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ON UNIQUENESS OF THE COMPLEX MONGE-AMPÈRE EQUATION ON COMPACT KÄHLER MANIFOLDS

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Abstract. We prove a partial uniqueness for solutions of the complex Monge-Ampère equation on a compact Kähler manifold in the class of quasiplurisubharmonic functions introduced recently by Guedj and Zeriahi.

1. Introduction

Let (X,ω) be a compact Kähler manifold of complex dimension n. $PSH(X,\omega)$ will denote the class of quasiplurisubharmonic functions φ on X satisfying $\omega_\varphi := \omega + dd^c\varphi \geq 0$. Guedj and Zeriahi [9] introduced the class $\mathcal{E}(X,\omega)$ of functions $\varphi \in PSH(X,\omega)$ satisfying $\int_{\{\varphi>-\infty\}} \omega_\varphi^n = \int_X \omega^n$ (the measure $\omega_\varphi^n = \omega_\varphi \wedge \dots \wedge \omega_\varphi$ is well defined on $\{\varphi > -\infty\}$ for any $\varphi \in PSH(X,\omega)$ by [1]). They showed in particular that for $\varphi \in \mathcal{E}(X,\omega)$ the measure ω_φ^n is well defined on X (with total mass $\int_X \omega^n$), vanishes on pluripolar sets, and is continuous (in the weak* topology) for decreasing sequences in $\mathcal{E}(X,\omega)$.

One of the main results in [9], building on earlier work of Yau [11], Kołodziej [10], and Cegrell [6] was the existence of a solution of the Dirichlet problem

$$\begin{cases} \varphi \in \mathcal{E}(X,\omega) \\ \omega_\varphi^n = \mu \\ \max\limits_X \varphi = 0, \end{cases} \tag{1}$$

provided that μ is a measure on X vanishing on pluripolar sets and with total mass $\int_X \omega^n$ (which are of course necessary conditions).

The uniqueness in (1) was posed as a problem in [9]. It had been proved in [3] for bounded φ, ψ (for $X = \mathbb{P}^n$ it had been earlier done in [2] with much more complicated methods). As observed in [9] the method from [3] actually gives the following result: if $\varphi \in \mathcal{E}(X,\omega)$ and $\psi \in \mathcal{E}^1(X,\omega)$ (where $\mathcal{E}^p(X,\omega) := \{\psi \in \mathcal{E}(X,\omega) : \int_X |\psi|^p \omega_\psi^n < \infty\}$, $p > 0$) are such that $\omega_\varphi^n = \omega_\psi^n$ then $\varphi - \psi = const$.

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The goal of this note is to prove the following improvement:

Theorem 1. *Assume $\varphi \in \mathcal{E}(X,\omega)$ and $\psi \in \mathcal{E}^p(X,\omega)$ for some $p > 1 - 2^{1-n}$. If $\omega_\varphi^n = \omega_\psi^n$ then $\varphi - \psi = const.$*

Let us mention that if $\varphi \in \mathcal{E}(X,\omega)$ are such that $\omega_\varphi^n = \omega_\psi^n =: \mu$ then we must have $\omega^n_{\max\{\varphi,\psi\}} = \mu$ (see [9], Proposition 3.4, we will make use of this result in the proof of Theorem 1) and also $\omega^n_{t\varphi+(1-t)\psi} = \mu$, $0 \le t \le 1$ (see [8]).

In [4] and [5] the class $\mathcal{D}$ of (germs of) plurisubharmonic functions was defined (it was shown that it is actually the same as the class $\mathcal{E}$ studied in [7]). It is the maximal subclass of the class of plurisubharmonic functions where the complex Monge-Ampère operator $(dd^c)^n$ can be defined (as a regular measure) so that it is continuous for decreasing sequences. It was shown in [4] that $\mathcal{D} = PSH \cap W^{1,2}_{loc}$ for $n = 2$ and it was characterized (similarly, but in a more complicated way) for $n \ge 3$ in [5].

