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Some fully non-linear elliptic equations in differential geometry

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ABSTRACT

Some fully non-linear elliptic equations in differential geometry

Two types of fully non-linear elliptic equations are studied in this thesis.

For the first type, we study the existence of geodesics in the space of volume forms associated with a real closed Riemannian manifold, which is a counterpart of the geodesic problem in the space of Kähler potentials. We show the existence of $C^{1,1}$ geodesics, provided that the sectional curvature of the manifold is non-negative.

As for the second type, we study the Dirichlet problem for complex Hessian equations on Hermitian manifolds with boundary. By establishing a priori estimates up to second order, we are able to solve the equation in a Euclidean ball in $\mathbb{C}^n$ of radius small enough. Based on this, we apply Perron envelope technique together with pluripotential theory to study the weak solutions following Bedford-Taylor. We show the existence of a continuous solution to the Dirichlet problem with the right hand side continuous, provided that there exists a subsolution and the Hermitian metric is locally conformal Kähler.

Contents

Chapter 1

Introduction

This thesis contains two parts. In each part we discuss some fully non-linear elliptic equations related to problems from differential geometry. In the first part, we deal with geometric problems in real Riemannian case, and the main result in Section 2.6 is a joint work with Z. Błocki [13]. In the second part, we deal with Hessian type equations in complex case, and the technical proof of Theorem 1.0.2 in Section 3.4 is a joint work with N. C. Nguyen [44]. We introduce them separately as follows.

1. Geodesics in the space of volume forms

It is well known that on any compact Kähler manifold, there is a natural infinite-dimensional Riemannian manifold structure with a Weil-Peterson metric in the space of Kähler potentials introduced by Mabuchi [69], Semmes [78] and Donaldson [32] separately. Moreover, this is a locally symmetric metric space of non-positive curvature. In [78], Semmes pointed out that the geodesic equation is equivalent to a homogeneous complex Monge-Ampère equation. In [32], Donaldson further conjectured that this space is geodesically convex and is a metric space. In [23], Chen proved that there always exist **weak geodesics** with bounded mixed complex derivatives, and from which he concluded that this space is indeed a metric space.

In [33], Donaldson introduced a corresponding situation in the case of real Riemannian manifolds. Precisely, for a closed Riemannian manifold (M^m, g), he considered the space of volume forms with fixed total volume. It seems natural since when M is 2-dimensional, it can be seen as a complex manifold of complex dimension 1, and then these two programs coincide exactly. He has also shown that this space admits an infinite-dimensional Riemannian manifold structure with non-positive sectional curvature. He also asked whether there is a smooth geodesic connecting any two points in this space. In fact, the existence of such geodesic segment is related to some other problems in partial differential equations such as Nahm's equations, regularity for some free boundary problems.

The geodesic equation related to the space of volume forms was investigated by Chen and He [24], where the authors applied similar strategy as in [23] by considering a perturbed equation. They then proved that there exists a smooth and unique solution to this perturbed equation. Based on this, they showed that this space is also a metric space. Actually, to solve the geodesic equation, they established the a priori *weak* C^2 estimates, that is, for the solution u, $\Delta u, u_{tt}$, and ∇u_t are bounded while boundedness of $\nabla^2 u$ remained open.

In Kähler setting, in general, for homogeneous complex Monge-Ampère equation, the solution, if exists, is at most $C^{1,1}$. So we may expect that Chen's regularity result is close to optimal. This is actually confirmed by Lempert, Vivas and Darvas in [27, 61], that is, they need not be C^2. In [11], Błocki showed that these geodesics are of class $C^{1,1}$, provided that the bisectional curvature of the manifold is non-negative. Inspired by this, in the real setting, we show the main result in the first part, obtained jointly with Z. Błocki in [13].

