# Hörmander's $\bar\partial$-estimate, Some Generalizations, and New Applications

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We will discuss applications of Hörmander's $L^2$-estimate for $\bar{\partial}$ in the following problems:

1. Suita Conjecture (1972) from potential theory
2. Optimal constant in the Ohsawa-Takegoshi extension theorem (1987)
3. Mahler Conjecture (1938) from convex analysis

## Suita Conjecture

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Green function for bounded domain $D$ in $\mathbb{C}$:

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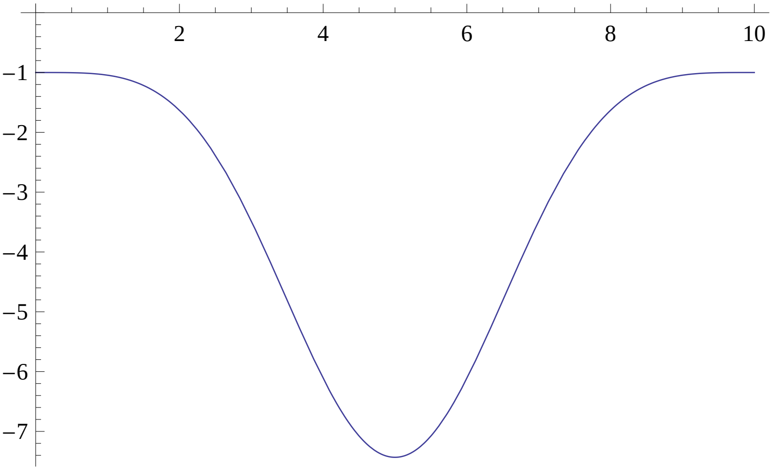
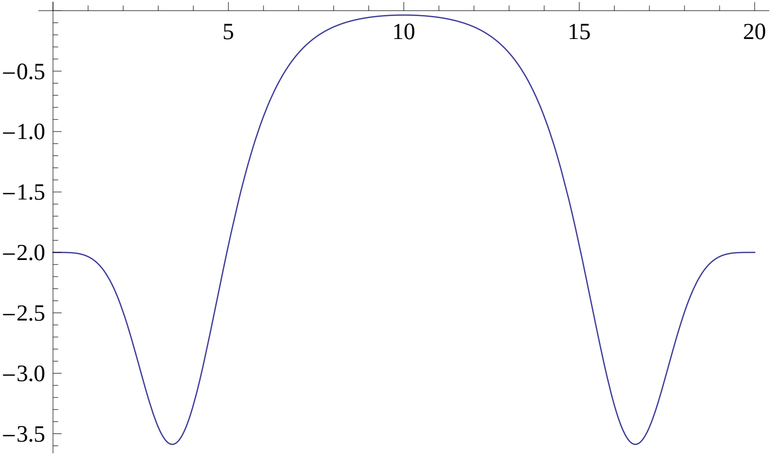
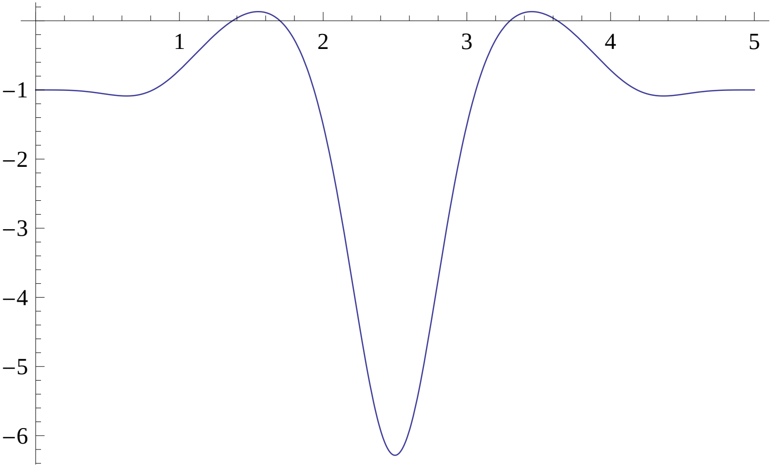
- "$=$" if $D$ is simply connected
- "$<$" if $D$ is an annulus (Suita)
- Enough to prove for $D$ with smooth boundary
- "$=$" on $\partial D$ if $D$ has smooth boundary




