Suita Conjecture and the Ohsawa-Takegoshi Extension Theorem

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D domain in $\mathbb{C}$

$c_D(z) := \exp \lim_{\zeta\to z} \left(G_D(\zeta, z) - \log|\zeta - z| \right)$

(logarithmic capacity of $\mathbb{C} \setminus D$ w.r.t. z)

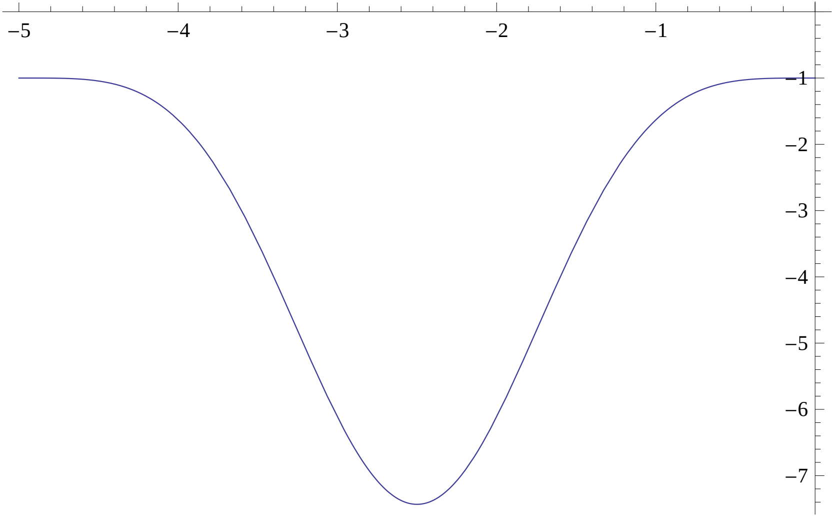
$c_D|dz|$ is an invariant metric (Suita metric)

$$Curv_{c_D|dz|} = -\frac{(\log c_D)_{z\bar{z}}}{c_D^2}$$

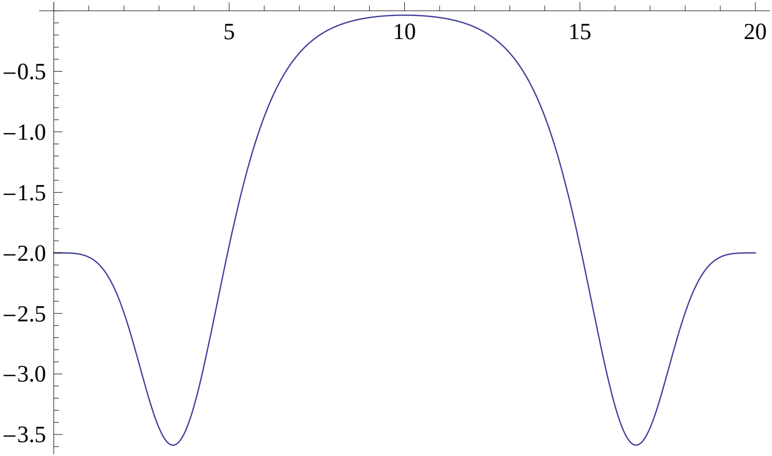
Suita Conjecture (1972): $\quad Curv_{c_D|dz|} \leq -1$

- "$=$" if D is simply connected
- "$<$" if D is an annulus (Suita)
- Enough to prove for D with smooth boundary
- "$=$" on ∂D if D has smooth boundary