Theorem 1.0.1. *Let (M^m, g) be a closed Riemannian manifold with non-negative sectional curvature. Then for any two points in the space of volume forms, there exists a unique $C^{1,1}$ geodesic segment connecting these two points.*

2. Complex Hessian equations on compact Hermitian manifolds with boundary

Let $(\bar{M}, \alpha)$ be a compact Hermitian manifold with smooth boundary ∂M, of complex dimension n. Denote $M := \bar{M} \setminus \partial M$. Let $1 \leq m \leq n$ be an integer. Fix a real $(1,1)$-form χ on $\bar{M}$. For a positive right-hand side $f \in C^\infty(\bar{M})$ and a smooth boundary data $\varphi \in C^\infty(\partial M)$, the classical Dirichlet problem for the complex Hessian equation is to find a real-valued function $u \in C^\infty(\bar{M})$, such that

$$\begin{aligned} &(\chi + dd^c u)^m \wedge \alpha^{n-m} = f\alpha^n, \\ &u = \varphi \quad \text{on } \partial M, \end{aligned} \tag{1.0.1}$$

where u is subjected to point-wise inequalities

$$(\chi + dd^c u)^k \wedge \alpha^{n-k} > 0, \quad k = 1, .., m. \tag{1.0.2}$$

We first solve the equation in a small ball.

Theorem 1.0.2. *Let $M = B(z, \delta) \subset\subset B(0,1)$ be a Euclidean ball of radius δ in the unit ball $B(0,1) \subset \mathbb{C}^n$. Assume that χ, α are smooth on $\overline{B(0,1)}$. Then, the classical Dirichlet problem* (1.0.1) *is uniquely solvable for δ small enough, which depends only on χ, α.*

A C^2 real-valued function satisfying inequalities (1.0.2) is called $(\chi, m) - \alpha$-subharmonic. These inequalities can be generalised to non-smooth functions to obtain the class of $(\chi, m) - \alpha$-subharmonic functions on M. Locally, the convolution of a function in this class with a smooth kernel, in general, will not belong to this class again. However, using the theorem above and an adapted potential theory, we prove the approximation property.

Corollary 1.0.1. *Any $(\chi, m) - \alpha$-subharmonic function on M is locally approximated by a decreasing sequence of smooth $(\chi, m) - \alpha$-subharmonic functions.*

Following Bedford-Taylor [4, 5, 6] and Kołodziej [55, 56, 57], the two results above allow us to use Perron's envelope together with pluripotential theory techniques to study weak solutions to this equation with continuous right hand sides. A Hermitian metric α is called a locally conformal

Kähler metric on M if at any given point on M, there exist a local chart Ω and a smooth real-valued function G such that $e^G\alpha$ is Kähler on Ω. This class of metrics is strictly larger than the Kähler one, and not every Hermitian metric is locally conformal Käher (see e.g. [15]). Our main result is

Theorem 1.0.3. *Assume that α is locally conformal Kähler. Let $0 \leq f \in C^0(\bar{M})$ and $\varphi \in C^0(\partial M)$. Assume that there is a C^2-subsolution ρ, i.e., ρ satisfies inequalities* (1.0.2) *and*

$$(\chi + dd^c\rho)^m \wedge \alpha^{n-m} \geq f\alpha^n \quad \textit{in } \bar{M}, \quad \rho = \varphi \quad \textit{on } \partial M.$$

Then, there exists a continuous solution to the Dirichlet problem (1.0.1) *in pluripotential theory sense.*

When $m = n$ we need not assume α is locally conformal Kähler. The Dirichlet problem for the Monge-Ampère equation on compact Hermitian manifolds with boundary has been studied extensively, in smooth category, by Cherrier-Hanani [25, 26], Guan-Li [41] and Guan-Sun [42]. Our theorem generalises the result in [41] to continuous datum.

When $1 < m < n$ and $\alpha = dd^c|z|^2$ is the Euclidean metric the Dirichlet problem for the complex Hessian equation in a domain in $\mathbb{C}^n$ has been studied by many authors [9, 22, 29, 62, 65, 67, 72]. To our best knowledge the classical Dirichlet problem (1.0.1) on a compact Hermitian (or Kähler) manifold with boundary still remains open. The difficulty lies in the C^1-estimate for a general Hermitian metric α. Here we only obtain such an estimate in a small ball (Theorem 1.0.2). Moreover, in our approach, the locally conformal Kähler assumption of α is needed to define the complex Hessian operator of bounded functions (Section 3.2).