A natural counterpart of the class $\mathcal{D}$ on a compact Kähler manifold is the class $\mathcal{D}(X,\omega)$ consisting of those $\varphi \in PSH(X,\omega)$ such that locally $\varphi + g \in \mathcal{D}$, where g is a local potential for ω (that is $\omega = dd^c g$). In particular, for $n = 2$ we get $\mathcal{D}(X,\omega) = \mathcal{P}SH(X,\omega) \cap W^{1,2}(X)$ (and $\subset$ for arbitrary n). The measure ω_φ^n is of course well defined for $\varphi \in \mathcal{D}(X,\omega)$. By $\mathcal{D}_a(X,\omega)$ denote the class of those $\varphi \in \mathcal{D}(X,\omega)$ for which ω_φ^n vanishes on pluripolar sets. It follows that $\mathcal{D}_a(X,\omega) \subset \mathcal{E}(X,\omega)$ but by Example 2.14 in [9] we don't have the equality in general.

By Lemma 5.14 in [7] the Dirichlet problem (1), where μ is a measure on X vanishing on pluripolar sets and with total mass $\int_X \omega^n$, always has a local solution in $\mathcal{D}_a$. This is therefore perhaps natural to ask whether it has a global solution belonging to $\mathcal{D}_a(X,\omega)$, which would be an improvement of Theorem A in [9]. However, using Theorem 1 we can show that this is not the case:

Theorem 2. *Let (X,ω) be the projective space $\mathbb{P}^n$ with the Fubini-Study metric. There exists a measure μ on X, vanishing on pluripolar sets and with total mass $\int_X \omega^n$, such that there is no $\varphi \in \mathcal{E}(X,\omega) \cap W^{1,2}(X)$ satisfying $\omega_\varphi^n = \mu$.*

In the proofs of Theorems 1 and 2 we will follow the notation from [9] and use various results proved in that article. We always assume that (X,ω) is a fixed Kähler manifold.

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Proofs

As the proof in dimension 2 is simpler and more transparent, we first prove Theorem 1 in this case.

Proof of Theorem 1 in dimension 2. If $\widetilde{\psi} := \max\{\varphi,\psi\}$ then $\widetilde{\psi} \ge \psi$, $\widetilde{\psi} \in \mathcal{E}^p(X,\omega)$ (by Lemma 2.3 in [9]), and by Proposition 3.4 in [9] we have $\omega^2_{\widetilde{\psi}} = \omega^2_\psi = \omega^2_\varphi$. We may thus assume that $\varphi \le \psi \le -1$. Then, if we set $\psi^j := \max\{\varphi, \psi - j\}$, we have $\psi - j \le \psi^j \le \psi$, $\psi^j \in \mathcal{E}^p(X,\omega)$, and ψ^j decreases to φ as $j \to \infty$. Without loss of generality we may thus assume that $0 \le \rho := \psi - \varphi \le C$; then both φ and ψ belong to $\mathcal{E}^p(X,\omega)$.

We now set $\varphi_j := \max\{\varphi, -j\}$, $\psi_j := \max\{\psi, -j\}$, $\rho_j := \psi_j - \varphi_j$, and $h_j := (\varphi_j + \psi_j)/2$. First, we claim that

$$\lim_{j\to\infty} \int_X d\rho_j \wedge d^c\rho_j \wedge \omega_{h_j} = 0. \tag{2}$$

We have

$$\int_X \rho_j(\omega^2_{\varphi_j} - \omega^2_{\psi_j}) = -2\int_X \rho_j dd^c\rho_j \wedge \omega_{h_j} = 2\int_X d\rho_j \wedge d^c\rho_j \wedge \omega_{h_j}.$$

On the other hand,

$$\begin{aligned}\left|\int_X \rho_j(\omega^2_{\varphi_j} - \omega^2_{\psi_j})\right| &= \left|\int_{\{\varphi\le -j\}} \rho_j(\omega^2_{\varphi_j} - \omega^2_{\psi_j})\right| \\ &\le C\left(\int_{\{\varphi\le -j\}} \omega^2_{\varphi_j} + \int_{\{\psi\le -j\}} \omega^2_{\psi_j}\right) \to 0.\end{aligned}$$

We thus get (2).