$Curv_{c_D|dz|}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $t = -2\log|z|$




$Curv_{K_D|dz|^2}$ for $D = \{e^{-10} < |z| < 1\}$ as a function of $t = -2\log|z|$




$Curv_{(\log K_D)_{z\bar{z}}|dz|^2}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $t = -2\log|z|$

$$\frac{\partial^2}{\partial z \partial \bar{z}}(\log c_D) = \pi K_D \qquad \text{(Suita)}$$

where $K_D$ is the Bergman kernel on the diagonal:

$$K_D(z) := \sup\{|f(z)|^2 : f \in \mathcal{O}(D),\ \int_D |f|^2 d\lambda \leq 1\}.$$

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with $C = 750$.

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$C = 2$ (B., 2007)
$C = 1.95388\ldots$ (Guan-Zhou-Zhu, 2011)

## Ohsawa-Takegoshi Extension Theorem (1987)

$\Omega$ - bounded pseudoconvex domain in $\mathbb{C}^n$, $\varphi$ - psh in $\Omega$
$H$ - complex affine subspace of $\mathbb{C}^n$
$f$ - holomorphic in $\Omega' := \Omega \cap H$
Then there exists a holomorphic extension $F$ of $f$ to $\Omega$ such that

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B.-Y. Chen (2011): Ohsawa-Takegoshi extension theorem can be deduced directly from Hörmander's estimate for $\bar\partial$-equation!

# Mahler Conjecture

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$$K' := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } x \in K\}$$

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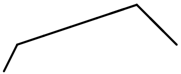

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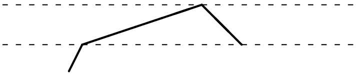

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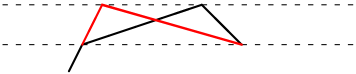

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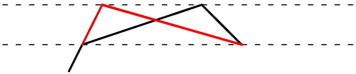

Bourgain-Milman (1987): There exists $c > 0$ such that

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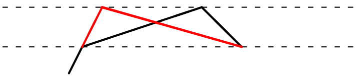

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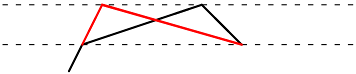

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Nazarov (2012): One can show the Bourgain-Milman inequality with $c = (\pi/4)^3$ using Hörmander's estimate.

## Hörmander's Estimate (1965)

$\Omega$ - pseudoconvex in $\mathbb{C}^n$, $\varphi$ - smooth, strongly psh in $\Omega$
$\alpha = \sum_j \alpha_j d\bar{z}_j \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
Then one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda.$$

Here $|\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k} \varphi^{j\bar{k}} \bar{\alpha}_j \alpha_k$, where $(\varphi^{j\bar{k}}) = (\partial^2\varphi/\partial z_j \partial\bar{z}_k)^{-1}$ is the length of $\alpha$ w.r.t. the Kähler metric $i\partial\bar\partial\varphi$.

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The estimate also makes sense for non-smooth $\varphi$: instead of $|\alpha|^2_{i\partial\bar\partial\varphi}$ one has to take any nonnegative $H \in L^\infty_{loc}(\Omega)$ with

$$i\bar\alpha \wedge \alpha \le H\, i\partial\bar\partial\varphi$$

(B., 2005).

## Donnelly-Fefferman (1982)

$\Omega$, $\alpha$, $\varphi$ as before
$\psi$ psh in $\Omega$ s.th. $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq 1$ (that is $i\partial\psi \wedge \bar\partial\psi \leq i\partial\bar\partial\psi$)
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Then, if $0 \leq \delta < 1$, one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{\delta\psi - \varphi} d\lambda \leq \frac{4}{(1-\delta)^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\delta\psi - \varphi} d\lambda.$$

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The above constant was obtained in B. 2004 and is optimal (B. 2012).
Therefore $C = 4$ is optimal in Donnelly-Fefferman.

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Berndtsson's estimate is not enough to obtain Ohsawa-Takegoshi (it would be if it were true for $\delta = 1$).

