ON THE REGULARITY OF THE COMPLEX MONGE-AMPÈRE OPERATOR

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Abstract. In this paper we show how to apply some results on fully nonlinear elliptic operators to the theory of the complex Monge-Ampère operator.

1. Introduction

If u is a smooth plurisubharmonic function, the complex Monge-Ampère operator on u is defined by

$$Mu := \det(u_{j\bar{k}}), \tag{1.1}$$

where $u_{j\bar{k}} = \partial^2 u/\partial z_j \partial \bar{z}_k$, $j, k = 1, \dots, n$. Bedford and Taylor [2] showed in particular that one can define Mu as a nonnegative Borel measure for any continuous plurisubharmonic function u in such a way that (1.1) holds if u is C^∞-smooth and if $u_j \longrightarrow u$ uniformly then $Mu_j \longrightarrow Mu$ weakly. Obviously this determines Mu uniquely for every u, since continuous plurisubharmonic functions can be locally uniformly approximated by smooth plurisubharmonic functions.

We see that $\det(u_{j\bar{k}})$ makes sense and is a nonnegative Borel measure if u is in $W^{2,n}$.

Proposition 1.1. *If u is plurisubharmonic, continuous and in $W^{2,n}$ then (1.1) holds.*

Proof. . Let $u^\varepsilon = u * \rho_\varepsilon$ denote the standard regularization of u. Then $u^\varepsilon_{j\bar{k}} = u_{j\bar{k}} * \rho_\varepsilon \longrightarrow u_{j\bar{k}}$ in L^n_{loc} as $\varepsilon \downarrow 0$. We have to show that $Mu^\varepsilon = \det(u^\varepsilon_{j\bar{k}})$ tends weakly to $\det(u_{j\bar{k}})$. It is enough to observe that if $f_j^\varepsilon \longrightarrow f_j$ in L^n_{loc}, $j = 1, \dots, n$, then $f_1^\varepsilon \dots f_n^\varepsilon \longrightarrow f_1 \dots f_n$ in L^1_{loc}. Indeed, write

$$f_1^\varepsilon \dots f_n^\varepsilon - f_1 \dots f_n = \sum_{k=1}^n f_1 \dots f_{k-1}(f_k^\varepsilon - f_k) f_{k+1}^\varepsilon \dots f_n^\varepsilon$$

and use the Hölder inequality. □

In this paper we discuss regularity of the operator M. Our basic question will be: under what conditions regularity of Mu implies regularity of u? For example, if $n = 1$ then $M = \Delta/4$ and for every $k = 0, 1, \dots$ and $0 < \alpha < 1$ $\Delta u \in C^{k,\alpha}$

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implies $u \in C^{k+2,\alpha}$ (see e.g. [12]). We want to find out what happens with this kind of regularity if $n \geq 2$.

First, we see that if for example u does not depend on one variable then Mu=0. Thus, we should always assume $Mu > 0$. Even then we have the following example.

Example. For $\beta > 0$ set
$$u(z) = (|z_1|^2 + 1)|z'|^{2\beta}$$
where $z' = (z_2, \dots, z_n)$. Then u is continuous and plurisubharmonic on $\mathbb{C}^n$ since $\log u$ is plurisubharmonic. Moreover u is C^∞ on the set $\{z' \neq 0\}$ and one can compute that

$$Mu = \beta^n (1 + |z_1|^2)^{n-2} |z'|^{2(\beta n - n + 1)} \tag{1.2}$$

there. However, since $\{z' = 0\}$ is in particular a pluripolar set, by [3] we have

$$\int_{\{z'=0\}} Mu = 0$$

and thus (1.2) holds in $\mathbb{C}^n$.

If we take $\beta = 1 - 1/n$ then $Mu \in C^\infty$, $Mu > 0$ in $\mathbb{C}^n$ but $u \notin C^{1,\alpha}$ for $\alpha > 1 - 2/n$ (if $n = 2$ then even $u \notin C^1$) and $u \notin W^{2,p}$ for $p \geq n(n-1)$.

The paper is organized as follows: in section 2 we show how to use the (real) theory of nonlinear elliptic operators to get results on the complex Monge-Ampère operator. Necessary facts from the matrix theory are collected in the appendix. In section 3 we recall known facts about corresponding problems for the real Monge-Ampère operator. Finally, section 4 is devoted to the problem of regularity of exhaustion plurisubharmonic functions in hyperconevex domains. So far, it has been solved only in the case of convex domains.

