SINGULAR SETS
OF SEPARATELY ANALYTIC FUNCTIONS

by
Zbigniew Błocki

Abstract. In this paper we complete the characterization of singular sets of separately analytic functions. In the case of functions of two variables it was earlier done by J. Saint Raymond and J. Siciak.

-1.Introduction. If Ω is an open subset of $\mathbf{R}^{n_1}\times\cdots\times\mathbf{R}^{n_s}$, then we say that a function $f:\Omega\longrightarrow\mathbf{C}$ is $p-$*separately analytic* ($1\le p<s$), if for every $x^0=\left(x_1^0,\dots,x_s^0\right)\in\Omega$ and for every sequence $1\le i_1<\cdots<i_p\le s$ the function

$$\left(x_{i_1},\dots,x_{i_p}\right)\longrightarrow f\left(x_1^0,\dots,x_{i_1},\dots,x_{i_p},\dots,x_s^0\right)$$

is analytic in a neighbourhood of $\left(x_{i_1}^0,\dots,x_{i_p}^0\right)$. For a $p-$separately analyitc function f in Ω let

$$\mathrm{A}\left(f\right):=\{x\in\Omega:f\text{ is analytic in a neighbourhood of }x\}$$

denote its *set of analycity*, and $\mathrm{S}\left(f\right):=\Omega\setminus\mathrm{A}\left(f\right)$ - its *singular set*.

If X and Y are any sets, $S\subset X\times Y$ and $\left(x^0,y^0\right)\in X\times Y$, then we denote $S\left(x^0,\bullet\right):=\left\{y\in Y:\left(x^0,y\right)\in S\right\}$, $S\left(\bullet,y^0\right):=\left\{x\in X:\left(x,y^0\right)\in S\right\}$.

The following theorems characterize singular sets of separately analytic functions:

Theorem A. *If f is $p-$separately analytic in Ω, then for every sequence $1\le j_1<\cdots<j_q\le s$, where $q:=s-p$, the projection of $S\left(f\right)$ on $\mathbf{R}^{n_{j_1}}\times\cdots\times\mathbf{R}^{n_{j_q}}$ is pluripolar (in $\mathbf{C}^{n_{j_1}}\times\cdots\times\mathbf{C}^{n_{j_q}}$).*

Theorem B. *Let S be a closed subset of Ω such that for every sequence $1\le j_1<\cdots<j_q\le s$, where $q:=s-p$, the projection of S on $\mathbf{R}^{n_{j_1}}\times\cdots\times\mathbf{R}^{n_{j_q}}$ is pluripolar. Then there exists $p-$separately anlytic function f in Ω such that $S=S\left(f\right)$.*

Theorem C. *Let f be a $p-$separately analytic in Ω. If $1\le k<s$, then for quasi almost all $x\in\mathbf{R}^{n_1}\times\cdots\times\mathbf{R}^{n_k}$ (that is for $x\in\mathbf{R}^{n_1}\times\cdots\times\mathbf{R}^{n_k}\setminus P$, where P is pluripolar) $S\left(f\left(x,\bullet\right)\right)=S\left(f\right)\left(x,\bullet\right)$.*

Theorems A and B in case $s=2$, $p=n_1=n_2=1$ were proved by Saint Raymond [2]. This result was generelized by Siciak [5], who proved theorem A for $p\ge s/2$ and theorem B. The aim of this paper is to give a proof of theorem C and, as a trivial consequence, we get theorem A.

0.Preliminaries. We need the following two theorems:

Siciak's theorem ([3]; see also [4], theorem 9.7). *Let for $j=1,\dots,s$ $D_j=D_j^1\times\cdots\times D_j^{n_j}$, D_j^t - open sets in $\mathbf{C}$, symmetric with respect to x_t-axis ($t=1,\dots,n_j$), $K_j=K_j^1\times\cdots\times K_j^{n_j}$, K_j^t - closed intervals in $D_j^t\cap\mathbf{R}$. Let f be a separately holomorphic function in*

$$X:=\bigcup_{j=1}^{s}K_1\times\cdots\times D_j\times\cdots\times K_s$$

(that is for every $(x_1, \dots, x_s) \in K_1 \times \cdots \times K_s$ and for every $j = 1, \dots, s$ the function $f(x_1, \dots, x_{j-1}, \bullet, x_{j+1}, \dots, x_s)$ is holomorphic in D_j). Then f can be extended to a holomorphic function in a neighbourhood of X.[1]

Bedford-Taylor theorem on negligible sets [1]. *If $\{u_j\}_{j\in J}$ is a family of plurisubharmonic functions locally bounded from above then the set*

$$\left\{ z \in D : u(z) := \sup_{j\in J} u_j(z) < u^*(z) \right\}$$

is pluripolar (u^* denotes the upper regularization of u).