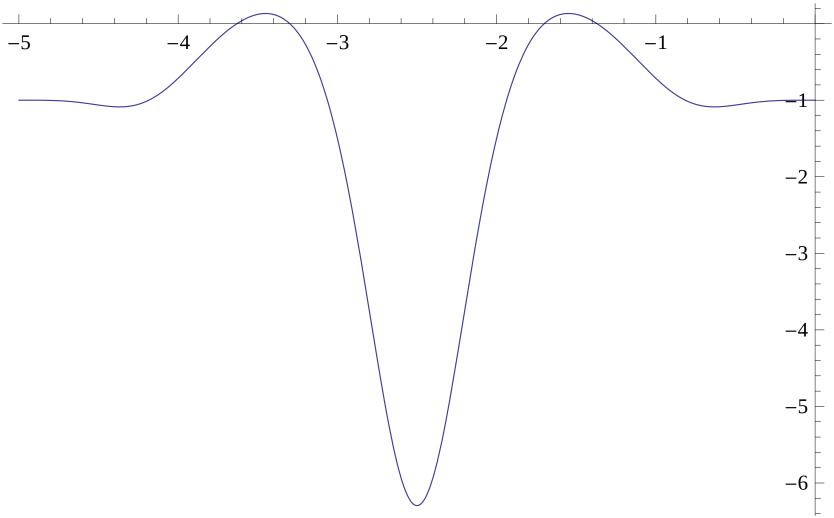
We are essentially asking whether the curvature of the Suita metric satisfies maximum principle.



$Curv_{c_D|dz|}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log|z|$



$Curv_{K_D|dz|^2}$ for $D = \{e^{-10} < |z| < 1\}$ as a function of $-2\log|z|$



$Curv_{(\log K_D)_{z\bar{z}}|dz|^2}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log |z|$

$$\frac{\partial^2}{\partial z \partial \bar{z}}(\log c_D) = \pi K_D \qquad \text{(Suita)},$$

$$K_D(z) = \sup\{|f(z)|^2 : \ f \in \mathcal{O}(D), \ \int_D |f|^2 d\lambda \leq 1\}.$$

Therefore the Suita conjecture is equivalent to

$$c_D^2 \leq \pi K_D.$$

Surprisingly, the only sensible approach to this problem turned out to be by several complex variables! Ohsawa (1995) observed that it is really an extension problem: for $z \in D$ find $f \in \mathcal{O}(D)$ such that $f(z) = 1$ and

$$\int_D |f|^2 d\lambda \leq \frac{\pi}{(c_D(z))^2}.$$

Using the methods of the Ohsawa-Takegoshi extension theorem he showed the estimate

$$c_D^2 \leq C \pi K_D$$

with $C = 750$.

$C = 2$	(B., 2007)
$C = 1.95388\ldots$	(Guan-Zhou-Zhu, 2011)

Ohsawa-Takegoshi Extension Theorem

Theorem (1987)
Ω bounded pscvx domain in $\mathbb{C}^n$, φ psh in Ω
H complex affine subspace of $\mathbb{C}^n$
f holomorphic in $\Omega' := \Omega \cap H$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq C\pi \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda',$$

where C depends only on n and the diameter of Ω.

Siu / Berndtsson (1996)
If $\Omega \subset \mathbb{C}^{n-1} \times \{|z_n| < 1\}$ and $H = \{z_n = 0\}$ then $C = 4$.

Problem Can we improve to $C = 1$?

B.-Y. Chen (2011) Ohsawa-Takegoshi extension theorem can be proved using directly Hörmander's estimate for $\bar{\partial}$-equation!

L^2-Estimates for $\bar\partial$

Hörmander (1965)

Ω pscvx in $\mathbb{C}^n$, φ smooth, strongly psh in Ω

$\alpha = \sum_j \alpha_j d\bar z_j \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$

Then one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda.$$

Here $|\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k} \varphi^{j\bar k}\bar\alpha_j\alpha_k$, where $(\varphi^{j\bar k}) = (\partial^2\varphi/\partial z_j\partial\bar z_k)^{-1}$, is the length of α w.r.t. the Kähler metric $i\partial\bar\partial\varphi$.

The estimate also makes sense for non-smooth psh φ: instead of $|\alpha|^2_{i\partial\bar\partial\varphi}$ one has to take any nonnegative $H \in L^\infty_{loc}(\Omega)$ with

$$i\bar\alpha \wedge \alpha \le H\, i\partial\bar\partial\varphi$$

(B., 2005).

