Estimates for the Bergman Kernel and the Suita Conjecture

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Suita Conjecture

D domain in $\mathbb{C}$

$c_D(z) := \exp \lim_{\zeta \to z} \left(G_D(\zeta, z) - \log|\zeta - z| \right)$

(logarithmic capacity of $\mathbb{C} \setminus D$ w.r.t. z)

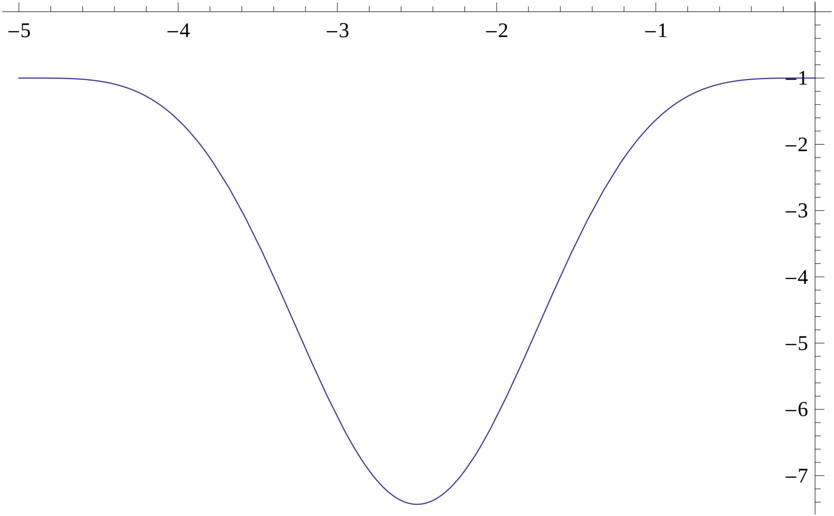
$c_D|dz|$ is an invariant metric (Suita metric)

$$Curv_{c_D|dz|} = -\frac{(\log c_D)_{z\bar{z}}}{c_D^2}$$

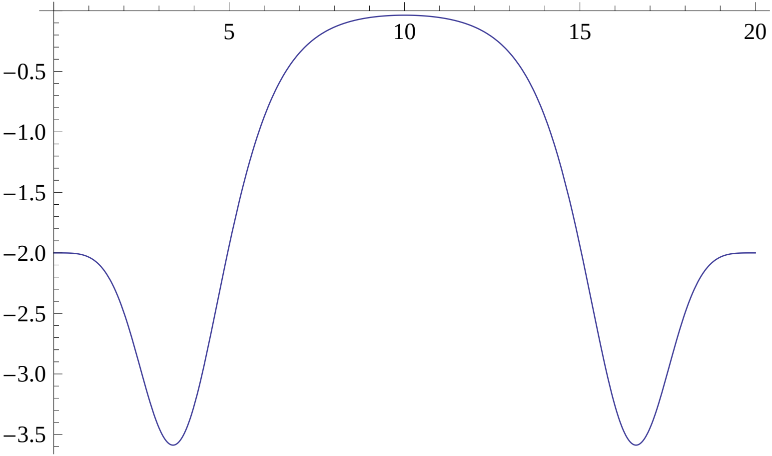
Suita Conjecture (1972): $\quad Curv_{c_D|dz|} \leq -1$

- "$=$" if D is simply connected
- "$<$" if D is an annulus (Suita)
- Enough to prove for D with smooth boundary
- "$=$" on ∂D if D has smooth boundary

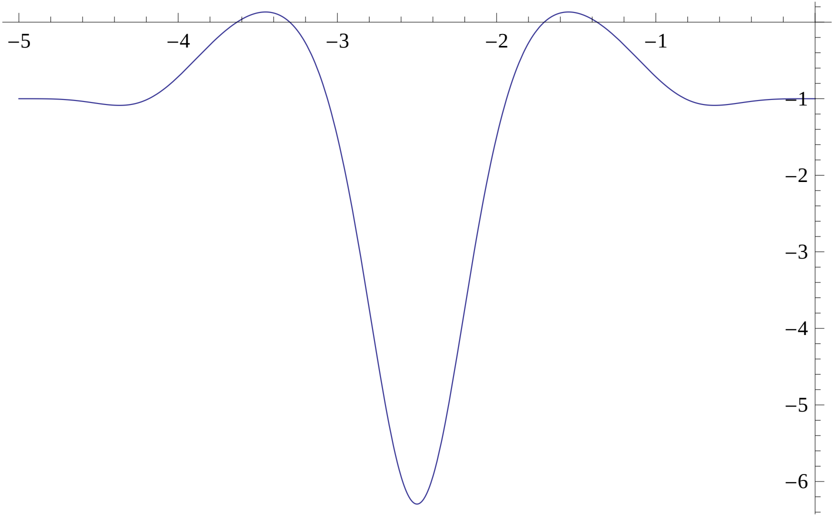
We are essentially asking whether the curvature of the Suita metric satisfies maximum principle.