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2. The complex Monge-Ampère operator as a nonlinear elliptic operator

Consider an equation of the form

$$F(D^2 u) = g(x) \tag{2.1}$$

where F is a function defined on the space of symmetric matrices from $\mathbb{R}^{m \times m}$. We always assume that
$$F \text{ is concave.}$$
We say that F is *elliptic* on a function u defined on $\Omega \subset \mathbb{R}^m$ if the matrix

$$(F_{pq}) = \left(\frac{\partial F}{\partial u_{x_p x_q}} \right)$$

is positive on $\{D^2u(x) : x \in \Omega\}$. We call F *uniformly elliptic* on u if there exist constants $0 < \lambda < \Lambda < \infty$ such that

$$\lambda \le \lambda_{\min}\left((F_{pq})\right) \le \lambda_{\max}\left((F_{pq})\right) \le \Lambda,$$

where $\lambda_{\min}(A)$ (resp. $\lambda_{\max}(A)$) denotes the minimal (resp. maximal) eigenvalue of A. For a detailed discussion of nonlinear elliptic operators see [12].

Now suppose that u is a function defined on $\Omega \subset \mathbb{C}^n$. Then we may write

$$D^2u = \begin{pmatrix} \left(u_{x_jx_k}\right) & \left(u_{x_jy_k}\right) \\ \left(u_{y_jx_k}\right) & \left(u_{y_jy_k}\right) \end{pmatrix}.$$

One can easily compute that

$$u_{z_j\bar{z}_k} = \frac{1}{4}\left(u_{x_jx_k} + u_{y_jy_k} + i\left(u_{x_jy_k} - u_{y_jx_k}\right)\right).$$

If $A \in \mathbb{R}^{2n\times 2n}$ then in the variables $(x, y) = (x_1, \dots, x_n, y_1, \dots, , y_n)$ we may write

$$A = \begin{pmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{pmatrix},$$

where $A_{xx}, A_{xy}, A_{yx}, A_{yy} \in \mathbb{R}^{n\times n}$. Let

$$H(A) := \frac{1}{4}\left(A_{xx} + A_{yy} + i\left(A_{xy} - A_{yx}\right)\right) \in \mathbb{C}^{n\times n} \tag{2.2}$$

so that $D^2_{\mathbb{C}}u = H(D^2u)$, where $D^2_{\mathbb{C}}u = \left(u_{z_j\bar{z}_k}\right)$ is the complex Hessian of u.

Consider an equation of the form

$$\widetilde{F}(D^2_{\mathbb{C}}u) = \psi(z).$$

We want to see when this equation is elliptic in the sense as above (that is as a real equation). We set

$$F(D^2u) := \widetilde{F}(D^2_{\mathbb{C}}u) = \widetilde{F}(H(D^2u)).$$

Consider matrices

$$(F_{pq}) = \begin{pmatrix} \left(F_{u_{x_jx_k}}\right) & \left(F_{u_{x_jy_k}}\right) \\ \left(F_{u_{y_jx_k}}\right) & \left(F_{u_{y_jy_k}}\right) \end{pmatrix}$$

and

$$(\widetilde{F}_{jk}) = \left(\widetilde{F}_{u_{z_j\bar{z}_k}}\right).$$

Proposition 2.1. *We have*

$$\lambda_{\min}((F_{pq})) = \frac{1}{4}\lambda_{\min}((\widetilde{F}_{jk}))$$
$$\lambda_{\max}((F_{pq})) = \frac{1}{4}\lambda_{\max}((\widetilde{F}_{jk}))$$
$$(\det(F_{pq}))^{1/2n} \ge \frac{1}{4}(\det(\widetilde{F}_{jk}))^{1/n}.$$

Proof. We claim that for a symmetric $A \in \mathbb{R}^{2n\times 2n}$ we have

$$\text{trace}\,\left((F_{pq})A^t\right) = \text{trace}\,\left((\widetilde{F}_{jk})H(A)^t\right). \tag{2.3}$$