Berndtsson (1996)

Ω, α, φ as before, $\psi \in PSH(\Omega)$ s.th. $i\partial\psi \wedge \bar\partial\psi \leq i\partial\bar\partial\psi$.
Then, if $0 \leq \delta < 1$, one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{\delta\psi-\varphi} d\lambda \leq \frac{4}{(1-\delta)^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\delta\psi-\varphi} d\lambda.$$

For $\delta = 0$ and $\varphi \equiv 0$ the estimate is due to Donnelly-Fefferman (1982).

The constant $4/(1-\delta)^2$ was obtained in B. 2004 (originally it was $4/(\delta(1-\delta)^2)$) and is optimal for every δ (B. 2012).

Berndtsson's estimate is not enough to obtain Ohsawa-Takegoshi (it would be if it were true for $\delta = 1$).

Theorem Ω, α, φ, ψ as above
Assume in addition that $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq a < 1$ on $\operatorname{supp}\alpha$.
Then there exists $u \in L^2_{loc}(\Omega)$ solving $\bar\partial u = \alpha$ with

$$\int_\Omega |u|^2 (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\psi}) e^{\psi-\varphi} d\lambda \leq \frac{1+\sqrt{a}}{1-\sqrt{a}} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\psi-\varphi} d\lambda.$$

From this estimate one can get Ohsawa-Takegoshi and Suita with $C = 1.95388\ldots$ (obtained earlier by Guan-Zhou-Zhu).

Theorem Ω pscvx in $\mathbb{C}^n$, φ psh in Ω, $\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
$\psi \in W^{1,2}_{loc}(\Omega)$ locally bounded from above, s.th.

$$|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \begin{cases} \leq 1 & \text{in } \Omega \\ \leq a < 1 & \text{on supp}\,\alpha \end{cases}$$

Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}) e^{2\psi-\varphi} d\lambda \leq \frac{1+\sqrt{a}}{1-\sqrt{a}} \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda.$$

Remarks 1. Setting $\psi \equiv 0$ we recover the Hörmander estimate.
2. This theorem also implies all previous estimates: for psh φ, ψ with $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq 1$ and $\delta < 1$ set $\widetilde{\varphi} := \varphi + \psi$ and $\widetilde{\psi} = \frac{1+\delta}{2}\psi$.
Then $2\widetilde{\psi} - \widetilde{\varphi} = \delta\psi - \varphi$ and $|\bar\partial\widetilde{\psi}|^2_{i\partial\bar\partial\widetilde{\varphi}} \leq \frac{(1+\delta)^2}{4} =: a$.
We will get Berndtsson's estimate with the constant

$$\frac{1+\sqrt{a}}{(1-\sqrt{a})(1-a)} = \frac{4}{(1-\delta)^2}.$$

For $\delta = 1$ we have $|\bar\partial\widetilde{\psi}|^2_{i\partial\bar\partial\widetilde{\varphi}} \leq |\bar\partial\psi|^2_{i\partial\bar\partial\psi}$.

Theorem Ω pscvx in $\mathbb{C}^n$, φ psh in Ω, $\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
$\psi \in W^{1,2}_{loc}(\Omega)$ locally bounded from above, s.th.

$$|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \begin{cases} \leq 1 & \text{in } \Omega \\ \leq a < 1 & \text{on } \operatorname{supp}\alpha \end{cases}$$

Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2(1 - |\bar\partial\psi|^2_{i\partial\bar\partial\varphi})e^{2\psi-\varphi}d\lambda \leq \frac{1+\sqrt{a}}{1-\sqrt{a}}\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi}e^{2\psi-\varphi}d\lambda.$$