Set $\chi(t) := -\sqrt{-t}$, $t \le -1$. We want to show the following improvement of (2)

$$\lim_{j\to\infty} \int_X \chi\circ h_j d\rho_j \wedge d^c\rho_j \wedge \omega_{h_j} = 0. \tag{3}$$

Similarly as above we have

$$\begin{aligned}\left|\int_X \chi\circ h_j\rho_j(\omega^2_{\varphi_j} - \omega^2_{\psi_j})\right| &= \left|\int_{\{\varphi\le -j\}} \chi\circ h_j\rho_j(\omega^2_{\varphi_j} - \omega^2_{\psi_j})\right| \\ &\le C|\chi(-j)|\left(\int_{\{\varphi\le -j\}} \omega^2_{\varphi_j} + \int_{\{\psi\le -j\}} \omega^2_{\psi_j}\right) \to 0\end{aligned} \tag{4}$$

because $\varphi, \psi \in \mathcal{E}^p(X,\omega)$ and $p > 1/2$. On the other hand,

$$\int_X \chi\circ h_j\rho_j(\omega^2_{\varphi_j} - \omega^2_{\psi_j}) = 2\int_X d(\chi\circ h_j\rho_j) \wedge d^c\rho_j \wedge \omega_{h_j}.$$

By (4) it is enough to estimate, using the Schwarz inequality,

$$\begin{aligned}&\left|\int_X \rho_j\chi'\circ h_j dh_j \wedge d^c\rho_j \wedge \omega_{h_j}\right| \\ &\qquad \le C\sqrt{\int_X \chi'\circ h_j\, d\rho_j \wedge d^c\rho_j \wedge \omega_{h_j}}\sqrt{\int_X \chi'\circ h_j\, dh_j \wedge d^c h_j \wedge \omega_{h_j}}.\end{aligned}$$

In order to show that the last integral is bounded in j we write

$$\begin{aligned}\int_X \chi'\circ h_j\, dh_j \wedge d^c h_j \wedge \omega_{h_j} &= -\int_X \chi\circ h_j\, dd^c h_j \wedge \omega_{h_j} \\ &\le -\int_X \chi\circ h_j\, \omega^2_{h_j} \\ &\le -4\int_X \chi\circ \varphi_j\, \omega^2_{\varphi_j}\end{aligned}$$

by Lemma 2.3 in [9]. Now from (2) (note that $0 \le \chi' \le 1$) we thus get (3).

Proceeding as in [3] we write

$$\int_X d\rho_j \wedge d^c\rho_j \wedge \omega = \int_X d\rho_j \wedge d^c\rho_j \wedge \omega_{h_j} - \int_X d\rho_j \wedge d^c\rho_j \wedge dd^c h_j,$$

so by (2) it is enough to estimate the last integral. We have

$$-\int_X d\rho_j \wedge d^c\rho_j \wedge dd^c h_j = \int_X d\rho_j \wedge d^c h_j \wedge dd^c\rho_j = \int_X d\rho_j \wedge d^c h_j \wedge (\omega_{\psi_j} - \omega_{\varphi_j}).$$

By the Schwarz inequality and since $\omega_{\psi_j} \le 2\omega_{h_j}$

$$\left|\int_X d\rho_j \wedge d^c h_j \wedge \omega_{\psi_j}\right| \le 2\sqrt{\int_X \frac{1}{\chi' \circ h_j}\, d\rho_j \wedge d^c\rho_j \wedge \omega_{h_j}}\sqrt{\int_X \chi' \circ h_j\, dh_j \wedge d^c h_j \wedge \omega_{h_j}}.$$

The last integral is bounded in j. In our case we also have $1/\chi' = 2\chi$ and by (3) we get

$$\lim_{j\to\infty} \int_X d\rho_j \wedge d^c h_j \wedge \omega_{\psi_j} = 0.$$

Similarly we show that

$$\lim_{j\to\infty} \int_X d\rho_j \wedge d^c h_j \wedge \omega_{\varphi_j} = 0,$$

and thus

$$\lim_{j\to\infty} \int_X d\rho_j \wedge d^c\rho_j \wedge \omega = 0. \quad \square$$

For the proof of Theorem 1 in arbitrary dimension we will need some preparatory results.