Motivations to study the Dirichlet problem for such equations come from recent developments of fully non-linear elliptic equations on compact complex manifolds. First, it is natural to consider this problem which has been raised in [80] after the complex Hessian equation was solved by Dinew-Kołodziej [30] on compact Kähler manifolds, and by Székelyhidi [80] and Zhang [88] on compact Hermitian manifolds. Second, on compact Hermitian manifolds, it is strongly related to the elementary symmetric positive cone with which several types of equations associated were studied by Székelyhidi-Tosatti-Weinkove [81], Tosatti-Weinkove [83, 84]. Our results may provide some tools to study these cones. In the case when α is Kähler (χ may be not), the Hessian type equations related to a Strominger system, which generalised Fu-Yau equations [37], have been studied recently by Phong-Picard-Zhang [74, 75, 76]. Lastly, the viscosity solutions of fully nonlinear elliptic equations on Riemannian and Hermitian manifolds have been also investigated by Harvey and Lawson [48, 49] in a more general frame work, and the existence of continuous solutions was proved under additional assumptions on the relation of the group structure of manifolds and given equations.

Chapter 2

Geodesics in the space of volume forms

Organisation. In Section 2.1, we provide some basic knowledge in differential geometry which will be needed in the following sections. In Section 2.2, we review the geodesic problem in the space of Kähler metrics and rewrite the proof of "interior C^2 estimate" in Błocki [11], so that we can compare with the proof of the main result in Section 2.6. In Section 2.3, we introduce $\mathcal{V}$, i.e., the space of volume forms together with some properties mainly based on Donaldson [33]. In Section 2.4, we discuss the solvability of the Dirichlet problem associated to the existence of geodesics in $\mathcal{V}$ following Chen and He [24]. In Section 2.5, we show that $\mathcal{V}$ is a metric space using results in Section 2.4 following [24]. In Section 2.6, we improve the regularity of weak geodesics in $\mathcal{V}$ under additional assumption that the sectional curvature of the manifold is non-negative.

2.1 Preliminaries

2.1.1 Basics of Riemannian geometry

Let M be a connected closed Riemannian manifold of dimension m and g a Riemannian metric on M, which is also denoted by an inner product $\langle \cdot, \cdot \rangle$. Throughout this article ∇ denotes the Levi-Civita connection (or covariant derivative) of (M^m, g), unless stated otherwise.

Let T be a (p, q)-tensor, that is, $T \in \otimes^{p,q} M = (\otimes^p TM) \otimes (\otimes^q TM^*)$, and $X, Z_1, ..., Z_q$ differential vector fields on M, then the covariant derivative of T is defined by

$$\nabla_X T(Z_1, ..., Z_q) := \nabla_X (T(Z_1, ..., Z_q)) - \sum_{i=1}^{q} T(Z_1, ..., \nabla_X Z_i, ..., Z_q),$$

where each term is an element of $\otimes^p TM$.

The covariant derivative can be considered as an operator

$$\nabla : C^\infty(\otimes^{p,q} M) \to C^\infty(\otimes^{p,q+1} M),$$

where

$$\nabla T(X, Z_1, ..., Z_q) := \nabla_X T(Z_1, ..., Z_q).$$

In this way we may define inductively $\nabla^2 T, \nabla^3 T$, etc. For example, the operator ∇^2 is given by

$$\begin{aligned}
\nabla^2 T(X, Y, Z_1, ..., Z_q) &= \nabla_X (\nabla T)(Y, Z_1, ..., Z_q) \\
&= \nabla_X (\nabla_Y T(Z_1, ..., Z_q)) - \nabla_{\nabla_X Y} T(Z_1, ..., Z_q) \\
&\quad - \sum_{i=1}^{q} \nabla_Y T(Z_1, ..., \nabla_X Z_i, ..., Z_q) \\
&= \nabla_X \nabla_Y T(Z_1, ..., Z_q) - \nabla_{\nabla_X Y} T(Z_1, ..., Z_q).
\end{aligned}$$

The Riemann curvature tensor is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It satisfies the first Bianchi identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

The Ricci tensor Rc is the trace of the Riemann curvature tensor

$$Rc(Y,Z) := trace(X \mapsto R(X,Y)Z).$$

Let $e_1, e_2, ..., e_m$ be a local frame of vector fields on M. We denote $g_{ij} = \langle e_i, e_j\rangle$, $(g^{ij}) = (g_{ij})^{-1}$, that is, $g^{ij}g_{jk} = \delta_{ik}$. The Christoffel symbols Γ_{ij}^k are defined by $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$.

