much simpler proof of it from [B2].

Proposition 1. *Let u, v be plurisubharmonic and locally bounded on an open subset of $\mathbb{C}^n$ and let $w := \max\{u, v\}$. If $2 \le p \le n$, then*

$$(dd^c w)^p = dd^c w \wedge \sum_{k=0}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-1-k} - \sum_{k=1}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-k}.$$

Proof. We may assume that u, v are smooth. A simple inductive argument reduces the proof to the case $p = 2$. For $\varepsilon > 0$ set $w_\varepsilon := \max\{u + \varepsilon, v\}$. In the open set $\{u + \varepsilon > v\}$ we have $w_\varepsilon - u = \varepsilon$, whereas $w - v = 0$ in $\{u < v\}$. Therefore $dd^c(w_\varepsilon - u) \wedge dd^c(w - v) = 0$ for every $\varepsilon > 0$ and taking the limit we conclude that $dd^c(w - u) \wedge dd^c(w - v) = 0$. ∎

Proposition 2. *Let u, v be locally bounded plurisubharmonic functions defined in open subsets $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$, respectively. Assume in addition that $(dd^c u)^n = 0$ and set*

$$w(z', z'') := \max\{u(z'), v(z'')\}, \qquad z' \in U,\ z'' \in V.$$

Then for a fixed $z'' \in V$ one has

$$(dd^c w)^{n+m} = (dd^c \max\{u, v(z'')\})^n \wedge (dd^c v)^m.$$

Proof. We will follow a method from [B1]. By Proposition 1 we have

$$(dd^c w)^{n+m} = dd^c w \wedge (dd^c u)^{n-1} \wedge (dd^c v)^m.$$

Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a convex function such that $\chi = 0$ on $(-\infty, -1]$ and $\chi(t) = t$ for $t \geq 1$. Set $\alpha := v - u$ (we now treat u and v as functions defined in $U \times V$) and

$$w_j := u + \frac{1}{j}\chi(j\alpha).$$

Then w_j is a sequence of locally bounded functions decreasing to w. We have

$$dd^c w_j = (1 - \chi'(j\alpha)) dd^c u + \chi'(j\alpha) dd^c v + j\chi''(j\alpha) d\alpha \wedge d^c\alpha$$

(which implies in particular that w_j are plurisubharmonic) and therefore

$$dd^c w_j \wedge (dd^c u)^{n-1} \wedge (dd^c v)^m = j\chi''(j\alpha) du \wedge d^c u \wedge (dd^c u)^{n-1} \wedge (dd^c v)^m.$$

Fix $z'' \in V$ and let $a := v(z'')$. Then (using again that $(dd^c u)^n = 0$)

$$j\chi''(j\alpha) du \wedge d^c u \wedge (dd^c u)^{n-1} = dd^c u_j \wedge (dd^c u)^{n-1},$$

where $u_j := a + \chi(j(u-a))/j$ decreases to $u_a := \max\{u, a\} = w(\cdot, z'')$. We thus get

$$(dd^c w)^{n+m} = dd^c u_a \wedge (dd^c u)^{n-1} \wedge (dd^c v)^m.$$

It remains to notice that $dd^c u_a \wedge (dd^c u)^{n-1} = (dd^c u_a)^n$ by Proposition 1. ∎

PROPOSITION 3. *For $u \in \mathcal{H}_+(\mathbb{C}^n)$ and $v \in \mathcal{L}_+(\mathbb{C}^m)$ set*

$$w(z', z'') := \max\{u(z'), v(z'')\}, \qquad z' \in \mathbb{C}^n,\ z'' \in \mathbb{C}^m.$$

Then

$$\int_{\mathbb{C}^{n+m}} w(dd^c w)^{n+m} = (2\pi)^n \int_{\mathbb{C}^m} v(dd^c v)^m.$$

Proof. Fix $z'' \in \mathbb{C}^m$ and set $a := v(z'')$, $u_a := \max\{u, a\}$. Since Monge–Ampère masses of locally bounded plurisubharmonic functions put no mass on pluripolar sets, we have

$$\int_{\{0\}\times\mathbb{C}^m} w(dd^c w)^{n+m} = 0.$$

Thus by Proposition 2 and the Fubini Theorem (note that $(dd^c u)^n = 0$ in $\mathbb{C}^n \setminus \{0\}$) it is enough to show that

$$\int_{\mathbb{C}^n} u_a(dd^c u_a)^n = (2\pi)^n a. \tag{4}$$

Indeed, for smooth (or even continuous) u (away from the origin) the measure $(dd^c u_a)^n$ is concentrated on the set $\{u = a\}$ and by (3) is of total mass $(2\pi)^n$. Since $u_a = a$ on the support of $(dd^c u_a)^n$, we clearly have (4) for continuous u. The general case follows since every element of $\mathcal{H}_+(\mathbb{C}^n)$ can be approximated by a decreasing sequence of smooth (away from the origin) functions from $\mathcal{H}_+(\mathbb{C}^n)$ (see [S3]). ∎

Proof of the product property. It was shown in [S2] that

$$V_{K\times L}(z',z'') = \max\{V_K(z'),V_L(z'')\}, \qquad z'\in\mathbb{C}^n,\ z''\in\mathbb{C}^m \tag{5}$$

(see [Ze] or [B2] for the proof using the Monge–Ampère operator). Without loss of generality we may assume that K, L are not pluripolar (in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively). By (5) we clearly have

$$\rho_{K\times L}(z',z'') = \max\{\rho_K(z'),\rho_L(z'')\}, \qquad z'\in\mathbb{C}^n,\ z''\in\mathbb{C}^m.$$

Therefore

$$\rho^{(p)}_{K\times L}(z',z'') = \begin{cases} \rho^{(p)}_K(z'), & p=0,1,\ldots,n-1,\\ \max\{\rho_K(z'),\rho_L^{(p-n)}(z'')\}, & p=n,n+1,\ldots,n+m-1,\end{cases}$$

and thus by Rumely's formula (2),

$$-(n+m)\log\delta(K\times L) = -n\log(K) + \sum_{q=0}^{m-1}\frac{1}{(2\pi)^{n+q}}\int\limits_{\mathbb{C}^{n+q}} w^{(q)}(dd^c w^{(q)})^{n+q},$$

where

$$w^{(q)}(z',z'') = \max\{\rho_K(z'),\rho_L^{(q)}(z'')\}.$$

From Proposition 3 we get

$$\int\limits_{\mathbb{C}^{n+q}} w^{(q)}(dd^c w^{(q)})^{n+q} = (2\pi)^n\int\limits_{\mathbb{C}^q}\rho_L^{(q)}(dd^c\rho_L^{(q)})^{n+q}$$

and it suffices to use Rumely's formula once again. ∎

Acknowledgements. We would like to thank Norm Levenberg for calling our attention to [BC1] and [CM]. Part of this research was done at the Erwin Schrödinger Institute in Vienna (Błocki and Edigarian). Błocki was also partially supported by the projects N N201 3679 33 and 189/6 PR EU/2007/7 of the Polish Ministry of Science and Higher Education, and Edigarian by the project N N201 3614 36.

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Zbigniew Błocki, Armen Edigarian, Józef Siciak
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: Zbigniew.Blocki@im.uj.edu.pl
Armen.Edigarian@im.uj.edu.pl
Jozef.Siciak@im.uj.edu.pl

Received 3.3.2010
and in final form 29.10.2010 (2178)