Uniqueness and Stability for the Complex Monge-Ampère Equation on Compact Kähler Manifolds

ZBIGNIEW BŁOCKI

ABSTRACT. We prove uniqueness of weak solutions of the Dirichlet problem for the complex Monge-Ampère equation on compact Kähler manifolds. A qualitative version of this result implies the $L^{2n/(n-1)} - L^1$ stability of solutions of this equation.

1. INTRODUCTION

Let M be a compact Kähler manifold of the complex dimension n with the Kähler form ω. We say that a function φ on M is *admissible* if it is upper semicontinuous, locally integrable and $\omega_\varphi := dd^c\varphi + \omega \geq 0$, where $d = \partial + \bar\partial$ and $d^c = \sqrt{-1}(\bar\partial - \partial)$. By [1], for bounded admissible φ on M one can well define the complex Monge-Ampère measure

$$\omega_\varphi^n = \omega_\varphi \wedge \cdots \wedge \omega_\varphi.$$

The main goal of this note is to show the following uniqueness result.

Theorem 1.1. *Let φ, ψ be bounded admissible functions on M such that $\omega_\varphi^n = \omega_\psi^n$. Then $\varphi - \psi$ is constant.*

For $M = \mathbb{P}^n$ Theorem 1.1 was proved in [2]. In this case it is equivalent to the uniqueness of the Dirichlet problem for the complex Monge-Ampère equation for entire plurisubharmonic functions with logarithmic growth. For arbitrary M, Calabi showed in the 1950's that the uniqueness holds in the case when φ, ψ are smooth and $\omega_\varphi, \omega_\psi > 0$. Recently in [5] Theorem 1.1 was proved under extra assumption that ω_φ^n and ω_ψ^n have a density belonging to $L^q(M)$ for some $q > 1$.

In fact, we prove the following stability result, which is a quantitative version of Theorem 1.1.

Indiana University Mathematics Journal ©, Vol. 52, No. 6 (2003)

Theorem 1.2. *If φ and ψ are bounded admissible on M, then*

$$\int_M d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega^{n-1}\leq C\Big(\int_M(\psi-\varphi)(\omega_\varphi^n-\omega_\psi^n)\Big)^{2^{1-n}},$$

where C is a positive constant depending only on n and upper bounds of $\|\varphi\|_{L^\infty(M)}$, $\|\psi\|_{L^\infty(M)}$ and the volume of M.

Theorem 1.2 and the Sobolev inequality give an $L^{2n/(n-1)}-L^1$ stability for the complex Monge-Ampère operator on compact Kähler manifolds (compare with [5]).

Theorem 1.3. *Let φ and ψ be bounded admissible functions on M such that*

$$\int_M\varphi\,\omega^n=\int_M\psi\omega^n$$

and $\omega_\varphi^n=f\omega^n$, $\omega_\psi^n=g\omega^n$ for some $f,g\in L^1(M)$. Then, if $n\geq 2$,

$$\|\varphi-\psi\|_{L^{2n/(n-1)}(M)}\leq C\|f-g\|_{L^1(M)}^{2-n},$$

where C is a positive constant depending on M and on upper bounds of $\|\varphi\|_{L^\infty(M)}$ and $\|\psi\|_{L^\infty(M)}$.

The proof of Theorem 1.2 is in a way quite elementary. Unlike in [5], it does not use the existence of solutions of the Dirichlet problem (given directly or indirectly by [6]).

Note that the uniqueness (Theorem 1.1) does not hold for unbounded solutions: consider for example $M=\mathbb{P}^n$ and for $z=(z_0,z_1,\dots,z_n)\in\mathbb{C}^{n+1}\setminus\{0\}$

$$\varphi([z_0:z_1:\cdots:z_n]):=\log\frac{\sqrt{|z_1|^2+\cdots+|z_n|^2}}{\sqrt{|z_0|^2+|z_1|^2+\cdots+|z_n|^2}},$$

$$\psi([z_0:z_1:\cdots:z_n]):=\log\frac{\max\{|z_1|,\dots,|z_n|\}}{\sqrt{|z_0|^2+|z_1|^2+\cdots+|z_n|^2}}.$$

One can then check that $\omega_\varphi^n=\omega_\psi^n=c_n\delta_{[1:0:\cdots:0]}$ but of course $\varphi-\psi$ is not constant.