The components of the curvature tensor are defined by

$$R(e_i, e_j)e_k := R_{ijk}^l e_l,$$

where $R_{ijk}^l := g^{lm}R_{ijkm}$, and $R_{ijkm} := \langle R(e_i, e_j)e_k, e_m\rangle$. The Riemann curvature tensor satisfies some symmetric properties

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}.$$

And the components of the Ricci tensor are given by

$$R_{jk} := Rc(e_j, e_k) = g^{im}R_{ijkm}.$$

The sectional curvature of a 2-plane P spanned by $\{e_i, e_j\}$ is defined by

$$K(P) := -\frac{\langle R(e_i, e_j)e_i, e_j\rangle}{\langle e_i, e_i\rangle\langle e_j, e_j\rangle - \langle e_i, e_j\rangle^2}.$$

Notations

For a differential function f defined on M, ∇f denotes the gradient of f, which means that $\langle \nabla f, X\rangle = Xf$, $X \in TM$, thus we can consider ∇f as a $(0,1)$-tensor, and $\nabla^2 f$ denotes the Hessian of f, which is given by $\nabla^2 f(e_i, e_j) = e_i e_j f - \Gamma_{ij}^k e_k f$. For simplicity, we denote the covariant derivative of a differential function f as follows

$$f_i = \nabla_i f,\ f_{ij} = \nabla^2 f(e_j, e_i),\ f_{ijk} = \nabla^3 f(e_k, e_j, e_i),\ \text{etc.}$$

In general, the commutation of covariant derivatives acting on tensors are expressed in terms of the curvature. For the (p,q)-tensor T, we recall the standard **commutation formulas**:

$$\begin{aligned}
&\nabla^2 T(e_i, e_j, e_{k_1}, ..., e_{k_q}; \omega^{l_1}, ..., \omega^{l_p}) - \nabla^2 T(e_j, e_i, e_{k_1}, ..., e_{k_q}; \omega^{l_1}, ..., \omega^{l_p})\\
=& -\sum_{h=1}^{q}\sum_{r=1}^{m} R^r_{ijk_h} T(e_{k_1}, ..., e_{k_{h-1}}, e_r, e_{k_{h+1}}, ..., e_{k_q}; \omega^{l_1}, ..., \omega^{l_p})\\
&+\sum_{h=1}^{p}\sum_{r=1}^{m} R^{l_h}_{ijr} T(e_{k_1}, ..., e_{k_q}; \omega^{l_1}, ..., \omega^{l_{h-1}}, \omega^r, \omega^{l_{h+1}}, ..., \omega^{l_p}),
\end{aligned}$$

where $\{\omega^i\}_{i=1}^m$ is the dual of $\{e_i\}_{i=1}^m$.

In particular, we have

$$f_{ij} = f_{ji},$$

$$f_{lkj} - f_{ljk} = R^m_{kjl} f_m. \tag{2.1.1}$$

Taking covariant derivative of (2.1.1) with respect to e_i, we have

$$f_{lkji} - f_{ljki} = \nabla_{e_i}(R^m_{kjl} f_m) = (\nabla_{e_i} R)^m_{kjl} f_m + R^m_{kjl} f_{mi}. \tag{2.1.2}$$

Also we have

$$f_{ljki} - f_{ljik} = R^m_{kij} f_{lm} + R^m_{kil} f_{mj}. \tag{2.1.3}$$

From (2.1.2) and (2.1.3) we have

$$f_{lkji} - f_{jilk} = (\nabla_{e_i} R)^m_{kjl} f_m + R^m_{kjl} f_{mi} + R^m_{kij} f_{lm} + R^m_{kil} f_{mj} + (\nabla_{e_k} R)^m_{lij} f_m + R^m_{lij} f_{mk}. \tag{2.1.4}$$

2.1.2 Basics of Kähler geometry

Let M be a complex manifold of complex dimension n and by $J : TM \to TM$ denote its complex structure. We fix a Hermitian metric g on M, i.e., g is compatible with J, such that

$$g(X,Y) = g(JX, JY), \quad X, Y \in TM.$$

We can then define a real 2-form ω on M by

$$\omega(X, Y) := g(JX, Y). \tag{2.1.5}$$

Usually we call such an ω the Kähler form of g.