## Berndtsson's Estimate

$\Omega$ - pseudoconvex

$\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar{\partial}\alpha = 0$

$\varphi$, $\psi$ - psh, $|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} \leq 1$

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**Theorem.** $\Omega$, $\alpha$, $\varphi$, $\psi$ as above

Assume in addition that $|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} \leq \delta < 1$ on $\operatorname{supp}\alpha$.

Then there exists $u \in L^2_{loc}(\Omega)$ solving $\bar{\partial}u = \alpha$ with

$$\int_\Omega |u|^2 (1 - |\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi}) e^{\psi - \varphi} d\lambda \leq \frac{1}{(1-\sqrt{\delta})^2} \int_\Omega |\alpha|^2_{i\partial\bar{\partial}\psi} e^{\psi - \varphi} d\lambda.$$

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**Theorem.** $\Omega$, $\alpha$, $\varphi$, $\psi$ as above
Assume in addition that $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq \delta < 1$ on $\operatorname{supp}\alpha$.
Then there exists $u \in L^2_{loc}(\Omega)$ solving $\bar\partial u = \alpha$ with

$$\int_\Omega |u|^2 (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\psi}) e^{\psi-\varphi} d\lambda \leq \frac{1}{(1-\sqrt{\delta})^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\psi-\varphi} d\lambda.$$

From this estimate one can obtain Ohsawa-Takegoshi and Suita with $C = 1.95388\ldots$ (obtained earlier by Guan-Zhou-Zhu).

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Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

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By Hörmander's estimate

$$\int_\Omega |v|^2 e^{-\varphi} d\lambda \leq \int_\Omega |\beta|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda.$$

Therefore

$$\int_\Omega |u|^2 e^{2\psi-\varphi} d\lambda \leq \int_\Omega |\alpha + u\,\bar\partial\psi|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda$$

Therefore

$$\int_\Omega |u|^2 e^{2\psi-\varphi} d\lambda \leq \int_\Omega |\alpha + u\,\bar\partial\psi|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda$$
$$\leq \int_\Omega \left( |\alpha|^2_{i\partial\bar\partial\varphi} + 2|u|\sqrt{H}|\alpha|_{i\partial\bar\partial\varphi} + |u|^2 H \right) e^{2\psi-\varphi} d\lambda,$$

where $H = |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}$.

Therefore

$$
\begin{aligned}
\int_\Omega |u|^2 e^{2\psi-\varphi} d\lambda &\le \int_\Omega |\alpha + u\,\bar\partial\psi|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda \\
&\le \int_\Omega \left(|\alpha|^2_{i\partial\bar\partial\varphi} + 2|u|\sqrt{H}|\alpha|_{i\partial\bar\partial\varphi} + |u|^2 H\right) e^{2\psi-\varphi} d\lambda,
\end{aligned}
$$

where $H = |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}$. For $t > 0$ we will get

$$
\begin{aligned}
&\int_\Omega |u|^2 (1-H) e^{2\psi-\varphi} d\lambda \\
&\qquad \le \int_\Omega \left[|\alpha|^2_{i\partial\bar\partial\varphi}\left(1 + t^{-1}\frac{H}{1-H}\right) + t|u|^2(1-H)\right] e^{2\psi-\varphi} d\lambda \\
&\qquad \le \left(1 + t^{-1}\frac{\delta}{1-\delta}\right)\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda \\
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We will obtain the required estimate if we take $t := 1/(\delta^{-1/2}+1)$.

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Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

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We will get Berndtsson's estimate with the constant

$$\frac{1+\sqrt{\widetilde\delta}}{(1-\sqrt{\widetilde\delta})(1-\widetilde\delta)} = \frac{4}{(1-\delta)^2}.$$

Theorem (Ohsawa-Takegoshi with optimal constant)
$\Omega$ - pseudoconvex in $\mathbb{C}^{n-1} \times D$, where $0 \in D \subset \mathbb{C}$,
$\varphi$ - psh in $\Omega$, $f$ - holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
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$\varepsilon > 0$

$$\alpha := \bar\partial\big(f(z')\chi(-2\log|z_n|)\big),$$

where $\chi(t) = 0$ for $t \leq -2\log\varepsilon$ and $\chi(\infty) = 1$.