$Curv_{c_D|dz|}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log|z|$



$Curv_{K_D|dz|^2}$ for $D = \{e^{-10} < |z| < 1\}$ as a function of $-2\log|z|$



$Curv_{(\log K_D)_{z\bar{z}}|dz|^2}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log|z|$

$$\frac{\partial^2}{\partial z \partial \bar{z}}(\log c_D) = \pi K_D \qquad \text{(Suita)},$$

$$K_D(z) = \sup\{|f(z)|^2 : \ f \in \mathcal{O}(D), \ \int_D |f|^2 d\lambda \leq 1\}.$$

Therefore the Suita conjecture is equivalent to

$$c_D^2 \leq \pi K_D.$$

Surprisingly, the only sensible approach to this problem turned out to be by several complex variables! Ohsawa (1995) observed that it is really an extension problem: for $z \in D$ find $f \in \mathcal{O}(D)$ such that $f(z) = 1$ and

$$\int_D |f|^2 d\lambda \leq \frac{\pi}{(c_D(z))^2}.$$

Using the methods of the Ohsawa-Takegoshi extension theorem he showed the estimate

$$c_D^2 \leq C \pi K_D$$

with $C = 750$.

$C = 2$	(B., 2007)
$C = 1.95388\ldots$	(Guan-Zhou-Zhu, 2011)

Theorem (Ohsawa-Takegoshi with optimal constant, B. 2013)
Ω pscvx in $\mathbb{C}^{n-1} \times D$, where $0 \in D \subset \mathbb{C}$,
φ psh in Ω, f holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Original solution of the L^2-extension problem with optimal constant.
Heavily influenced by Bo-Yong Chen's proof of the Ohsawa-Takegoshi theorem directly from Hörmander's estimate.
For $n = 1$ and $\varphi \equiv 0$ we obtain the Suita conjecture.

Crucial ODE Problem Find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ s.th. $h' < 0$, $h'' > 0$,

$$\lim_{t\to\infty} (g(t) + \log t) = \lim_{t\to\infty} (h(t) + \log t) = 0$$

and

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1.$$

Solution
$$h(t) := -\log(t + e^{-t} - 1)$$
$$g(t) := -\log(t + e^{-t} - 1) + \log(1 - e^{-t}).$$

Guan-Zhou recently gave another proof of the Ohsawa-Takegoshi with optimal constant (and obtained some generalizations) but used essentially the same ODE with two unknowns (with essentially the same solutions).

They also answered the following, more detailed problem posed by Suita:

Theorem (Guan-Zhou, 2013) For any Riemann surface M which is not biholomorphic to a disc with a polar subset removed and which admits the Green function one has strict inequality in the Suita conjecture.

Another Approach to Suita Conjecture

$\Omega \subset \mathbb{C}^n$, $w \in \Omega$

$$\begin{aligned} G_w(z) &= G_\Omega(z, w) \\ &= \sup\{u(z) : u \in PSH^-(\Omega) : \overline{\lim_{z \to w}} \left(u(z) - \log|z - w|\right) < \infty\} \end{aligned}$$

(pluricomplex Green function)

Theorem 0 Assume Ω is pseudoconvex in $\mathbb{C}^n$. Then for $w \in \Omega$ and $t \leq 0$

$$K_\Omega(w) \geq \frac{1}{e^{-2nt}\lambda(\{G_w < t\})}.$$

Optimal constant: "$=$" if $\Omega = B(w, r)$.