1.Proofs.

Theorem C $\Rightarrow$ theorem A: We may assume that $(j_1, \dots, j_q) = (1, \dots, q)$. Then it is enough to take $k = q$ and see that for $x \in \mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_k}$ $\mathrm{S}(f(x, \bullet)) = \emptyset$.

Proof of theorem C: We can write

$$\mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_s} = (\mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_p}) \times \cdots \times (\mathbf{R}^{n_{ap+1}} \times \cdots \times \mathbf{R}^{n_k}) \\ \times (\mathbf{R}^{n_{k+1}} \times \cdots \times \mathbf{R}^{n_{k+p}}) \times \cdots \times (\mathbf{R}^{n_{k+bp+1}} \times \cdots \times \mathbf{R}^{n_s}),$$

where $a = [k/p]$, $b = [(s-k)/p]$. Then f is separately analytic (that is $1-$separately analytic) with respect to such variables. Therefore it is enough to prove theorem C for $p = 1$. Let $\{X_\nu \times Y_\nu\}_{\nu\in\mathbf{N}}$ be a countable family of closed intervals in $(\mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_k}) \times (\mathbf{R}^{n_{k+1}} \times \cdots \times \mathbf{R}^{n_s})$ such that $\bigcup_{\nu=1}^{\infty} X_\nu \times Y_\nu = \Omega$. It is clear that the set

$$\left\{ x \in \mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_k} : \mathrm{S}(f(x, \bullet)) \subsetneq \mathrm{S}(f)(x, \bullet) \right\}$$

is contained in

$$\bigcup_{\nu=1}^{\infty} \left\{ x \in X_\nu : \mathrm{S}(f(x, \bullet)) \cap Y_\nu \subsetneq \mathrm{S}(f)(x, \bullet) \cap Y_\nu \right\}.$$

Hence we may assume that f is separately analytic in a closed interval $I_1 \times \cdots \times I_s \subset \mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_s}$ (that is analytic in some open neighbourhood of this interval).

To prove theorem C we have to show that the set

$$Z_{f,k} := \left\{ x \in I_1 \times \cdots \times I_k : \mathrm{S}(f(x, \bullet)) \subsetneq \mathrm{S}(f)(x, \bullet) \right\}$$

is pluripolar.

For $(x, y) \in (I_1 \times \cdots \times I_k) \times (I_{k+1} \times \cdots \times I_s)$ such that $y \in \mathrm{A}(f(x, \bullet))$ define

$$Q_{f,k}(x, y) := \sup_{|\alpha|\geq 1} \left| \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial y^\alpha}(x, y) \right|^{1/|\alpha|}$$

[1] In fact we use the Siciak's theorem under additional assumption that f is bounded. In this case the proof of the theorem is much simpler - it can be deduced from theorem 2a in [3].

(of course $Q_{f,k}(x,y) < +\infty$ and $f(x,\bullet)$ is holomorphic in the polydisc $P(y, 1/Q_{f,k}(x,y))$).

For $y \in I_{k+1} \times \cdots \times I_s$ let

$$F_{f,k}(y) := \{x \in \mathrm{A}(f)(\bullet, y) : Q_{f,k}(\bullet, y) \text{ is not upper semicontinuous at } x\}.$$

Theorem C is proved by induction with respect to k. First assume that $k = 1$.

