REARRANGEMENTS
AND THE MONGE-AMPÈRE EQUATIONS

ZBIGNIEW BŁOCKI

Abstract. We show that the direct counterpart of the Talenti symmetrization estimate for the Laplacian does not hold neither for the complex nor real Monge-Ampère equations. We also use this Talenti result to improve some known estimates for subharmonic functions in $\mathbb{C}$, where the constant depends on the area of the domain, instead of the diameter.

1. Introduction

For a measurable $A \subset \mathbb{R}^n$ with $\lambda(A) < \infty$, where λ denotes the Lebesgue measure, its Schwarz symmetrization (or rearrangement) A^* is the ball centered at the origin such that $\lambda(A) = \lambda(A^*)$. For a measurable function f, $f \geq 0$, its symmetrization is the radially-symmetric function f^* (i.e. $f^*(x)$ depends only on $|x|$) satisfying $\lambda(\{f^* > t\}) = \lambda(\{f > t\})$ for $t > 0$ (we assume that f is such that $\lambda(\{f > t\}) < \infty$ for $t > 0$). If $u \leq 0$ then we set $u^* := -(-u)^*$, or equivalently require that $\lambda(\{u^* < t\}) = \lambda(\{u < t\})$ for $t < 0$.

The basic property is that for bijective increasing $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ we have

$$\int_{A^*} \gamma \circ f^* \, d\lambda = \int_A \gamma \circ f \, d\lambda. \tag{1}$$

In particular, $||f^*||_{L^p} = ||f||_{L^p}$. This makes it a very useful tool for various optimal estimates for PDEs, often reducing the problem to radially-symmetric functions. We refer to [7] for a good introduction to the topic of rearrangements.

The main result from the viewpoint of PDEs is the following theorem due to Talenti [13]: if Ω is a bounded domain in $\mathbb{R}^n$, $f \geq 0$ and u, v solve the following Dirichlet problems:

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} \Delta v = f^* & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega^*, \end{cases}$$

then $v \leq u^*$. The main tool in the proof is the isoperimetric inequality. One should note that in general u^* need not be subharmonic: as noticed in [3] if u is the Green

2020 *Mathematics Subject Classification.* Primary: 32W20, 32U05, secondary: 35J96, 31A05.

Key words and phrases. Symmetrization, Monge-Ampère equations, convex functions, subharmonic functions, plurisubharmonic functions, isoperimetric inequalities.

function for a disc with pole away from the center then u^* is not subharmonic (this example is due to P. Thomas).

The Talenti estimate has many applications, it for example immediately gives various optimal bounds for subharmonic functions. We present them in Section 2, including improved estimates of Brezis-Merle [6] and Benelkourchi-Jennane-Zeriahi [2].

One could ask a similar question for the complex Monge-Ampère equation (CMA): assume that Ω is a bounded domain in $\mathbb{C}^n$, $f \in C(\overline{\Omega})$, $f \geq 0$, and u, v solve the following Dirichlet problems:

$$\begin{cases} u \in PSH \cap C(\Omega) \\ (dd^c u)^n = f\, d\lambda \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega \end{cases} \qquad \begin{cases} v \in PSH \cap C(\Omega^*) \\ (dd^c v)^n = f^*\, d\lambda \ \text{ in } \Omega^* \\ v = 0 \ \text{ on } \partial\Omega^*. \end{cases} \tag{2}$$

Of course, in such a case Ω has to be pseudoconvex, or even hyperconvex (see [1], and also [4], for the result on existence of weak solutions to this Dirichlet problem).