Proof (Some ideas going back to Berndtsson and B.-Y. Chen.)
By approximation we may assume that φ is smooth up to the boundary and strongly psh, and ψ is bounded.
u minimal solution to $\bar\partial u = \alpha$ in $L^2(\Omega, e^{\psi-\varphi})$
$\Rightarrow u \perp \ker\bar\partial$ in $L^2(\Omega, e^{\psi-\varphi})$
$\Rightarrow v := ue^{\psi} \perp \ker\bar\partial$ in $L^2(\Omega, e^{-\varphi})$
$\Rightarrow v$ minimal solution to $\bar\partial v = \beta := e^{\psi}(\alpha + u\bar\partial\psi)$ in $L^2(\Omega, e^{-\varphi})$

$$\text{Hörmander} \quad \Rightarrow \quad \int_\Omega |v|^2e^{-\varphi}d\lambda \leq \int_\Omega |\beta|^2_{i\partial\bar\partial\varphi}e^{-\varphi}d\lambda$$

Therefore

$$\begin{aligned}\int_\Omega |u|^2 e^{2\psi-\varphi} d\lambda &\le \int_\Omega |\alpha + u\,\bar\partial\psi|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda \\ &\le \int_\Omega \left(|\alpha|^2_{i\partial\bar\partial\varphi} + 2|u|\sqrt{H}|\alpha|_{i\partial\bar\partial\varphi} + |u|^2 H\right) e^{2\psi-\varphi} d\lambda,\end{aligned}$$

where $H = |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}$. For $t > 0$ we will get

$$\begin{aligned}&\int_\Omega |u|^2(1-H) e^{2\psi-\varphi} d\lambda \\ &\qquad \le \int_\Omega \left[|\alpha|^2_{i\partial\bar\partial\varphi}\left(1 + t^{-1}\frac{H}{1-H}\right) + t|u|^2(1-H)\right] e^{2\psi-\varphi} d\lambda \\ &\qquad \le \left(1 + t^{-1}\frac{a}{1-a}\right)\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda \\ &\qquad\qquad + t\int_\Omega |u|^2(1-H) e^{2\psi-\varphi} d\lambda.\end{aligned}$$

We will obtain the required estimate if we take $t := 1/(a^{-1/2}+1)$.

Theorem (Ohsawa-Takegoshi with optimal constant, B. 2013)
Ω pscvx in $\mathbb{C}^{n-1} \times D$, where $0 \in D \subset \mathbb{C}$,
φ psh in Ω, f holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Original solution of the L^2-extension problem with optimal constant.
For $n = 1$ and $\varphi \equiv 0$ we obtain the Suita conjecture.

Crucial ODE Problem Find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ s.th. $h' < 0$, $h'' > 0$,

$$\lim_{t\to\infty} (g(t) + \log t) = \lim_{t\to\infty} (h(t) + \log t) = 0$$

and

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1.$$

Solution
$$h(t) := -\log(t + e^{-t} - 1)$$
$$g(t) := -\log(t + e^{-t} - 1) + \log(1 - e^{-t}).$$

Guan-Zhou recently gave another proof of the Ohsawa-Takegoshi with optimal constant (and obtained various generalizations) but used essentially the same ODE with two unknowns (with essentially the same solutions).

They also answered the following, more detailed problem posed by Suita:

Theorem (Guan-Zhou, 2013) For any Riemann surface M which is not biholomorphic to a disc with a polar subset removed and which admits the Green function one has strict inequality in the Suita conjecture.