2. Preliminaries

We will now recall some facts that will be used in the proof of Theorem 2. They were proved mostly in [1] (see also [3], [4]). We assume that M is just a complex manifold of dimension n. Let T be a nonnegative (and thus in particular of order zero, or, in other words, representable by integration) closed complex (p,p) current on M ($p\leq n-1$) and let η, ρ be functions defined on M which locally can be written in the form $\eta = u_1 - u_2$, $\rho = u_3 - u_4$, where $u_1, \dots, u_4$ are bounded and plurisubharmonic. One can then well define currents

$$\eta\, dd^c\rho \wedge T, \quad d\eta \wedge d^c\rho \wedge T,$$

and they are of order zero. Moreover, we always have

$$d\rho \wedge d^c\rho \wedge T \geq 0$$

and, if ρ is plurisubharmonic,

$$dd^c\rho \wedge T \geq 0.$$

Since currents of the form $dx_i \wedge d^c\rho \wedge T$ are well defined, so are $d\rho \wedge T$ and $d^c\rho \wedge T$ (but they are not necessarily of order zero).

If $p = n - 1$, then the Schwarz inequality gives

$$\left| \int_M d\eta \wedge d^c\rho \wedge T \right| \leq \left(\int_M d\eta \wedge d^c\eta \wedge T \right)^{1/2} \left(\int_M d\rho \wedge d^c\rho \wedge T \right)^{1/2}.$$

If we take $T = (dd^c|z|^2)^{n-1}$ in some chart of M, it follows that $d\rho \wedge d^c\rho \wedge T$ has a locally bounded mass and therefore $\rho \in W^{1,2}_{\mathrm{loc}}(M)$.

Finally, by the Stokes theorem, if U is a real $2n - 1$ current (with complex coefficients) on M such that dU is of order zero, and M is compact, then

$$\int_M dU = 0.$$

3. Proof of Theorem 1.2

By C we will denote possibly different constants depending only on the required quantities. Set $\rho = \varphi - \psi$. For $k = 0, 1, \dots, n-1$ we will prove inductively that

$$\int_M d\rho \wedge d^c\rho \wedge \omega_\varphi^i \wedge \omega_\psi^j \wedge \omega^k \leq C a^{2^{-k}}, \tag{3.1}$$

where

$$a = \int_M (\psi - \varphi)(\omega_\varphi^n - \omega_\psi^n) = \int_M d\rho \wedge d^c\rho \wedge T,$$

$$T = \sum_{l=0}^{n-1} \omega_\varphi^\ell \wedge \omega_\psi^{n-1-\ell},$$

and i, j are such that $i + j + k = n - 1$. For $k = n - 1$ we will then obtain the desired estimate.

If $k = 0$, then

$$\int_M d\rho \wedge d^c\rho \wedge \omega_\varphi^i \wedge \omega_\psi^j \leq \int_M d\rho \wedge d^c\rho \wedge T = a.$$

Assume that (3.1) holds for $0, 1, \ldots, k-1$. We have

$$\omega_\varphi^i \wedge \omega_\psi^j \wedge \omega^k = \omega_\varphi^{i+k} \wedge \omega_\psi^j - dd^c\varphi \wedge \alpha,$$

where

$$\alpha = \omega_\varphi^i \wedge \omega_\psi^j \wedge \sum_{l=0}^{k-1} \omega_\varphi^\ell \wedge \omega^{k-1-\ell}.$$