We will show that a direct counterpart of the Talenti result does not hold for CMA:

Theorem 1. *If u, v satisfy (2) then it is not always true that $v \leq u^*$.*

The way to prove Theorem 1 is to show that the symmetrization result for (2) would in fact be equivalent to the following complex isoperimetric inequality: for bounded strongly pseudoconvex domains Ω in $\mathbb{C}^n$ one would have

$$\int_{\partial\Omega} K^{1/(n+1)}\, d\sigma \geq a_n \,(\lambda(\Omega))^{n/(n+1)}, \tag{3}$$

where K denotes the Levi curvature of $\partial\Omega$, $d\sigma$ is the surface measure on $\partial\Omega$, and a_n is a constant depending only on n such that the equality holds for balls (in fact $a_n = 2n\omega_{2n}^{1/(n+1)}$, where ω_m denotes the volume of the unit ball in $\mathbb{R}^m$). Note that for $n = 1$ we have $K \equiv 1$ and then (3) is the classical isoperimetric inequality. For $n \geq 2$ however we will find a counterexample to (3). But this example does not contradict a complex isopermetric inequality conjectured in [5]:

$$\int_{\partial\Omega} K\, d\sigma \geq 2n\sqrt{\omega_{2n}\lambda(\Omega)},$$

it still remains open.

A similar statement to Theorem 1 could be proved for the real Monge-Ampère equation (RMA): if Ω is a bounded domain in $\mathbb{R}^n$, $f \in C(\overline{\Omega})$, $f \geq 0$, and u, v solve the following Dirichlet problems

$$\begin{cases} u \in CVX(\Omega) \\ \det D^2 u = f \ \text{in } \Omega \\ u = 0 \ \text{ on } \partial\Omega \end{cases} \qquad \begin{cases} v \in CVX(\Omega^*) \\ \det D^2 v = f^* \ \text{ in } \Omega^* \\ v = 0 \ \text{ on } \partial\Omega^* \end{cases} \tag{4}$$

(see e.g. [12] existence of weak solutions):

Theorem 2. *If u, v satisfy* (4) *then it is not always true that $v \le u^*$.*

In this case however, we have the following slightly weaker symmetrization result for RMA due to Talenti [14] for $n = 2$ and to Tso [15] for arbitrary n. Instead of the Lebesgue measure we symmetrize with respect to the quermassintegral

$$W_{n-1}(\Omega) = V(\Omega, \mathbb{B}, \dots, \mathbb{B}),$$

where V denotes the mixed volume and $\mathbb{B}$ the unit ball in $\mathbb{R}^n$. If we denote the symmetrization of Ω with respect to W_{n-1} by $\widetilde{\Omega}$ and u, v solve

$$\text{(5)} \qquad \begin{cases} u \in CVX(\Omega) \\ \det D^2 u = f \text{ in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \qquad \begin{cases} v \in CVX(\widetilde{\Omega}) \\ \det D^2 v = f^* \quad \text{in } \widetilde{\Omega} \\ v = 0 \quad \text{on } \partial\widetilde{\Omega} \end{cases}$$

(we extend f^* to $\widetilde{\Omega} \setminus \Omega^*$ by zero) then $v \le \widetilde{u}$. The main tool in the proof is the Alexandrov-Fenchel isoperimetric inequality (it also immediately implies that $\Omega^* \subset \widetilde{\Omega}$ and that $\widetilde{u} \le u^*$ in Ω^*). Interestingly, this kind of result is still open for the real Hessian equation, the reason is the lack of an analogous isoperimetric inequality in this case - it is only known for star-shaped domains, see [8].

This Talenti-Tso result easily implies that if Ω is a ball in $\mathbb{R}^n$ and u, v satisfy (4) then $v \le u^*$. We conjecture that analogous result holds for (2). It would have far-reaching consequences for CMA, in particular it would easily imply Kołodziej's estimates [9, 10] and would give optimal constants for a ball.

2. Optimal bounds for Subharmonic Functions

The following estimate was proved by Brezis-Merle [6] but the constant depended on the diameter of Ω instead of the volume.