1^0 *The projection of* $S(f)$ *on* $I_2 \times \cdots \times I_s$ *is nowhere dense in* $\mathbf{R}^{n_2} \times \cdots \times \mathbf{R}^{n_s}$*, that is there exists* U *- open, dense subset of* $I_2 \times \cdots \times I_s$ *such that* $I_1 \times U \subset A(f)$*. In particular* $A(f)$ *is dense in* $I_1 \times \cdots \times I_s$*.*

■ Induction with respect to s. The same proof applies to the case $s = 2$ and to the step $s-1 \Rightarrow s$. We have

$$I_1 = [a_1, b_1] \times \cdots \times [a_{n_1}, b_{n_1}].$$

Define for $m \in \mathbf{N}$

$$I_1^m := \left\{ z \in \mathbf{C}^{n_1} : \max_{1 \le t \le s} \operatorname{dist}(z_t, [a_t, b_t]) < 1/m \right\},$$

$$E_m := \left\{ y_1 \in I_2 \times \cdots \times I_s : f(\bullet, y_1) \text{ is holomorphic in } I_1^m, \sup_{z \in I_1^m} |f(z, y_1)| \le m \right\}.$$

We have $E_m \subset E_{m+1}$, $\bigcup_{m=1}^{\infty} E_m = I_2 \times \cdots \times I_s$. First we want to show that the set $U_1 := \bigcup_{m=1}^{\infty} \operatorname{int} E_m$ is dense in $I_2 \times \cdots \times I_s$. Let Y' be a closed interval in $I_2 \times \cdots \times I_s$, and $\mathcal{H}$ - a family of closed intervals which form a countable base of topology in Y'. For $x_1 \in I_1$ the set $\mathrm{A}(f(x_1, \bullet))$ is dense: this is trivial if $s = 2$ and follows from the inductive assumption if $s \ge 3$. Therefore, if for $H \in \mathcal{H}$ we denote

$$A_H := \{x_1 \in I_1 : f(x_1, \bullet) \text{ is analytic in } H\},$$

it follows that $\bigcup_{H \in \mathcal{H}} A_H = I_1$. We claim that there exists $H_0 \in \mathcal{H}$ such that the set A_{H_0} is determinig for functions holomorphic in a complex neighbourhood of I_1. Indeed, suppose it is not so. Then all the sets A_H $(H \in \mathcal{H})$ are nowhere dense in I_1 and by the Baire theorem we get a contradiction. Hence, by the Montel's lemma, the sets $E_m \cap H_0$ $(m \in \mathbf{N})$ are closed, and, again by the Baire theorem, $U_1 \cap H_0 \neq \emptyset$. Therefore U_1 is open, dense in $I_2 \times \cdots \times I_s$. Analogously to I_1^m and U_1 we define sets I_j^m and U_j $(j = 2, \ldots, s,\ m \in \mathbf{N})$. Let us take a closed interval $K_2 \times \cdots \times K_s \subset U_1$. Since U_j are dense we can find closed intervals $\widetilde{K}_1 \subset I_1$, $\widetilde{K}_j \subset K_j$ $(j = 2, \ldots, s)$ and $m \in \mathbf{N}$ such that for $j = 1, \ldots s$

$$\widetilde{K}_1 \times \cdots \times \widetilde{K}_{j-1} \times \widetilde{K}_{j+1} \times \cdots \times \widetilde{K}_s \subset U_j$$

and is f separately holomorphic and bounded by m in the set

$$\bigcup_{j=1}^{s} \widetilde{K}_1 \times \cdots \times I_j^m \times \cdots \times \widetilde{K}_s.$$

Hence, by the Siciak's theorem, $I_1 \times \widetilde{K}_2 \times \cdots \times \widetilde{K}_s \subset \mathrm{A}(f)$. ∎

2^0 *For $y_1 \in U$ the set $F_{f,1}(y_1)$ is pluripolar.*

∎ Since $I_1 \times \{y_1\} \subset \mathrm{A}(f)$ we see that there exist D - complex neighbourhood of I_1 and B - complex neighbourhood of y_1 such that f is holomorphic in $D \times B$. By the Bedford-Taylor theorem

$$N := \left\{ z \in D : \varphi(z) := \sup_{|\alpha| \geq 1} \left| \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial y_1^{\alpha}}(z, y_1) \right|^{1/|\alpha|} < \varphi^*(z) \right\}$$

is pluripolar, and of course $F_{f,1} \subset N$. ∎

3^0 *If V is a countable and dense subset of U then $Z_{f,1} \subset \bigcup_{y_1 \in V} F_{f,1}(y_1)$.*