Indeed, write $H(A) = (h_{jk})$ and

$$\begin{aligned}\text{trace}\,\left((F_{pq})A^t\right) &= \sum_{p,q} F_{pq}a_{pq}\\ &= \frac{d}{dt}F(D^2u + tA)\Big|_{t=0}\\ &= \frac{d}{dt}\widetilde{F}(D^2_{\mathbb{C}}u + tH(A))\Big|_{t=0}\\ &= \sum_{j,k}\widetilde{F}_{jk}h_{jk}\\ &= \text{trace}\,\left((\widetilde{F}_{jk})H(A)^t\right).\end{aligned}$$

If we take $A = (a_pa_q)$, where

$$a = (a_1, \dots, a_{2n}) = (a_{x_1}, \dots, a_{x_n}, a_{y_1}, \dots, a_{y_n}),$$

then $h_{jk} = (a_{x_j} + ia_{y_j})\overline{(a_{x_j} + ia_{y_j})}/4$ and by (2.3)

$$\sum_{p,q} F_{pq}a_pa_q = \frac{1}{4}\sum_{j,k}\widetilde{F}_{jk}\left(a_{x_j} + iay_j\right)\overline{\left(a_{x_j} + iay_j\right)}.$$

This shows the first two equalities. To prove the last inequality we use Lemma A1 and (2.3) again:

$$(\det(F_{pq}))^{1/2n} = \frac{1}{2n}\inf_A \text{trace}\,\left((F_{pq})A^t\right) = \frac{1}{2n}\inf_A \text{trace}\,\left((\widetilde{F}_{jk})H(A)^t\right),$$

the infimum being taken over symmetric, positive $A \in \mathbb{R}^{2n\times 2n}$ with $\det A \geq 1$. For such A by Lemma A4 we have $(\det H(A))^{1/n} \geq 1/2$ and the desired estimate follows from Lemma A1. □

Now we write the complex Monge-Ampère equation in the form

$$F(D^2u) = \widetilde{F}(D^2_{\mathbb{C}}u) = (\det(D^2_{\mathbb{C}}u))^{1/n} = \psi(z), \tag{2.4}$$

where $\psi > 0$ and u is plurisubharmonic and in $W^{2,n}$. Assume that u is such that

$$\lambda|w|^2 \leq \sum_{j,k} u_{z_j\bar{z}_k}w_j\overline{w}_k \leq \Lambda|w|^2.$$

Then $\widetilde{F}_{jk} = \psi^{1-n}M_{jk}/n = \psi\left((D^2_{\mathbb{C}}u)^{-1}\right)^t/n$, where M_{jk} is a cominor of the matrix $D^2_{\mathbb{C}}u$. By Proposition 2.1

$$\begin{aligned}\lambda_{\min}((F_{pq})) &\geq \frac{1}{4n}\frac{\psi}{\Lambda}\\ \lambda_{\max}((F_{pq})) &\leq \frac{1}{4n}\frac{\psi}{\lambda}\\ (\det(F_{pq}))^{1/2n} &\geq \frac{1}{4n}.\end{aligned} \tag{2.5}$$

We shall now invoke a few results from the theory of nonlinear elliptic operators and use them to obtain results on local regularity of the complex Monge-Ampère operator. From the standard elliptic theory it follows that if u is a C^2 solution of (2.1), F, g are in $C^{k,\alpha}$ for some $k = 1, 2, \dots$, $0 < \alpha < 1$ and F is uniformly elliptic on u then $u \in C^{k+2,\alpha}$ (see [12], Lemma 17.16).

Theorem 2.2. *If u is plurisubharmonic and C^2, $Mu \in C^{k,\alpha}$ for some $k = 1, 2, \dots$, $0 < \alpha < 1$ and $Mu > 0$ then $u \in C^{k+2,\alpha}$.*

We want to relax the assumption that u must be C^2. We do this using two results due to Trudinger [15]:

Theorem 2.3. *Let $u \in W^{2,m}(\Omega)$, Ω open in $\mathbb{R}^m$, be a solution of (2.1). Assume that F is elliptic on u, $\det(F_{pq}(D^2u)) \geq 1$ and $F_{pq}(D^2u) \in L^s(\Omega)$, $p, q = 1, \dots, m$, for some $s > m$. If $g \in W^{2,m}(\Omega)$ then $u \in C^{1,1}$.*

Theorem 2.4. *Assume that F is uniformly elliptic on $u \in W^{2,m}$, a solution of (2.1). If $g \in W^{2,m}$ then $u \in C^{2,\alpha}$ for some $0 < \alpha < 1$.*

They yield the following fact about the complex Monge-Ampère operator.