Theorem 3. *Let u be a subharmonic function in a bounded domain Ω in $\mathbb{C}$ such that $u = 0$ on $\partial\Omega$. Then, if $0 < \delta < 4\pi$ and $a := \int_\Omega \Delta u$,*

$$\int_\Omega e^{\frac{-(4\pi-\delta)}{a} u} d\lambda \le \frac{4\pi}{\delta} \lambda(\Omega).$$

Proof. By approximation we may assume that Ω and u are smooth. Set $f := \Delta u$, we can solve the Dirichlet problem

$$\begin{cases} \Delta v = f^* & \text{in } \Omega^* = D(0, R) \\ v = 0 & \text{on } \partial\Omega^*. \end{cases}$$

We then have

$$a = \int_\Omega \Delta u = \int_\Omega f \, d\lambda = \int_{\Omega^*} f^* \, d\lambda = \int_{D(0,R)} \Delta v = 2\pi R \eta'(R),$$

where $v(z) = \eta(|z|)$. Since η is log-convex (that is $\eta(e^t)$ is convex with respect to t),

$$\eta(r) \geq R\eta'(R) \log \frac{r}{R} = \frac{a}{2\pi} \log \frac{r}{R}.$$

Talenti's theorem implies that $u^* \geq v$, hence

$$u^* \geq \frac{a}{2\pi} \log \frac{|z|}{R}, \quad 0 \leq r \leq R. \tag{6}$$

For $p = \dfrac{4\pi - \delta}{a}$ we then have

$$\int_\Omega e^{-pu}\, d\lambda = \int_{\Omega^*} e^{-pu^*}\, d\lambda \leq \int_{D(0,R)} e^{-\frac{ap}{2\pi} \log \frac{|z|}{R}}\, d\lambda = \frac{4\pi^2 R^2}{\delta} = \frac{4\pi}{\delta} \lambda(\Omega).$$

$\square$

We can also improve (and simplify) an estimate due to Benelkourchi-Jennane-Zeriahi [2]:

Theorem 4. *Let Ω, u and a be as in Theorem 3. Then*

$$\lambda(\{u < t\}) \leq \lambda(\Omega) e^{4\pi t/a}, \quad t < 0.$$

Proof. Let R be as in the proof of Theorem 3. Again by Talenti's theorem we obtain (6). Using this we get

$$\lambda(\{u < t\}) = \lambda(\{u^* < t\}) \leq \lambda\left(D\left(0, Re^{2\pi t/a}\right)\right) = \pi R^2 e^{4\pi t/a} = \lambda(\Omega) e^{4\pi t/a}.$$

$\square$

3. Proofs of Theorems 1 and 2

Take Ω, f, u, v satisfying (2). By approximation we may assume that Ω is smooth strongly pseudoconvex, f, u are smooth up to the boundary, and that 0 is a regular value for u. Write $v(z) = \eta(|z|)$, $u^*(z) = \gamma(|z|)$, and $\Omega^* = B(0, R)$.

Suppose we have $v \leq u^*$, then $\eta'(R) \geq \gamma'(R)$ (since $\eta(R) = \gamma(R) = 0$). We have

$$\int_\Omega f\, d\lambda = \int_{\Omega^*} f^*\, d\lambda = \int_{B(0,R)} (dd^c v)^n = b_n \left(R\eta'(R)\right)^n.$$

On the other hand,

$$\int_\Omega f\, d\lambda = \int_\Omega (dd^c u)^n = c_n \int_{\partial\Omega} K |\nabla u|^n d\sigma,$$

where K is the Levi curvature of $\partial\Omega$ (see e.g. [11]). Differentiating the equation

$$\lambda(\{u < \gamma(r)\}) = \lambda(\{u^* < \gamma(r)\}) = \lambda(B(0, r)) = \omega_{2n} r^{2n}$$

and using the co-area formula we will get

$$\gamma'(r)\int_{\{u=\gamma(r)\}}\frac{d\sigma}{|\nabla u|}=2n\omega_{2n}r^{2n-1}.$$

Therefore $\eta'(R)\geq\gamma'(R)$ implies

$$\int_{\partial\Omega}K|\nabla u|^n d\sigma\geq d_n\frac{(\lambda(\Omega))^n}{\left(\displaystyle\int_{\partial\Omega}\frac{d\sigma}{|\nabla u|}\right)^n}. \tag{7}$$