For $n = 1$ letting $a \to \infty$ this gives the Suita conjecture:

$$K_\Omega(w) \geq \frac{c_\Omega(w)^2}{\pi}.$$

Proof 1 Using Donnelly-Fefferman's estimate for $\bar\partial$ one can prove

$$K_\Omega(w) \geq \frac{1}{c(n,t)\lambda(\{G_w < t\})}, \tag{1}$$

where

$$c(n,t) = \left(1 + \frac{C}{Ei(-nt)}\right)^2, \quad Ei(a) = \int_a^\infty \frac{ds}{se^s}$$

(B. 2005). Now use the tensor power trick: $\widetilde{\Omega} = \Omega \times \cdots \times \Omega \subset \mathbb{C}^{nm}$, $\widetilde{w} = (w, \ldots, w)$ for $m \gg 0$. Then

$$K_{\widetilde{\Omega}}(\widetilde{w}) = (K_\Omega(w))^m, \quad \lambda(\{G_{\widetilde{w}} < t\}) = (\lambda(\{G_w < t\}))^m,$$

and by (1) for $\widetilde{\Omega}$

$$K_\Omega(w) \geq \frac{1}{c(nm,t)^{1/m}\lambda(\{G_w < t\})}.$$

But $\lim_{m\to\infty} c(nm,t)^{1/m} = e^{-2nt}$. $\square$

Proof 2 (Lempert) By Maitani-Yamaguchi / Berndtsson's result on log-(pluri)subharmonicity of the Bergman kernel for sections of a pseudoconvex domain it follows that $\log K_{\{G_w<t\}}(w)$ is convex for $t \in (-\infty, 0]$. Therefore

$$t \longmapsto 2nt + \log K_{\{G_w<t\}}(w)$$

is convex and bounded, hence non-decreasing. It follows that

$$K_\Omega(w) \geq e^{2nt} K_{\{G_w<t\}}(w) \geq \frac{e^{2nt}}{\lambda(\{G_w < t\})}. \quad \square$$

This way we have two additional proofs of the Suita conjecture, the first making effective use of arbitrarily many complex variables and the second one using two complex variables. The initial proof (using ODE) could be done entirely in one variable.

Berndtsson-Lempert Proof 2 can be improved to obtain the Ohsawa-Takegoshi extension theorem with optimal constant (one has to use Berndtsson's positivity of direct image bundles).

What happens with $e^{-2nt}\lambda(\{G_w < t\})$ as $t \to -\infty$ for arbitrary n? For convex Ω using Lempert's theory one can get

Proposition If Ω is bounded, smooth and strongly convex in $\mathbb{C}^n$ then for $w \in \Omega$

$$\lim_{t\to-\infty} e^{-2nt}\lambda(\{G_w < t\}) = \lambda(I_\Omega^K(w)),$$

where $I_\Omega^K(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta,\Omega),\ \varphi(0) = w\}$ (Kobayashi indicatrix).

Corollary If $\Omega \subset \mathbb{C}^n$ is convex then

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega^K(w))}, \quad w \in \Omega.$$

For general Ω one can prove

Theorem (B.-Zwonek) If Ω is bounded and hyperconvex in $\mathbb{C}^n$ and $w \in \Omega$ then

$$\lim_{t\to-\infty} e^{-2nt}\lambda(\{G_w < t\}) = \lambda(I_\Omega^A(w)),$$

where $I_\Omega^A(w) = \{X \in \mathbb{C}^n : \overline{\lim}_{\zeta\to 0}\left(G_w(w+\zeta X) - \log|\zeta|\right) \leq 0\}$
(Azukawa indicatrix)

Corollary (SCV version of the Suita conjecture) If $\Omega \subset \mathbb{C}^n$ is pseudoconvex and $w \in \Omega$ then

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega^A(w))}.$$

Conjecture 1 For Ω pseudoconvex and $w \in \Omega$ the function

$$t \longmapsto e^{-2nt}\lambda(\{G_w < t\})$$

is non-decreasing in t.

It would follow if the function $t \longmapsto \log \lambda(\{G_w < t\})$ was convex on $(-\infty, 0]$. Fornæss: this doesn't have to be true even for $n = 1$.

Theorem (B.-Zwonek) Conjecture 1 is true for $n = 1$.

Proof It is be enough to prove that $f'(t) \geq 0$ where

$$f(t) := \log \lambda(\{G_w < t\}) - 2t$$

and t is a regular value of G_w. By the co-area formula

$$\lambda(\{G_w < t\}) = \int_{-\infty}^{t} \int_{\{G_w = s\}} \frac{d\sigma}{|\nabla G_w|} ds$$

and therefore

$$f'(t) = \frac{\displaystyle\int_{\{G_w = t\}} \frac{d\sigma}{|\nabla G_w|}}{\lambda(\{G_w < t\})} - 2.$$