Theorem 2.5. *Let u be plurisubharmonic and $u \in W^{2,p}$ for some $p > 2n(n-1)$. If $Mu \in W^{2,2n}$, $Mu > 0$ then u is $C^{2,\alpha}$ for some $0 < \alpha < 1$.*

Proof. Consider (2.4). We may write

$$F_{p'q'}(D^2u) = \frac{1}{n}\psi^{1-n}P(D^2u)$$

where P is a polynomial of degree $n-1$. Therefore $F_{p'q'}(D^2u) \in L^{p/(n-1)}$ and $p/(n-1) > 2n$ which is the real dimension of $\mathbb{C}^n$. By (2.5) and Theorem 2.3, $u \in C^{1,1}$. By Theorem 2.4 it remains to show that the operator given by (2.4) is uniformly elliptic on u. Since u is $C^{1,1}$, we may take $\Lambda = \sup |D^2u|$ and $\lambda = \Lambda^{1-n} \inf Mu$. □

Theorems 2.2 and 2.5 give

Theorem 2.6. *If u is plurisubharmonic and $u \in W^{2,p}$ for some $p > 2n(n-1)$ then*

$$Mu \in C^\infty, \ Mu > 0 \textit{ implies } u \in C^\infty. \tag{2.6}$$

A function u is called *strongly plurisubharmonic* in an open set Ω in $\mathbb{C}^n$ if for every $\Omega' \Subset \Omega$ there exists $\lambda > 0$ such that

$$\sum_{j,k} u_{j\bar{k}} w_j \overline{w}_k \geq \lambda |w|^2, \quad w \in \mathbb{C}^n, \tag{2.7}$$

in Ω'. The following result shows that (2.6) holds for strongly plurisubharmonic functions.

Theorem 2.7. *Let u be a function satisfying (2.7) and such that $Mu \in L^\infty$, $Mu \le K$. Then*

$$\sum_{j,k} u_{j\bar{k}} w_j \overline{w}_k \le \frac{K}{\lambda^{n-1}}|w|^2, \quad w \in \mathbb{C}^n.$$

In particular, $\Delta u \in L^\infty$ and thus $u \in W^{2,p}$ for every $p < \infty$.

Proof. The result is clear if we already know that $u \in W^{2,n}$ - then $Mu = \det(u_{j\bar{k}})$ and

$$\lambda_{\max}((u_{j\bar{k}})) \le \frac{\det(u_{j\bar{k}})}{\left(\lambda_{\min}((u_{j\bar{k}}))\right)^{n-1}} \le \frac{K}{\lambda^{n-1}}.$$

For arbitrary u set $u^\varepsilon = u * \rho_\varepsilon$ and take a nonnegative test function ϕ. Then for $w \in \mathbb{C}^n$ we have

$$\begin{aligned}
\int \phi \sum u_{j\bar{k}} w_j \overline{w}_k &= \lim_{\varepsilon \to 0} \int \phi \sum u^\varepsilon_{j\bar{k}} w_j \overline{w}_k \\
&\le \lim_{\varepsilon \to 0} \int \phi \frac{Mu^\varepsilon}{\lambda^{n-1}} |w|^2 \\
&= \int \phi \frac{Mu}{\lambda^{n-1}} |w|^2 \\
&\le \int \phi \frac{K}{\lambda^{n-1}} |w|^2
\end{aligned}$$

and the theorem follows. □

3. Regularity of the real Monge-Ampère operator

If u is a smooth convex function in $\Omega \subset \mathbb{R}^n$ then

$$M_{\mathbb{R}} u = \det\left(u_{x_j x_k}\right)$$

and similarly as in the complex case one can define $M_{\mathbb{R}}u$ for arbitrary convex u. Another way to see this is to treat convex functions as plurisubharmonic functions of $x + iy$ not depending on y. Then $M_{\mathbb{R}}u = 4^n Mu$. However, more classical way to define $M_{\mathbb{R}}u$ for arbitrary u is a geometric one - see [13] and the references given there.

The following example is due to Pogorelov.