Another Approach to Suita Conjecture

$$K_\Omega(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 d\lambda \leq 1\}$$

(Bergman kernel)

$$G_\Omega(\cdot, w) = G_w = \sup\{v \in PSH^-(\Omega),\ \overline{\lim_{z\to w}}(v(z) - \log|z-w|) < \infty\}$$

(pluricomplex Green function)

Theorem Assume Ω is pscvx in $\mathbb{C}^n$. Then for $a \geq 0$ and $w \in \Omega$

$$K_\Omega(w) \geq \frac{1}{e^{2na}\lambda(\{G_\Omega(\cdot, w) < -a\})}.$$

Optimal constant: "=" if $\Omega = B(w, r)$

For $n = 1$ letting $a \to \infty$ this gives the Suita conjecture:

$$K_\Omega(w) \geq \frac{c_\Omega(w)^2}{\pi}.$$

Sketch of proof Using Donnelly-Fefferman's estimate for $\bar\partial$ with

$$\varphi = 2nG_w, \quad \psi = -\log(-G_w), \quad \alpha = \bar\partial(\chi \circ G_w)$$

one can prove

$$K_\Omega(w) \geq \frac{1}{c(n,t)\lambda(\{G_w < t\})}, \tag{1}$$

where

$$c(n,t) = \left(1 + \frac{C}{Ei(-nt)}\right)^2, \quad Ei(a) = \int_a^\infty \frac{ds}{se^s}$$

(B. 2005). Now use the tensor power trick: $\widetilde\Omega = \Omega \times \cdots \times \Omega \subset \mathbb{C}^{nm}$, $\widetilde w = (w, \dots, w)$ for $m \gg 0$. Then

$$K_{\widetilde\Omega}(\widetilde w) = (K_\Omega(w))^m, \quad \lambda(\{G_{\widetilde w} < t\}) = (\lambda(\{G_w < t\}))^m,$$

and by (1) for $\widetilde\Omega$

$$K_\Omega(w) \geq \frac{1}{c(nm,t)^{1/m}\lambda(\{G_w < t\})}.$$

But $\lim_{m\to\infty} c(nm,t)^{1/m} = e^{-2nt}$. $\square$

Proof 2 (Lempert) By Maitani-Yamaguchi / Berndtsson's result on log-(pluri)subharmonicity of the Bergman kernel for sections of a pseudoconvex domain it follows that $\log K_{\{G_w<t\}}(w)$ is convex for $t \in (-\infty, 0]$. Therefore

$$t \longmapsto 2nt + \log K_{\{G_w<t\}}(w)$$

is convex and bounded, hence non-decreasing. It follows that

$$K_\Omega(w) \geq e^{2nt} K_{\{G_w<t\}}(w) \geq \frac{e^{2nt}}{\lambda(\{G_w < t\})}. \quad \square$$

Berndtsson-Lempert: This method can be improved to obtain the Ohsawa-Takegoshi extension theorem with optimal constant (one has to use Berndtsson's positivity of direct image bundles).

What happens with $e^{-2nt}\lambda(\{G_w < t\})$ as $t \to -\infty$ for arbitrary n? For convex Ω using Lempert's theory one can get

Proposition If Ω is bounded, smooth and strongly convex in $\mathbb{C}^n$ then for $w \in \Omega$

$$\lim_{t\to-\infty} e^{-2nt}\lambda(\{G_w < t\}) = \lambda(I^K_\Omega(w)),$$

where $I^K_\Omega(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$ (Kobayashi indicatrix).

Corollary If $\Omega \subset \mathbb{C}^n$ is convex then

$$K_\Omega(w) \geq \frac{1}{\lambda(I^K_\Omega(w))}, \quad w \in \Omega.$$

For general Ω one can prove

Theorem (B.-Zwonek) If Ω is bounded and hyperconvex in $\mathbb{C}^n$ and $w \in \Omega$ then

$$\lim_{t\to-\infty} e^{-2nt}\lambda(\{G_w < t\}) = \lambda(I^A_\Omega(w)),$$

where $I^A_\Omega(w) = \{X \in \mathbb{C}^n : \overline{\lim}_{\zeta\to 0}\left(G_w(w+\zeta X) - \log|\zeta|\right) \leq 0\}$

(Azukawa indicatrix)

Corollary (SCV version of the Suita conjecture) If $\Omega \subset \mathbb{C}^n$ is pseudoconvex and $w \in \Omega$ then

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega^A(w))}.$$

Conjecture 1 For Ω pseudoconvex and $w \in \Omega$ the function

$$t \longmapsto e^{-2nt}\lambda(\{G_w < t\})$$

is non-decreasing in t.