Therefore

$$\begin{aligned} d\rho \wedge d^c\rho \wedge \omega_\varphi^i \wedge \omega_\psi^j \wedge \omega^k &\leq d\rho \wedge d^c\rho \wedge (T - dd^c\varphi \wedge \alpha) \\ &= d(\rho d^c\rho \wedge T - d^c\varphi \wedge \alpha \wedge d\rho \wedge d^c\rho) \\ &\qquad - \rho dd^c\rho \wedge T - d\rho \wedge d^c\varphi \wedge \alpha \wedge dd^c\rho. \end{aligned}$$

This means that

$$\int_M d\rho \wedge d^c\rho \wedge \omega_\varphi^i \wedge \omega_\varphi^j \wedge \omega^k \leq a - \int_M d\rho \wedge d^c\varphi \wedge \alpha \wedge dd^c\rho.$$

We have

$$-\int_M d\rho \wedge d^c\varphi \wedge \alpha \wedge dd^c\rho \leq \left| \int_M d\rho \wedge d^c\varphi \wedge \alpha \wedge \omega_\varphi \right| + \left| \int_M d\rho \wedge d^c\varphi \wedge \alpha \wedge \omega_\psi \right|.$$

If η is equal to φ or ψ, the Schwarz inequality gives

$$\left| \int_M d\rho \wedge d^c\varphi \wedge \alpha \wedge \omega_\eta \right| \leq \left(\int_M d\rho \wedge d^c\rho \wedge \alpha \wedge \omega_\eta \right)^{1/2} \left(\int_M d\varphi \wedge d^c\varphi \wedge \alpha \wedge \omega_\eta \right)^{1/2}.$$

By the inductive assumption (and since $a \leq C$) it remains to show that

$$\int_M d\varphi \wedge d^c\varphi \wedge \alpha \wedge \omega_\eta \leq C.$$

But

$$\begin{aligned} \int_M d\varphi \wedge d^c\varphi \wedge \alpha \wedge \omega_\eta &= -\int_M \varphi \wedge dd^c\varphi \wedge \alpha \wedge \omega_\eta \\ &\leq \left| \int_M \varphi\, \omega \wedge \alpha \wedge \omega_\eta \right| + \left| \int_M \varphi\, \omega_\varphi \wedge \alpha \wedge \omega_\eta \right| \\ &\leq 2k \|\varphi\|_{L^\infty(M)} \operatorname{vol}(M). \end{aligned}$$

The proof is complete. □

Remark 3.1. Approximating φ and ψ by smooth admissible functions and using the continuity theorems for the complex Monge-Ampère operator from [1], one can reduce the proof of Theorem 1.2 to smooth φ and ψ.

Acknowledgement. The author was partially supported by KBN Grant #2 P03A 028 19.

The author would like to thank S. Kołodziej for his assistance and discussions on [2].

References

[1] E. Bedford and B.A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math **149** (1982), 1-41.

[2] ———, *Uniqueness for the complex Monge-Ampère equation for functions of logarithmic growth*, Indiana Univ. Math. J. **38** (1989), 455-469.

[3] J.-P. Demailly, *Monge-Ampère operators, Lelong numbers and intersection theory*, Complex Analysis and Geometry, Univ. Ser. Math. Plenum, New York, 1993, pp. 115-193.

[4] M. Klimek, *Pluripotential Theory*, Clarendon Press, 1991.

[5] S. Kołodziej, *Stability of solutions to the complex Monge-Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J. **52** (2003), 667-686.

[6] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, I Comm. Pure Appl. Math. **31** (1978), 339-411.

Jagiellonian University, Institute of Mathematics, Reymonta 4, 30-059, Kraków, Poland.
blocki@im.uj.edu.pl

Key words and phrases: complex Monge-Ampère equation; compact Kähler manifolds.
2000 Mathematics Subject Classification: Primary: 32W20; secondary: 32Q15.
Received: *August 1st, 2002; revised: March 26th, 2003.*