By the Schwarz inequality

$$\int_{\{G_w = t\}} \frac{d\sigma}{|\nabla G_w|} \geq \frac{(\sigma(\{G_w = t\}))^2}{\displaystyle\int_{\{G_w = t\}} |\nabla G_w| d\sigma} = \frac{(\sigma(\{G_w = t\}))^2}{2\pi}.$$

The isoperimetric inequality gives

$$(\sigma(\{G_w = t\}))^2 \geq 4\pi\lambda(\{G_w < t\})$$

and we obtain $f'(t) \geq 0$. $\square$

Conjecture 1 for arbitrary n is equivalent to the following *pluricomplex isoperimetric inequality* for smooth strongly pseudoconvex Ω (then $G_w \in C^{1,1}(\bar{\Omega} \setminus \{w\})$, B.Guan / B., 2000)

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G_w|} \geq 2\lambda(\Omega).$$

Conjecture 1 also turns out to be closely related to the problem of symmetrization of the complex Monge-Ampère equation.

What about corresponding upper bound in the Suita conjecture?
Not true in general:

Proposition (B.-Zwonek) Let $\Omega = \{r < |z| < 1\}$. Then

$$\frac{K_\Omega(\sqrt{r})}{(c_\Omega(\sqrt{r}))^2} \geq \frac{-2\log r}{\pi^3}.$$

It would be interesting to find un upper bound of the Bergman kernel for domains in $\mathbb{C}$ in terms of logarithmic capacity which would in particular imply the $\Rightarrow$ part in the well known equivalence

$$K_\Omega > 0 \;\Leftrightarrow\; c_\Omega > 0$$

($c_\Omega^2 \leq \pi K_\Omega$ being a quantitative version of $\Leftarrow$).

The upper bound for the Bergman kernel holds for convex domains:

Theorem (B.-Zwonek) For a convex Ω and $w \in \Omega$ set

$$F_\Omega(w) := \big(K_\Omega(w)\lambda(I_\Omega^K(w))\big)^{1/n}.$$

Then $F_\Omega(w) \leq 4$. If Ω is in addition symmetric w.r.t. w then $F_\Omega(w) \leq 16/\pi^2 = 1.621\ldots$.

Sketch of proof Denote $I := int\, I_\Omega^K(w)$ and assume that $w = 0$. One can show that $I \subset 2\,\Omega$ ($I \subset 4/\pi\,\Omega$ if Ω is symmetric). Then

$$K_\Omega(0)\lambda(I) \leq K_{I/2}(0)\lambda(I) = \frac{\lambda(I)}{\lambda(I/2)} = 4^n. \quad \square$$

For convex domains F_Ω is thus a biholomorphically invariant function satisfying $1 \leq F_\Omega \leq 4$. Can we find an example with $F_\Omega(w) > 1$?

Theorem (B.-Zwonek) Define

$$\Omega = \{z \in \mathbb{C}^n : |z_1| + \cdots + |z_n| < 1\}.$$

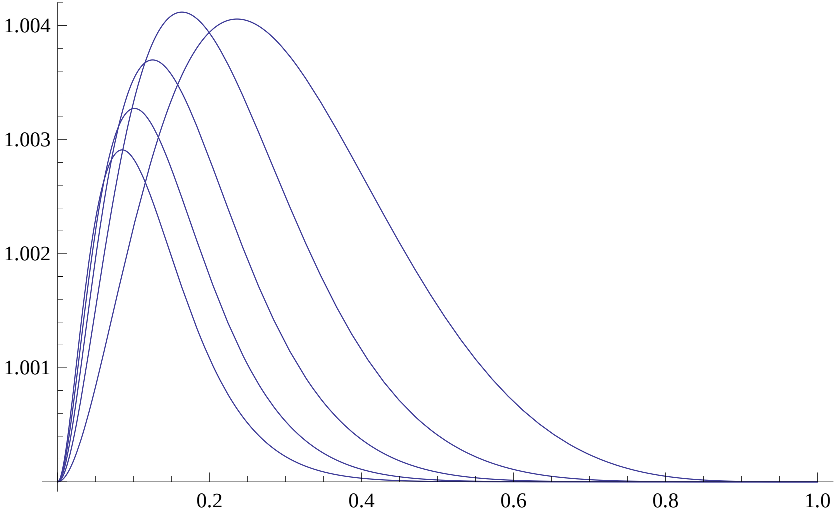
Then for $w = (b, 0, \ldots, 0)$, where $0 < b < 1$, one has

$$\begin{aligned} K_\Omega(w)\lambda(I_\Omega^K(w)) &= 1 + (1-b)^{2n} \frac{(1+b)^{2n} - (1-b)^{2n} - 4nb}{4nb(1+b)^{2n}} \\ &= 1 + \frac{(1-b)^{2n}}{(1+b)^{2n}} \sum_{j=1}^{n-1} \frac{1}{2j+1} \binom{2n-1}{2j} b^{2j}. \end{aligned}$$