Lemma 3. *For $p > 0$, $k = 1, \dots, n$, and $\varphi \in PSH(X,\omega) \cap L^\infty(X)$ with $\varphi \le -1$ we have*

$$\int_X (-\varphi)^p \omega_\varphi^{n-k} \wedge \omega^k \le \int_X (-\varphi)^p \omega_\varphi^n \tag{5}$$

and

$$\int_X (-\varphi)^{p-1} d\varphi \wedge d^c\varphi \wedge \omega_\varphi^{n-k} \wedge \omega^{k-1} \le \frac{1}{p}\int_X (-\varphi)^p \omega_\varphi^n. \tag{6}$$

Proof. Set $T := \omega_\varphi^{n-k} \wedge \omega^{k-1}$. Then

$$\int_X (-\varphi)^{p-1} d\varphi \wedge d^c\varphi \wedge T = -\frac{1}{p}\int_X d((-\varphi)^p) \wedge d^c\varphi \wedge T = \frac{1}{p}\int_X (-\varphi)^p dd^c\varphi \wedge T.$$

Therefore the last integral is nonnegative and thus

$$\int_X (-\varphi)^p \omega \wedge T \le \int_X (-\varphi)^p \omega_\varphi \wedge T,$$

so by induction on k we get (5). We also obtain

$$\int_X (-\varphi)^{p-1} d\varphi \wedge d^c\varphi \wedge T \le \frac{1}{p}\int_X (-\varphi)^p \omega_\varphi \wedge T$$

which, by virtue of (5), gives (6). $\square$

Lemma 4. *For $k = 0, 1, \dots, n-1$ set $p_k := 1 - 2^{-k}$. Assume that $\varphi, \psi \in PSH(X,\omega) \cap L^\infty(X)$ are ≤ -1 and denote $\rho := \psi - \varphi$, $h := (\varphi + \psi)/2$. Then for $p \geq p_{n-1}$*

$$\int_X (-h)^{p-p_k} d\rho \wedge d^c\rho \wedge \omega_h^{n-1-k} \wedge \omega^k \leq C \left(\int_X (-h)^p d\rho \wedge d^c\rho \wedge \omega_h^{n-1} \right)^{2^{-k}},$$

where C is a positive constant depending only on n and on upper bounds for $\int_X (-h)^p \omega_h^n$ and $\int_X (-h)^p d\rho \wedge d^c\rho \wedge \omega_h^{n-1}$.

Proof. We use induction on k. For $k = 0$ there is nothing to prove and we assume the estimate holds for $k-1$. We may write the left-hand side as

$$\int_X (-h)^{p-p_k} d\rho \wedge d^c\rho \wedge \omega_h \wedge T - \int_X (-h)^{p-p_k} d\rho \wedge d^c\rho \wedge dd^c h \wedge T,$$

where $T = \omega_h^{n-1-k} \wedge \omega^{k-1}$. The first integral is now estimated by the inductive assumption (and since $h \leq -1$), so it is enough to bound the second term from above. Note that for $q \geq 0$ we have

$$-(-h)^q dd^c h = \frac{1}{q+1} dd^c((-h)^{q+1}) - q(-h)^{q-1} dh \wedge d^c h \leq \frac{1}{q+1} dd^c((-h)^{q+1}).$$

Therefore

$$\begin{aligned} - \int_X &(-h)^{p-p_k} d\rho \wedge d^c\rho \wedge dd^c h \wedge T \\ &\leq \frac{1}{p-p_k+1} \int_X d\rho \wedge d^c\rho \wedge dd^c((-h)^{p-p_k+1}) \wedge T \\ &= -\frac{1}{p-p_k+1} \int_X d((-h)^{p-p_k+1}) \wedge d^c\rho \wedge dd^c\rho \wedge T \\ &= \int_X (-h)^{p-p_k} dh \wedge d^c\rho \wedge (\omega_\psi - \omega_\varphi) \wedge T. \end{aligned}$$

Since $\omega_\psi \leq 2\omega_h$, by the Schwarz inequality we get

$$\begin{aligned} &\left| \int_X (-h)^{p-p_k} dh \wedge d^c\rho \wedge \omega_\psi \wedge T \right| \\ &\quad \leq 2 \sqrt{\int_X (-h)^{p-1} dh \wedge d^c h \wedge \omega_h \wedge T} \sqrt{\int_X (-h)^{p-p_{k-1}} d\rho \wedge d^c\rho \wedge \omega_h \wedge T}. \end{aligned}$$