$G := G_D(\cdot, 0)$
$\widetilde{\varphi} := \varphi + 2G + \eta(-2G)$
$\psi := \gamma(-2G)$

$F := f(z')\chi(-2\log|z_n|) - u$, where $u$ is a solution of $\bar\partial u = \alpha$ given by the previous thm.

## Crucial ODE Problem

Find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ such that $h' < 0$, $h'' > 0$,

$$\lim_{t\to\infty}(g(t) + \log t) = \lim_{t\to\infty}(h(t) + \log t) = 0$$

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Guan-Zhou recently gave another proof of the Ohsawa-Takegoshi with optimal constant (and obtained some generalizations) but used essentially the same ODE.

# Another approach: general lower bound for the Bergman kernel

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$$K_\Omega(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 d\lambda \leq 1\}$$ (Bergman kernel)

$$G_\Omega(\cdot, w) = \sup\{v \in PSH^-(\Omega),\ \overline{\lim_{z \to w}}(v(z) - \log|z - w|) < \infty\}$$

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Theorem. Assume $\Omega$ is pseudoconvex in $\mathbb{C}^n$. Then for $a \ge 0$ and $w \in \Omega$

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**Corollary 1.** If $n = 1$ then

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**Corollary 2.** If $\Omega$ is convex in $\mathbb{C}^n$ then for $w \in \Omega$

$$K_\Omega(w) \ge \frac{1}{\lambda_{2n}(I_\Omega(w))},$$

where $I_\Omega(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$ (Kobayashi indicatrix).

Theorem. Assume $\Omega$ is pseudoconvex in $\mathbb{C}^n$. Then for $a \geq 0$ and $w \in \Omega$

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Sketch of proof. May assume that $\Omega$ is bounded, smooth and strongly pseudoconvex. $G := G_{\Omega,w}$. Using Donnelly-Fefferman with

$$\varphi := 2nG, \quad \psi := -\log(-G),$$

$$\alpha := \bar\partial(\chi \circ G) = \chi' \circ G\, \bar\partial G,$$

$$\chi(t) := \begin{cases} 0 & t \geq -a, \\ \displaystyle\int_a^{-t} \frac{e^{-ns}}{s}\, ds & t < -a, \end{cases}$$

$$f := \chi \circ G - u \in \mathcal{O}(\Omega)$$

we will get

$$K_\Omega(w) \geq \frac{|f(w)|^2}{||f||^2} \geq \frac{c_{n,a}}{\lambda(\{G < -a\})},$$

where

$$c_{n,a} = \frac{\mathrm{Ei}(na)^2}{(\mathrm{Ei}(na) + \sqrt{C})^2}, \qquad \mathrm{Ei}(a) = \int_a^\infty \frac{e^{-s}}{s}\, ds.$$

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$$\lim_{m\to\infty} c_{nm,a}^{1/m} = e^{-2na}.$$

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$K$ - convex symmetric body in $\mathbb{R}^n$
Nazarov: consider the tube domain $T_K := \mathrm{int}K + i\mathbb{R}^n \subset \mathbb{C}^n$. Then

$$\left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda_n(K))^2} \leq K_{T_K}(0) \leq \frac{n!}{\pi^n} \frac{\lambda_n(K')}{\lambda_n(K)}. \tag{1}$$

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Therefore

$$\lambda_n(K)\lambda_n(K') \geq \left(\frac{\pi}{4}\right)^{3n} \frac{4^n}{n!}.$$

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**Conjecture.** $\lambda_{2n}(I) \leq \left(\dfrac{4}{\pi}\right)^{n} (\lambda_n(K))^2$

$$K_{T_K}(0) \geq \left(\frac{\pi}{4}\right)^{n} \frac{1}{(\lambda_n(K))^2} \quad \text{(equality for cubes)}$$

By the Lempert theory, if $K$ is smooth, symmetric, strongly convex in $\mathbb{R}^n$,

$$\nu : \partial K \to S^{n-1}$$

is the Gauss map, then $\partial I$ is parametrized by

$$\frac{1}{4}\int_0^{2\pi} e^{it}\,\nu^{-1}\left(\frac{\mathrm{Re}\,(e^{it}\bar{w})}{|\mathrm{Re}\,(e^{it}\bar{w})|}\right) dt, \quad w \in S^{2n-1}.$$