We will need the following

Lemma 5. *For a bounded, smooth, strongly pseudoconvex Ω in $\mathbb{C}^n$ and positive $\rho\in C^\infty(\partial\Omega)$ there exists $u\in C^\infty(\overline{\Omega})$, strongly plurisubharmonic (psh) in Ω such that $u=0$ and $|\nabla u|=\rho$ on $\partial\Omega$.*

Proof. Let $\psi\in C^\infty(\overline{\Omega})$ be a psh defining function for Ω, that is $\psi=0$ and $\nabla\psi\neq 0$ on $\partial\Omega$. Then $|\nabla\psi|=\psi_n$ on $\partial\Omega$, where ψ_n denotes the outer normal derivative. Let $\varphi\in C^\infty(\overline{\Omega})$ be such that $\varphi=\rho/\psi_n$ on $\partial\Omega$. Set

$$v:=(\varphi+A\psi)\psi,$$

where $A\gg 0$ will be determined later. We have $v_n=\varphi\psi_n=\rho$ on $\partial\Omega$ and we claim that $(v_{j\bar{k}})>0$ on $\partial\Omega$ for A sufficiently large. For $\zeta\in\mathbb{C}^n$ with $|\zeta|=1$ we will use the notation

$$u_\zeta=\sum_j\zeta_j u_j,\quad u_{\zeta\bar{\zeta}}=\sum_{j,k}\zeta_j\bar{\zeta}_k u_{j\bar{k}}.$$

On $\partial\Omega$ we have

$$v_{\zeta\bar{\zeta}}=\varphi\psi_{\zeta\bar{\zeta}}+2\mathrm{Re}\left(\varphi_\zeta\psi_{\bar{\zeta}}\right)+2A|\psi_\zeta|^2.$$

We can find positive C and ε, independent of ζ, such that $|\varphi_\zeta|\leq C$ and $\varphi\psi_{\zeta\bar{\zeta}}\geq\varepsilon$ on $\partial\Omega$. Take $z_0\in\partial\Omega$, we may assume that $(1,0,\dots,0)$ is the outer normal to $\partial\Omega$ at z_0. Then at z_0

$$v_{\zeta\bar{\zeta}}\geq\varepsilon-C|\zeta_1|\psi_n+\frac{A}{2}|\zeta_1|^2\psi_n^2.$$

We can then find $A>0$, depending only on ε and C, such that $v_{\zeta\bar{\zeta}}\geq\varepsilon/2$.

We have constructed v with required properties except that it is strongly psh only near $\partial\Omega$, instead of entire $\overline{\Omega}$. To obtain the right u we can take the regularized maximum of v and $\widetilde{\varepsilon}\big(|z|^2-\widetilde{C}\big)$ for sufficiently large $\widetilde{C}$ and small $\widetilde{\varepsilon}>0$. $\square$

Choosing u with $|\nabla u|=K^{-1/(n+1)}$ in (7) we would obtain the isoperimetric inequality (3). But this inequality cannot hold in general. For $\tau\geq 0$ consider an *elongated* *ball*:

$$\begin{aligned}\Omega :=&\{z\in\mathbb{C}^n\colon \operatorname{Re} z_1\le 0,\ |z|<1\}\\ &\cup\{z\in\mathbb{C}^n\colon 0\le \operatorname{Re} z_1\le\tau,\ (\operatorname{Im} z_1)^2+|z_2|^2+\dots+|z_n|^2<1\}\\ &\cup\{z\in\mathbb{C}^n\colon \operatorname{Re} z_1\ge\tau,\ (\operatorname{Re} z_1-\tau)^2+(\operatorname{Im} z_1)^2+|z_2|^2+\dots+|z_n|^2<1\}\end{aligned}$$