Similarly we can deal with the term involving ω_φ and the required estimate follows from Lemma 3. □

Lemma 4 easily gives Theorem 1:

Proof of Theorem 1 for arbitrary n. Using the same notation as previously we can similarly as in the proof of (3) show that

$$\lim_{j \to \infty} \int_X (-h_j)^{1-2^{1-n}} d\rho_j \wedge d^c\rho_j \wedge \omega_{h_j}^{n-1} = 0.$$

Lemma 4 applied for $k = n - 1$, together with Lemma 2.3 in [9], now give

$$\lim_{j\to\infty} \int_X d\rho_j \wedge d^c\rho_j \wedge \omega^{n-1} = 0. \quad \square$$

For the proof of Theorem 2 we will need the following quantitative version of Example 2.14 in [9]:

Proposition 5. *Assume that $\psi \in PSH(X,\omega)$ is negative and $0 < \alpha < 1$. Then $-(-\psi)^\alpha \in \mathcal{E}^p(X,\omega)$ for $p < (1-\alpha)/\alpha$.*

Proof. Without loss of generality we may assume that $\psi \le -1$. Set $\varphi := -(-\psi)^\alpha$. We have

$$\begin{aligned}\omega_\varphi &= \alpha(1-\alpha)|\psi|^{\alpha-2} d\psi \wedge d^c\psi + \alpha|\psi|^{\alpha-1}\omega_\psi + (1-\alpha|\psi|^{\alpha-1})\omega \\ &\le \alpha(1-\alpha)|\psi|^{\alpha-2} d\psi \wedge d^c\psi + \alpha|\psi|^{\alpha-1}\omega_\psi + \omega\end{aligned}$$

and (for bounded ψ)

$$\begin{aligned}\omega_\varphi^n/C \le &\sum_{k=0}^{n-1} |\psi|^{\alpha-2+k(\alpha-1)} d\psi \wedge d^c\psi \wedge \omega_\psi^k \wedge \omega^{n-1-k} \\ &+ \sum_{l=1}^{n} |\psi|^{l(\alpha-1)}\omega_\psi^l \wedge \omega^{n-l} + \omega^n,\end{aligned}$$

where C is a positive constant depending on α and n. For $a > 0$ and $T = \omega_\psi^k \wedge \omega^{n-1-k}$ we get

$$\begin{aligned}\int_X (-\psi)^{-a-1} d\psi \wedge d^c\psi \wedge T &= \frac{1}{a}\int_X d(-\psi)^{-a} \wedge d^c\psi \wedge T \\ &= -\frac{1}{a}\int_X (-\psi)^{-a} dd^c\psi \wedge T \\ &\le \frac{1}{a}\int_X (-\psi)^{-a}\omega \wedge T \\ &\le \frac{1}{a}\int_X \omega \wedge T \\ &= \frac{1}{a}\int_X \omega^n.\end{aligned}$$

Therefore for $b < 1 - \alpha$ we obtain

$$\int_X |\psi|^b \omega_\varphi^n \le C(n,\alpha,b)\left(1 + \int_X |\psi|^b \omega^n\right)$$

and approximating arbitrary ψ by $\max\{\psi, -j\}$ the proposition follows. $\square$

Proof of Theorem 2. For $z \in \mathbb{C}^n$ set

$$\psi(z) := \log|z_1| - \frac{1}{2}\log(1 + |z|^2) - 1.$$

It can be extended to a function from $PSH(X,\omega)$. Then $\widetilde{\psi} := -\sqrt{-\psi} \notin W^{1,2}(X)$ and by Proposition 5 $\widetilde{\psi} \in \mathcal{E}^p(X,\omega)$ for $p < 1$. Using Theorem 1 we then conclude that for any $\varphi \in \mathcal{E}(X,\omega)$ satisfying $\omega_\varphi^n = \omega_{\widetilde{\psi}}^n$ we have $\varphi = \widetilde{\psi} + const \notin W^{1,2}(X)$. $\square$

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