It would follow if the function $t \longmapsto \log \lambda(\{G_w < t\})$ was convex on $(-\infty, 0]$. Fornæss: this doesn't have to be true even for $n = 1$.

Theorem (B.-Zwonek) Conjecture 1 is true for $n = 1$.

Proof It is be enough to prove that $f'(t) \geq 0$ where

$$f(t) := \log \lambda(\{G_w < t\}) - 2t$$

and t is a regular value of G_w. By the co-area formula

$$\lambda(\{G_w < t\}) = \int_{-\infty}^{t} \int_{\{G_w = s\}} \frac{d\sigma}{|\nabla G_w|} ds$$

and therefore

$$f'(t) = \frac{\displaystyle\int_{\{G_w = t\}} \frac{d\sigma}{|\nabla G_w|}}{\lambda(\{G_w < t\})} - 2.$$

By the Schwarz inequality

$$\int_{\{G_w = t\}} \frac{d\sigma}{|\nabla G_w|} \geq \frac{(\sigma(\{G_w = t\}))^2}{\displaystyle\int_{\{G_w = t\}} |\nabla G_w| d\sigma} = \frac{(\sigma(\{G_w = t\}))^2}{2\pi}.$$

The isoperimetric inequality gives

$$(\sigma(\{G_w = t\}))^2 \geq 4\pi\lambda(\{G_w < t\})$$

and we obtain $f'(t) \geq 0$. $\square$

Conjecture 1 for arbitrary n is equivalent to the following *pluricomplex isoperimetric inequality* for smooth strongly pseudoconvex Ω (then $G_w \in C^{1,1}(\bar{\Omega} \setminus \{w\})$, B.Guan / B., 2000)

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G_w|} \geq 2\lambda(\Omega).$$

Conjecture 1 also turns out to be closely related to the problem of symmetrization of the complex Monge-Ampère equation.

Theorem (B.-Zwonek) For a convex Ω and $w \in \Omega$ set

$$F_\Omega(w) := \left(K_\Omega(w)\lambda(I_\Omega^K(w))\right)^{1/n}.$$

Then $F_\Omega(w) \le 4$. If Ω is in addition symmetric w.r.t. w then $F_\Omega(w) \le 16/\pi^2 = 1.621\dots$.

For convex domains F_Ω is thus a biholomorphically invariant function satisfying $1 \le F_\Omega \le 4$. Can we find an example with $F_\Omega(w) > 1$? Using Jarnicki-Pflug-Zeinstra's formula for geodesics in convex complex ellipsoids (which is based on Lempert's theory) one can show the following