The proof uses the formula for the Bergman kernel for this ellipsoid

$$K_\Omega((b, 0, \ldots, 0)) = \frac{2n-1}{4\pi\omega b}\left((1-b)^{-2n} - (1+b)^{-2n}\right),$$

where $\omega = \lambda(\{z \in \mathbb{C}^{n-1} : |z_1| + \cdots + |z_{n-1}| < 1\})$, obtained from the deflation method of Boas-Fu-Straube (1999). To compute $\lambda(I_\Omega^K(w))$ (main part of the proof) the Jarnicki-Pflug-Zeinstra (1993) formula for geodesics in convex complex ellipsoids (which is based on Lempert's theory) is applied.



$F_\Omega(b, 0, \ldots, 0)$ in $\Omega = \{|z_1| + \cdots + |z_n| < 1\}$ for $n = 2, 3, \ldots, 6$.

Theorem (B.-Zwonek) For $m \geq 1/2$ set $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ and $w = (b, 0)$, $0 < b < 1$. Then

$$K_\Omega(w)\lambda(I_\Omega^K(w)) = P\frac{m(1-b^2)+1+b^2}{2(1-b^2)^3(m-2)m^2(m+1)(3m-2)(3m-1)},$$

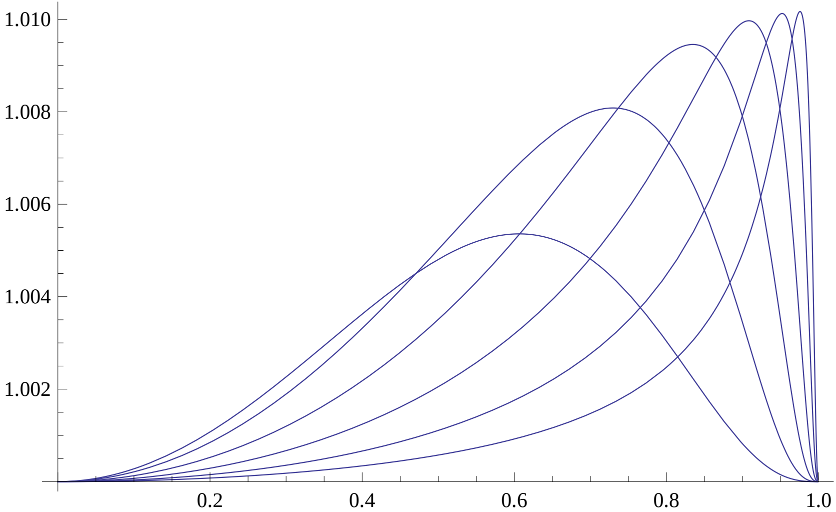
where

$$\begin{aligned} P =& b^{6m+2}\left(-m^3+2m^2+m-2\right) + b^{2m+2}\left(-27m^3+54m^2-33m+6\right) \\ &+ b^6m^2\left(3m^2+2m-1\right) + 6b^4m^2\left(3m^3-5m^2-4m+4\right) \\ &+ b^2\left(-36m^5+81m^4+10m^3-71m^2+32m-4\right) \\ &+ 2m^2\left(9m^3-27m^2+20m-4\right). \end{aligned}$$

Formulas obtained using *Mathematica*.

In this domain all values of F_Ω are attained for $(b, 0)$, $0 < b < 1$.

The Kobayashi function for this ellipsoid was computed implicitly by Blank-Fan-Klein-Krantz-Ma-Pang (1992) (this had only sufficed for numeric computations of $\lambda(I_\Omega^K(w))$).



$$F_\Omega(b,0) \text{ in } \Omega = \{|z_1|^{2m} + |z_2|^2 < 1\} \text{ for } m = 4, 8, 16, 32, 64, 128.$$

$$\sup_{0<b<1} F_\Omega(b,0) \to 1.010182\ldots \text{ as } m \to \infty$$

What is the highest value of F_Ω for convex Ω?

What can be said the function $w \longmapsto -\log \lambda(I_\Omega^A(w))$?

Is it plurisubharmonic?