∎ Take $x_1^0 \in Z_{f,1}$. We can find $y_1^0 \in I_2 \times \cdots \times I_s$ such that $\left(x_1^0, y_1^0\right) \in \mathrm{S}(f)$, but $y_1^0 \in \mathrm{A}\left(f\left(x_1^0, \bullet\right)\right)$. Hence $f\left(x_1^0, \bullet\right)$ is holomorphic in the polydisc $P\left(y_1^0, 1/Q_{f,1}(x_1^0, y_1^0)\right) \subset \mathbf{C}^N$, where $N := n_2 + \cdots + n_s$. Let λ be such that $0 < \lambda \leq 1/4$ and $(1-\lambda)^{-1-N} < 2$ and let $r := \min\left\{1, 1/Q_{f,1}\left(x_1^0, y_1^0\right)\right\}$. For $y_1 \in \vartheta := P\left(y_1^0, \lambda r\right) \subset \mathbf{C}^N$ we have

$$f\left(x_1^0, y_1\right) = \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial y^{\alpha}}\left(x_1^0, y_1^0\right) \ \left(y_1 - y_1^0\right)^{\alpha}.$$

We deduce that

$$\left| \frac{1}{\beta!} \frac{\partial^{|\beta|} f}{\partial y_1^{\beta}}\left(x_1^0, y_1\right) \right| \leq Q_{f,1}\left(x_1^0, y_1^0\right)^{|\beta|} \sum_{\alpha} \frac{(\alpha+\beta)!}{\alpha!\,\beta!} \lambda^{|\alpha|}$$
$$= Q_{f,1}\left(x_1^0, y_1^0\right)^{|\beta|} \ (1-\lambda)^{-|\beta|-N},$$

hence

$$Q_{f,1}\left(x_1^0, y_1\right) \leq (1-\lambda)^{-1-N} Q_{f,1}\left(x_1^0, y_1^0\right) < 2/r .$$

By 1^0 there exists $\widetilde{y}_1 \in \vartheta \cap V$. It is enough to show that $x_1^0 \in F_{f,1}(\widetilde{y}_1)$. Assume it is not so, that is $Q_{f,1}(\bullet, \widetilde{y})$ is upper semicontinuous at x_1^0. Therefore there exists a closed interval K - neighbourhood of x_1^0 in I_1 such that for $x_1 \in K$

$$Q_{f,1}(x_1, \widetilde{y}) < 2/r .$$

The function $f(x_1, \bullet)$ is holomorphic in a neighbourhood of $\widetilde{y}_1$ (because $\widetilde{y}_1 \in U$, hence $(x_1, \widetilde{y}_1) \in \mathrm{A}(f)$) and so is holomorphic in the polydisc $P(\widetilde{y}_1, 1/Q_{f,1}(x_1, \widetilde{y}_1))$. We have

$$P(\widetilde{y}_1, 1/Q_{f,1}(x_1, \widetilde{y}_1)) \supset P(\widetilde{y}_1, r/2) \supset \vartheta,$$

hence for $x_1 \in K$ $f(x_1, \bullet)$ is holomorphic in ϑ. Moreover for $y_1 \in \vartheta$ we have

$$|f(x_1, y_1)| \leq \sum_{\alpha} Q_{f,1}(x_1, y_1)^{|\alpha|} (\lambda r)^{|\alpha|} \leq \sum_{\alpha} \left(\frac{1}{2}\right)^{|\alpha|} = 2^N.$$

Let U_1 and I_1^m be as in the proof of 1^0. Take a closed interval $H \subset \vartheta \cap U_1$. We can find m such that f is separately holomorphic (as a function of two variables: $x_1 \in I_1$ and $y_1 \in I_2 \times \cdots \times I_s$) and bounded by m in $K \times \vartheta \cup I_1^m \times H$. By the Siciak's theorem $\left(x_1^0, y_1^0\right) \in \mathrm{A}\left(f\right)$ - contradiction. ■

By 2^0 and 3^0 we deduce that $Z_{f,1}$ is pluripolar. Thus we have proved the first inductive step: we have shown that theorem C is true for $k = 1$ and any $s \geq 2$. Now let $k \geq 2$ and assume that theorem C is true for $k - 1$ and any $s \geq k$.