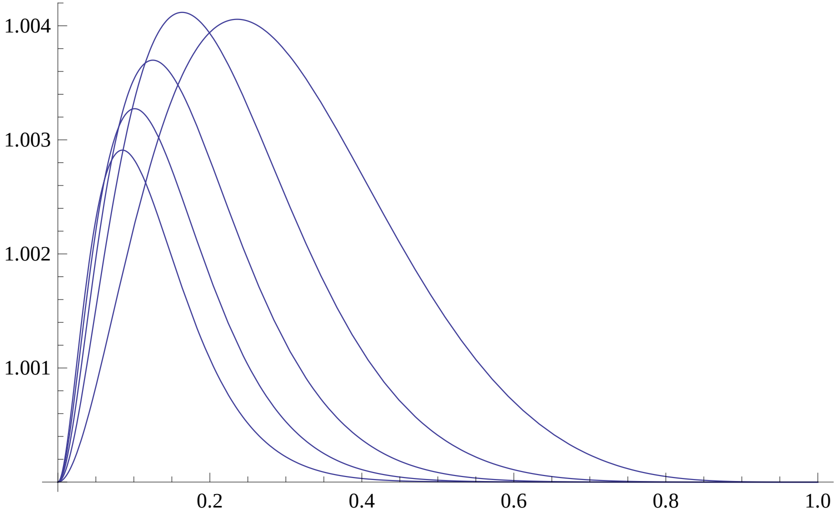
Theorem (B.-Zwonek) Define

$$\Omega = \{z \in \mathbb{C}^n : |z_1| + \dots + |z_n| < 1\}.$$

Then for $w = (b, 0, \dots, 0)$, where $0 < b < 1$, one has

$$K_\Omega(w)\lambda(I_\Omega^K(w)) = 1 + (1-b)^{2n}\frac{(1+b)^{2n} - (1-b)^{2n} - 4nb}{4nb(1+b)^{2n}}$$

$$= 1 + \frac{(1-b)^{2n}}{(1+b)^{2n}}\sum_{j=1}^{n-1}\frac{1}{2j+1}\binom{2n-1}{2j}b^{2j}.$$



$F_\Omega(b, 0, \dots, 0)$ in $\Omega = \{|z_1| + \cdots + |z_n| < 1\}$ for $n = 2, 3, \dots, 6$.

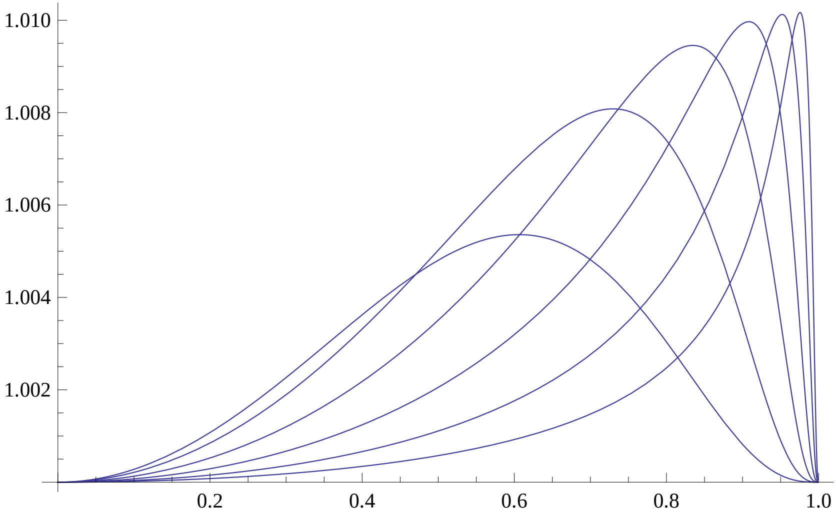
Theorem (B.-Zwonek) For $m \geq 1/2$ set $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ and $w = (b, 0)$, $0 < b < 1$. Then

$$K_\Omega(w)\lambda(I_\Omega^K(w)) = P\frac{m(1-b^2)+1+b^2}{2(1-b^2)^3(m-2)m^2(m+1)(3m-2)(3m-1)},$$

where

$$\begin{aligned} P =& b^{6m+2}\left(-m^3+2m^2+m-2\right) + b^{2m+2}\left(-27m^3+54m^2-33m+6\right) \\ &+ b^6m^2\left(3m^2+2m-1\right) + 6b^4m^2\left(3m^3-5m^2-4m+4\right) \\ &+ b^2\left(-36m^5+81m^4+10m^3-71m^2+32m-4\right) \\ &+ 2m^2\left(9m^3-27m^2+20m-4\right). \end{aligned}$$

In this domain all values of F_Ω are attained for $(b, 0)$, $0 < b < 1$.



$F_\Omega(b,0)$ in $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ for $m = 4, 8, 16, 32, 64, 128$.

$$\sup_{0<b<1} F_\Omega(b,0) \to 1.010182\ldots \text{ as } m \to \infty$$

Thank you!