It does not have to be C^2:

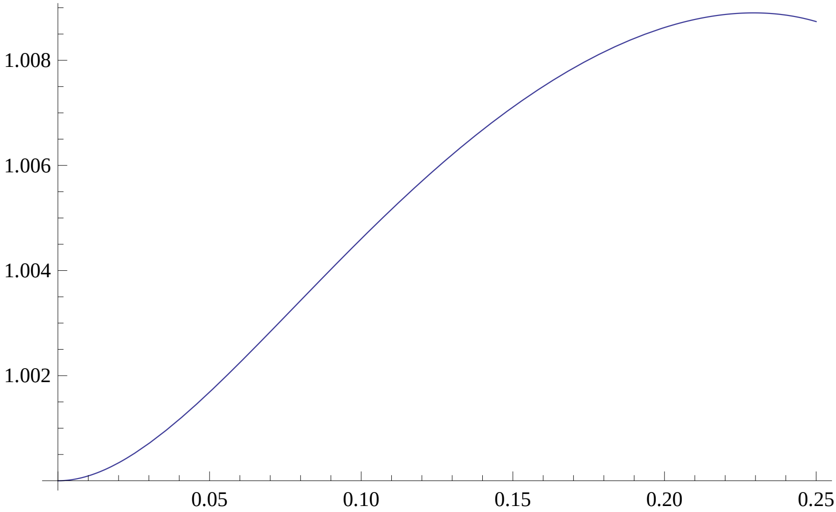
Theorem (B.-Zwonek) If $\Omega = \{|z_1| + |z_2| < 1\}$ and $0 < b \le 1/4$,

$$\lambda(I_\Omega^K((b,b))) \\ = \frac{\pi^2}{6}\left(30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1\right).$$

$\lambda(I_\Omega^K((b,b)))$ is not C^2 at $b = 1/4$.

It is known (Hahn-Pflug) that for $0 < b < 1/2$:

$$K_\Omega((b,b)) = \frac{2\left(8b^4 - 6b^2 + 3\right)}{\pi^2(1-4b^2)^3}.$$



$F_\Omega(b, b)$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $0 < b \leq 1/4$.

Mahler Conjecture

K - convex symmetric body in $\mathbb{R}^n$

$$K' := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } x \in K\}$$

Mahler volume $:= \lambda(K)\lambda(K')$

Mahler volume is an invariant of the Banach space defined by K: it is independent of linear transformations and of the choice of inner product.

Santaló Inequality (1949) Mahler volume is maximized by balls

Mahler Conjecture (1938) Mahler volume is minimized by cubes

Hansen-Lima bodies: starting from an interval they are produced by taking products of lower dimensional HL bodies and their duals.
$n = 2$: square
$n = 3$: cube & octahedron
$n = 4$: ...

Bourgain-Milman (1987) There exists $c > 0$ such that

$$\lambda(K)\lambda(K') \geq c^n \frac{4^n}{n!}.$$

Mahler Conjecture: $c = 1$

G. Kuperberg (2006) $c = \pi/4$

Nazarov (2012)

- equivalent SCV formulation of the Mahler Conjecture via the Fourier transform and the Paley-Wiener theorem
- proof of the Bourgain-Milman Inequality ($c = (\pi/4)^3$) using Hörmander's estimate for $\bar{\partial}$

K - convex symmetric body in $\mathbb{R}^n$
Nazarov: consider $T_K := \mathrm{int} K + i\mathbb{R}^n \subset \mathbb{C}^n$. Then

$$\left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda_n(K))^2} \le K_{T_K}(0) \le \frac{n!}{\pi^n} \frac{\lambda_n(K')}{\lambda_n(K)}.$$

Therefore

$$\lambda_n(K)\lambda_n(K') \ge \left(\frac{\pi}{4}\right)^{3n} \frac{4^n}{n!}.$$

To show the lower bound we can use the SCV version of the Suita conjecture for convex domains:

$$K_{T_K}(0) \ge \frac{1}{\lambda_{2n}(I_{T_K}(0))}.$$

Proposition $I_{T_K}(0) \subset \frac{4}{\pi}(K + iK)$

In particular, $\lambda_{2n}(I_{T_K}(0)) \le \left(\frac{4}{\pi}\right)^{2n} (\lambda_n(K))^2$

Conjecture $\lambda_{2n}(I_{T_K}(0)) \le \left(\frac{4}{\pi}\right)^{n} (\lambda_n(K))^2$

Thank you!