Although its boundary is only $C^{1,1}$ smooth, by approximation we see that (3) would still hold for Ω. The Levi curvature of $\partial\Omega$ is equal to 1 on the sphere parts of the boundary and to $1/2$ on the cylinder part. Therefore, (3) would imply that for $\tau\ge 0$

$$2n\omega_{2n}+2^{-1/(n+1)}\omega_{2n-2}\tau\ge 2n\omega_{2n}^{1/(n+1)}\big(\omega_{2n}+\omega_{2n-1}\tau\big)^{n/(n+1)}.$$

Comparing the derivatives at $\tau=0$ we would get

$$2^{-1/(n+1)}\omega_{2n-2}\ge 2n\omega_{2n-1}$$

and one can check that this is false for every $n\ge 2$. This finishes the proof of Theorem 1.

Remark. If Ω is bounded and smooth in $\mathbb{C}^n$, u is smooth up to the boundary with $u=0$ on $\partial\Omega$ then

$$\int_\Omega (dd^cu)^n=b_n\int_{\partial\Omega}K|\nabla u|^n d\sigma.$$

On the other hand, by the Stokes theorem we immediately get

$$\int_\Omega (dd^cu)^n=\int_{\partial\Omega}d^cu\wedge(dd^cu)^{n-1}.$$

In general, it is not true however that

$$d^cu\wedge(dd^cu)^{n-1}=b_nK|\nabla u|^n d\sigma$$

on $\partial\Omega$. If it were true then for $n=2$ and smooth ρ on $\overline{\Omega}$ we would have

$$d^c(\rho u)\wedge dd^c(\rho u)=\rho^2 d^cu\wedge dd^cu$$

on $\partial\Omega$. This would mean that

$$\rho d^c\rho\wedge du\wedge d^cu=0$$

on $\partial\Omega$. If a piece of $\partial\Omega$ is of the form $\{\operatorname{Re} z_1=0\}$ then we have $u_{z_2}=0$ there and

$$\rho d^c\rho\wedge du\wedge d^cu=2\rho|u_{z_1}|^2 i dz_1\wedge d\overline{z}_1\wedge i\big(\rho_{\overline{z}_2}d\overline{z}_2-\rho_{z_2}dz_2\big).$$

If we choose u, ρ with $\rho\ne 0$, $\rho_{z_2}\ne 0$ and $u_{z_1}\ne 0$ then this term does not vanish. Note however that the Stokes theorem easily gives

$$\int_{\partial\Omega}\rho d^c\rho\wedge du\wedge d^cu=0.$$

The proof of Theorem 2 is similar. We assume that Ω is bounded, smooth and strongly convex in $\mathbb{R}^n$, u, v, f are smooth up to the boundary, satisfy (4) and that 0 is a regular value of u. We write $v(x) = \eta(|x|)$, $u^*(x) = \gamma(|x|)$, and $\Omega^* = B(0,R)$. If $v \le u^*$ then $\eta'(R) \ge \gamma'(R)$. We have

$$\int_\Omega f\,d\lambda = \int_{\Omega^*} f^*\,d\lambda = \int_{B(0,R)} \det D^2 v\,d\lambda = \omega_n\,(\eta'(R))^n$$

and

$$\int_\Omega f\,d\lambda = \int_\Omega \det D^2 u\,d\lambda = \frac{1}{n}\int_{\partial\Omega} H|\nabla u|^n d\sigma,$$

where H is the Gauss curvature of $\partial\Omega$. Differentiating

$$\lambda(\{u < \gamma(r)\}) = \lambda(\{u^* < \gamma(r)\}) = \lambda(B(0,r)) = \omega_n r^n$$

and using the co-area formula we get

$$\gamma'(r)\int_{\{u=\gamma(r)\}} \frac{d\sigma}{|\nabla u|} = n\omega_n r^{n-1}.$$

Therefore $\eta'(R) \ge \gamma'(R)$ implies that

$$\int_{\partial\Omega} H|\nabla u|^n d\sigma \ge n^{n+1}\omega_n^2 \frac{(\lambda(\Omega))^{n-1}}{\left(\displaystyle\int_{\partial\Omega}\frac{d\sigma}{|\nabla u|}\right)^n}.$$