By $T_{\mathbb{C}}M$ denote the complexification of TM and extend J, ω, and ∇(the unique Levi-Civita connection determined by g) to $T_{\mathbb{C}}M$ in a $\mathbb{C}$-linear way. On a local coordinate chart if $(z_1, \ldots, z_n)$ are complex analytic coordinates and $z_k = x_k + iy_k$, here $i = \sqrt{-1}$, then $(x_1, y_1, \ldots, x_n, y_n)$ define real coordinates on this chart, and the complex structure is given by $J(\frac{\partial}{\partial x_k}) = \frac{\partial}{\partial y_k}$, $J(\frac{\partial}{\partial y_k}) = -\frac{\partial}{\partial x_k}$.

For simplicity, set

$$\partial_j := \frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}), \quad \partial_{\bar{j}} := \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}), \quad 1 \leq j \leq n.$$

It is easy to see that the complexified tangent space $T_{\mathbb{C}}M$ splits into conjugate complex subspaces $T^{1,0}M \oplus T^{0,1}M$ associated to the eigenvalues i and $-i$ with respect to J,

$$J(\partial_j) = i\partial_j, \quad J(\partial_{\bar{j}}) = -i\partial_{\bar{j}}.$$

Set

$$g_{j\bar{k}} := g(\partial_j, \partial_{\bar{k}})\big(= g(\partial_{\bar{k}}, \partial_j)\big).$$

Then $\overline{g_{j\bar{k}}} = g_{k\bar{j}}$ and $g(\partial_j, \partial_k) = g(\partial_{\bar{j}}, \partial_{\bar{k}}) = 0$.

By (2.1.5), we have

$$\omega = i\sum_{j,k} g_{j\bar{k}} dz_j \wedge d\bar{z}_k,$$

which is a real form of type (1,1).

The following lemma is well known.

Lemma 2.1.1. *For a Hermitian metric g on M, the following are equivalent*

(1) $\nabla J = 0$;

(2) $d\omega = 0$*;*

(3) $\omega = i\partial\bar{\partial}g$ *locally for some smooth real-valued function* g*.*

Proof. (1)$\Leftrightarrow$(2): By definition,

$$\begin{aligned} d\omega(X,Y,Z) =& \nabla_X\omega(Y,Z) - \nabla_Y\omega(X,Z) + \nabla_Z\omega(X,Y) \\ =& X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y) - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y). \end{aligned}$$

Since $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, we deduce from above that

$$d\omega(X,Y,Z) = g((\nabla_X J)Y,Z) + g((\nabla_Y J)Z,X) + g((\nabla_Z J)X,Y).$$

Using the facts that

$$J^2 = -I, \quad \text{and} \quad g(J\cdot,\cdot) + g(\cdot,J\cdot) = 0,$$

we have

$$\begin{aligned} & d\omega(JX,Y,Z) + d\omega(X,JY,Z) \\ =& 2g((\nabla_Z J)X,JY) + g((\nabla_X J)JY - (\nabla_Y J)JX + (\nabla_{JX}J)Y - (\nabla_{JY}J)X, Z) \\ =& 2g((\nabla_Z J)X,JY) - g(N(X,Y),Z), \end{aligned}$$

where $N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$ is the Nijenhuis tensor. Here M is complex, thus J is integrable, i.e. $N(X,Y)$ vanishes. It follows that if $d\omega = 0$, then $\nabla J = 0$, and viceversa.

(2)$\Leftrightarrow$(3): Since (3)$\Rightarrow$(2) is obvious, we only need to show (2)$\Rightarrow$(3). By $d\omega = 0$, locally there is a real 1-form γ such that $\omega = d\gamma$. We may write $\gamma = \alpha + \bar{\alpha}$, where α is a $(1,0)$-form. Then $\omega = \partial\alpha + \partial\bar{\alpha} + \bar{\partial}\alpha + \bar{\partial}\bar{\alpha}$. It follows that $\partial\alpha = \bar{\partial}\bar{\alpha} = 0$ since ω is a $(1,1)$-form. Therefore locally there is a smooth complex-valued function f such that $\alpha = \partial f$, which implies that $\omega = i\partial\bar{\partial}(-2Im\ f)$. We can thus take $g = -2Im\ f$. □

We call g a *Kähler* metric and (M,g) a Kähler manifold if g satisfies equivalent conditions in Lemma 2.1.1. From now on we assume that (M,g) is a Kähler manifold unless otherwise stated.