Example. For $\beta \ge 1/2$ let

$$u(x) = (x_1^2 + 1)|x'|^{2\beta}$$

where $x' = (x_2, \dots, x_n)$. Then u is convex with respect to the variables x_1 and x' and one can compute that on the set $\{x' \ne 0\}$ we have

$$M_{\mathbb{R}} u = 2^n \beta^{n-1} (1 + x_1^2)^{n-2} ((2\beta - 1) - (2\beta + 1)x_1^2)|x'|^{2(\beta n + 1 - n)}. \tag{3.1}$$

Thus u is convex in a neighborhood of the origin if $\beta > 1/2$. Moreover,

$$\int_{\{x'=0\}} M_{\mathbb{R}} u = \text{volume}\left(\nabla u(\{x' = 0\})\right) = 0$$

because $\partial u/\partial x_1 = 0$ if $x' = 0$, therefore (3.1) holds everywhere where u is convex. If $\beta = 1 - 1/n$ then Mu is C^∞ but $u \notin C^{1,\alpha}$ for $\alpha > 1 - 2/n$ and $u \notin W^{2,p}$ for $p \geq n(n-1)/2$.

The above example works only if $n \geq 3$ because we have to assume $\beta = 1-1/n > 1/2$. It is an old result due to Aleksandrov [1] that in $\mathbb{R}^2$ $M_{\mathbb{R}}u > 0$ implies that u is strictly convex (that is the graph of u contains no line segment). The example shows that it is not the case if $n \geq 3$. (See [6] for a related result.)

The following theorem is due to Urbas [16].

Theorem 3.1. *If u is convex and either $u \in C^{1,\alpha}$ for some $\alpha > 1-2/n$ or $u \in W^{2,p}$ for some $p > n(n-1)/2$ then*

$$M_{\mathbb{R}}u \in C^\infty,\ M_{\mathbb{R}}u > 0 \text{ implies } u \in C^\infty. \tag{3.2}$$

The proof of Theorem 3.1 makes use of the following result due to Pogorelov (see [9] and [10] for proofs without gaps).

Theorem 3.2. *Let u be a convex function in a bounded convex domain Ω in $\mathbb{R}^n$ such that $\lim_{x\to\partial\Omega} u(x) = 0$. Then (3.2) holds in Ω.*

Theorem 3.2 also easily implies the following fact.

Corollary 3.3. *(3.2) holds for strictly convex functions.*

Together with the result of Aleksandrov it means that if $n = 2$ then (3.2) holds for every convex u without any extra assumption. However, the example given in the introduction shows that there is nothing like that for the complex Monge-Ampère operator in $\mathbb{C}^2$.

4. Regularity in hyperconvex domains

A bounded domain Ω in $\mathbb{C}^n$ is called *hyperconvex* if there exists a bounded plurisubharmonic exhaustion function. The main question of this section is whether a counterpart of Theorem 3.2 holds for the complex Monge-Ampère operator and hyperconvex domains. By [7] and [14] it is enough to find an interior gradient estimate for smooth solutions of the complex Monge-Ampère equation vanishing on the boundary. In [5] it is done for convex domains. Together with a solution of the Dirichlet problem in hyperconvex domains (see [4]) one can get the following result.

Theorem 4.1. *Let Ω be a bounded convex domain in $\mathbb{C}^n$. Assume that $\psi \in C^\infty(\Omega)$ is positive and $|D\psi^{1/n}|$ is bounded in Ω. Then there exists a unique $u \in C^\infty(\Omega)$ which is plurisubharmonic, $\lim_{z\to\partial\Omega} u(z) = 0$ and $Mu = \psi$ in Ω.*

This gives a very partial counterpart of Corollary 3.3.

Corollary 4.2. *If u is a strictly convex function on an open set in $\mathbb{C}^n$ (thus u is in particular continuous and plurisubharmonic) then (2.6) holds.*

Appendix

For the convenience of the reader we collect here some elementary results from the matrix theory. Some of them can be found for example in [11] and [8].

Lemma A1. *If H is a hermitian, nonnegative matrix in $\mathbb{C}^{n\times n}$ then*

$$(\det H)^{1/n} = \frac{1}{n}\inf_G \operatorname{trace}(HG^t),$$

the infimum being taken over all hermitian, nonnegative G with $\det G \geq 1$.