Since a result corresponding to Lemma 5 also holds in the real case, we can find appriopriate u with $|\nabla u| = H^{-1/(n+1)}$ on $\partial\Omega$. This would imply the following isoperimetric inequality

$$\int_{\partial\Omega} H^{1/(n+1)} d\sigma \ge n\omega_n^{2/(n+1)}\big(\lambda(\Omega)\big)^{\frac{n-1}{n+1}}. \tag{8}$$

By approximation, it would also hold for bounded convex domains with $C^{1,1}$ boundary. To see that it is not true in general consider an elongated ball again:

$$\begin{aligned}\Omega :=& \{x \in \mathbb{R}^n \colon x_1 \le 0,\ |x| < 1\}\\ &\cup \{x \in \mathbb{R}^n \colon 0 \le \operatorname{Re} x_1 \le \tau,\ x_2^2 + \cdots + x_n^2 < 1\}\\ &\cup \{x \in \mathbb{R}^n \colon x_1 \ge \tau,\ (x_1-\tau)^2 + x_2^2 + \cdots + x_n^2 < 1\}\end{aligned}$$

for $\tau \ge 0$. Then $H = 1$ on the sphere parts but it vanishes on the cylinder part. Therefore it is clear that (8) does not hold for any $\tau > 0$.

References

[1] E. Bedford, B.A. Taylor, *The Dirichlet problem for a complex Monge–Ampère equation*, Invent. Math. 37 (1976), 1–44

[2] S. Benelkourchi, B. Jennane, A. Zeriahi, *Polya's inequalities, global uniform integrability and the size of plurisubharmonic functions*, Ark. Mat. 43 (2005), 85–112

[3] R. Berman, B. Berndtsson, *Symmetrization of plurisubharmonic and convex functions*, Indiana Univ. Math. J. 63 (2014), 345–365

[4] Z. Błocki, *The complex Monge-Ampère operator in hyperconvex domains*, Ann. Scuola Norm. Sup. Pisa 23 (1996), 721–747

[5] Z. Błocki, *Cauchy-Riemann meet Monge-Ampère*, Bull. Math. Sci. 4 (2014), 433–480

[6] H. Brezis, F. Merle, *Uniform estimates and blow–up behavior for solutions of $\Delta u = V(x)e^u$ in two dimensions*, Comm. PDE 16 (1991), 1223–1253

[7] A. Burchard, *A short course on rearrangement inequalities*, lecture notes, 2009, available at `http://www.math.utoronto.ca/almut/rearrange.pdf`

[8] P. Guan, J. Li, *The quermassintegral inequalities for k-convex starshaped domains*, Adv. Math. 221 (2009), 1725—1732

[9] S. Kołodziej, *Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge-Ampère operator*, Ann. Pol. Math. 65 (1996), 11–21

[10] S. Kołodziej, *The complex Monge-Ampère equation*, Acta Math. 180 (1998), 69–117

[11] V. Martino, A. Montanari, *Integral formulas for a class of curvature PDE's and applications to isoperimetric inequalities and to symmetry problems*, Forum Math. 22 (2010), 255–267, Erratum: Forum Math. 35 (2023), 1469–1470

[12] J. Rauch, B.A. Taylor, *The Dirichlet problem for the multidimensional Monge-Ampère equation*, Rocky Mountain Math. J. 7 (1977), 345–364

[13] G. Talenti, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa 3 (1976), 697–718

[14] G. Talenti, *Some estimates of solutions to Monge-Ampère type equations in dimension two*, Ann. Scuola Norm. Sup. Pisa 8 (1981), 183–230

[15] K. Tso, *On symmetrization and Hessian equations*, J. Analyse Math. 52 (1989), 94–106

Uniwersytet Jagielloński
Instytut Matematyki
Łojasiewicza 6
30-348 Kraków
Poland
Zbigniew.Blocki@im.uj.edu.pl