4^0 *The set*

$$W := \left\{y \in I_{k+1} \times \cdots \times I_s : S\left(f\left(\bullet, y\right)\right) = S\left(f\right)\left(\bullet, y\right)\right\}$$

is dense in $I_{k+1} \times \cdots \times I_s$.

■ As we have just shown theorem C is true for $k = 1$. Using this k times for any $k > 1$ we see that for quasi almost all $x_s \in I_s$, ... , for quasi almost all $x_{k+1} \in I_{k+1}$ we have

$$\mathrm{S}\left(f\left(\bullet, x_{k+1}, \ldots, x_s\right)\right) = \mathrm{S}\left(f\right)\left(\bullet, x_{k+1}, \ldots, x_s\right).$$

In particular W is dense. ■

5^0 *For* $y \in W$ *the set* $F_{f,k}\left(y\right)$ *is pluripolar.*

■ If $L \subset\subset \mathrm{A}\left(f\right)\left(\bullet, y\right)$, then in the same way as in the proof of 2^0 we show that $F_{f,k}\left(y\right) \cap L$ is pluripolar. ■

6^0 *If* W' *is a countable and dense subset of* W, *then the set*

$$R := Z_{f,k} \setminus \bigcup_{y \in W'} \left(S\left(f\left(\bullet, y\right)\right) \cup F_{f,k}\left(y\right)\right)$$

is pluripolar.

■ Take any $x^0 \in R$. By the definition of $Z_{f,k}$ we can find $y^0 \in I_{k+1} \times \cdots \times I_s$ such that $\left(x^0, y^0\right) \in \mathrm{S}\left(f\right)$, but $y^0 \in \mathrm{A}\left(f\left(x^0, \bullet\right)\right)$. Denote $g := f\left(x_1^0, \ldots, x_{k-1}^0, \bullet\right)$. First we want to show that $\left(x_k^0, y^0\right) \in \mathrm{A}\left(g\right)$. Assume $\left(x_k^0, y^0\right) \in \mathrm{S}\left(g\right)$. We have $y^0 \in \mathrm{A}\left(g\left(x_k^0, \bullet\right)\right)$, therefore $x_k^0 \in Z_{g,1}$. By 3^0 we can find $y \in W'$ such that $x_k^0 \in F_{g,1}\left(y\right)$, that is $Q_{g,1}\left(\bullet, y\right)$ is not upper semicontinuous at x_k^0. By the definition of R and W we have

$$x^0 \in \mathrm{A}\left(f\left(\bullet, y\right)\right) \setminus F_{f,k}\left(y\right) = \mathrm{A}\left(f\right)\left(\bullet, y\right) \setminus F_{f,k}\left(y\right),$$

whence $Q_{f,k}\left(\bullet, y\right)$ is upper semicontinuous at x_k^0. In particular $Q_{f,k}\left(x_1^0, \ldots, x_{k-1}^0, \bullet, y\right)$ $= Q_{g,1}\left(\bullet, y\right)$ is upper semicotinuous at x^0 - contradiction. Thus we have $\left(x_k^0, y^0\right) \in \mathrm{A}\left(g\right)$, hence

$$\left(x_k^0, y^0\right) \in \mathrm{S}\left(f\right)\left(x_1^0, \ldots, x_{k-1}^0, \bullet\right) \setminus \mathrm{S}\left(f\left(x_1^0, \ldots, x_{k-1}^0, \bullet\right)\right),$$

and so $\left(x_1^0, \ldots, x_{k-1}^0\right) \in Z_{f,k-1}$. We have shown that the projection of R on $I_1 \times \cdots \times I_{k-1}$ is contained in $Z_{f,k-1}$ which is, by the inductive assumption, pluripolar. In particular R is pluripolar. ■

By the inductive assumption theorem C is true for any separately analytic function of k variables, hence for such functions theorem A is true as well. In particular for $y \in I_{k+1} \times \cdots \times I_s$ the set $\mathrm{S}\,(f\,(\bullet, y))$ is pluripolar. Therefore, by 4^0, 5^0 and 6^0, $Z_{f,k}$ is pluripolar. The proof of theorem C is complete.

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Jagiellonian University
Institute of Mathematics
Reymonta 4
30-059 Kraków, Poland