Proof. If H and G are hermitian and nonnegative then so is HG^t and we may find a unitary matrix P so that $P^{-1}HG^tP$ is a diagonal matrix. Then from the inequality between geometric and arithmetic means we obtain

$$(\det(HG^t))^{1/n} = (\det(P^{-1}HG^tP))^{1/n} \leq \frac{1}{n}\operatorname{trace}(P^{-1}HG^tP) = \frac{1}{n}\operatorname{trace}(HG^t).$$

Thus we have "$\leq$". To show the reverse inequality let Q be a unitary matrix such that $Q^{-1}HQ = (\lambda_j\delta_{jk})$. Then it is enough to take $G = (g_j\delta_{jk})$, where $g_j = 1$ if $\lambda_j = 0$ and $g_j = (\lambda_1 \dots \lambda_n)^{1/n}/\lambda_j$ otherwise. $\square$

Lemma A2. *If $H, G \in \mathbb{C}^{n\times n}$ are hermitian and nonnegative then*

$$(\det(H+G))^{1/n} \geq (\det H)^{1/n} + (\det G)^{1/n}.$$

Proof. By Lemma A1

$$\begin{aligned}(\det(H+G))^{1/n} &= \frac{1}{n}\inf_K \operatorname{trace}((H+G)K^t)\\ &\geq \frac{1}{n}\inf_K \operatorname{trace}(HK^t) + \frac{1}{n}\inf_K \operatorname{trace}(GK^t)\\ &= (\det H)^{1/n} + (\det G)^{1/n}. \quad \square\end{aligned}$$

Lemma A3. *Let $X, Y \in \mathbb{R}^{n\times n}$. Suppose that $\lambda_1, \dots, \lambda_n$ are all eigenvalues of the matrix $X + iY \in \mathbb{C}^{n\times n}$. Then eigenvalues of*

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$$

are precisely $\lambda_1, \overline{\lambda}_1, \dots, \lambda_n, \overline{\lambda}_n$. In particular

$$\det\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = |\det(X+iY)|^2.$$

Proof. Let λ be an eigenvalue of $X + iY$ and let $z \in \mathbb{C}^n$ be the corresponding eigenvector. Then

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}\begin{pmatrix} iz \\ z \end{pmatrix} = \begin{pmatrix} i(x+iy)z \\ (iY+X)z \end{pmatrix} = \lambda\begin{pmatrix} iz \\ z \end{pmatrix}$$

and thus

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}\overline{\begin{pmatrix} iz \\ z \end{pmatrix}} = \overline{\lambda}\overline{\begin{pmatrix} iz \\ z \end{pmatrix}}.$$

It remains to show that if vectors $z^1, \dots, z^n$ form a basis of $\mathbb{C}^n$ then the vectors

$$\begin{pmatrix} iz^1 \\ z^1 \end{pmatrix}, \overline{\begin{pmatrix} iz^1 \\ z^1 \end{pmatrix}}, \dots, \begin{pmatrix} iz^n \\ z^n \end{pmatrix}, \overline{\begin{pmatrix} iz^n \\ z^n \end{pmatrix}}$$

form a basis of $\mathbb{C}^{2n}$. $\square$

Lemma A4. *Let $A \in \mathbb{R}^{2n\times 2n}$ be a symmetric matrix such that $H(A) \geq 0$, where $H(A)$ is defined by (2.2). Then*

$$\lambda_{\min}(H(A)) \geq \frac{1}{2}\lambda_{\min}(A)$$
$$\lambda_{\max}(H(A)) \leq \frac{1}{2}\lambda_{\max}(A)$$
$$(\det H(A))^{1/n} \geq \frac{1}{2}(detA)^{1/2n}.$$

Proof. By Lemma A3 $4H(A)$ has the same eigenvalues as the matrix

$$\begin{pmatrix} A_{xx}+A_{yy} & A_{yx}-A_{xy} \\ A_{xy}-A_{yx} & A_{xx}+A_{yy} \end{pmatrix} = \begin{pmatrix} A_{xx} & A_{yx} \\ A_{xy} & A_{yy} \end{pmatrix} + \begin{pmatrix} A_{yy} & -A_{xy} \\ -A_{yx} & A_{xx} \end{pmatrix} = A^t + P^{-1}A^tP,$$

where

$$P = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Of course A^t and $P^{-1}A^tP$ have the same eigenvalues as A and thus the first two estimates follow. The third one is a consequence of Lemma A2. □

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