Christoffel symbols

As in the real case, we define the Christoffel symbols Γ^k_{ij} by

$$\nabla_{\partial_j}\partial_k = \Gamma^l_{jk}\partial_l + \Gamma^{\bar{l}}_{jk}\partial_{\bar{l}},$$

and

$$\nabla_{\partial_j}\partial_{\bar{k}} = \Gamma^l_{j\bar{k}}\partial_l + \Gamma^{\bar{l}}_{j\bar{k}}\partial_{\bar{l}}.$$

Since $\nabla J = 0$, we have $i\nabla_{\partial_j}\partial_k = \nabla_{\partial_j}(J\partial_k) = J\nabla_{\partial_j}\partial_k$, it follows that $\Gamma^{\bar{l}}_{jk} = 0$. Similarly, $\Gamma^l_{j\bar{k}} = \Gamma^{\bar{l}}_{j\bar{k}} = 0$, so the only possible non-zero terms are Γ^k_{ij} and $\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \overline{\Gamma^k_{ij}}$. Moreover,

$$\frac{\partial g_{j\bar{k}}}{\partial z_i} = \frac{\partial}{\partial z_i} g(\partial_j, \partial_{\bar{k}}) = g(\nabla_{\partial_i}\partial_j, \partial_{\bar{k}}) = \Gamma^l_{ij} g_{l\bar{k}},$$

and hence

$$\Gamma^l_{ij} = g^{l\bar{k}}\frac{\partial g_{j\bar{k}}}{\partial z_i} = g^{l\bar{k}}\frac{\partial g_{i\bar{k}}}{\partial z_j},$$

where $g^{l\bar{k}}$ is determined by $g^{j\bar{p}}g_{k\bar{p}} = \delta_{jk}$.

(Symplectic) gradient

Let ϕ, ψ be any two differential real-valued function ϕ on M. Recall the gradient $\nabla\phi$ is defined by the relation $\langle \nabla\phi, X\rangle = X\phi$, $X \in TM$. Therefore, in local coordinates,

$$\nabla\phi = g^{j\bar{k}}(\phi_{\bar{k}}\partial_j + \phi_j\partial_{\bar{k}}), \text{ and } |\nabla\phi|^2 = 2g^{j\bar{k}}\phi_j\phi_{\bar{k}},$$

where $\phi_j = \frac{\partial\phi}{\partial z_j}$, $\phi_{\bar{k}} = \frac{\partial\phi}{\partial \bar{z}_k}$. The **symplectic gradient** of ϕ, say $grad_\omega\phi$, with respect to the form ω is defined by the relation $\omega(grad_\omega\phi, X) = -X\phi$, $X \in TM$. It is easy to see that $grad_\omega\phi = J\nabla\phi$. In particular, the vector field $grad_\omega\phi$ is **Hamiltonian**, i.e. the 1-form $i_{grad_\omega\phi}\omega := \omega(grad_\omega\phi, \cdot)$ is exact, since $i_{grad_\omega\phi}\omega = -d\phi$.

The **Poisson bracket** is defined by

$$\{\phi, \psi\}\omega^n := n d\phi \wedge d\psi \wedge \omega^{n-1}.$$

Equivalently, $\{\phi, \psi\} = \omega(grad_\omega\phi, grad_\omega\psi)$. Actually, the correspondence $\phi \mapsto grad_\omega\phi$ is a Lie algebra homomorphism since $grad_\omega(\{\phi, \psi\}) = [grad_\omega\phi, grad_\omega\psi]$.

Curvature of Kähler metrics

The Riemannian curvature tensor can also be extended in a $\mathbb{C}$-linear way to $T_{\mathbb{C}}M$. Since $\nabla J = 0$, we have

$$R(X,Y)JZ = JR(X,Y)Z,$$

thus by definition,

$$R(X,Y,JZ,JW) = R(X,Y,Z,W),$$

from which we can deduce that $R(X,Y,Z,W) = 0$ unless Z and W are of different type. In local coordinates $(z_1,\ldots,z_n)$, this means that the only possible non-zero terms are

$$R_{i\bar{j}k\bar{l}} := R(\partial_i, \partial_{\bar{j}}, \partial_k, \partial_{\bar{l}}).$$

By definition, we compute

$$R_{i\bar{j}k\bar{l}} = g(-\nabla_{\partial_{\bar{j}}}\nabla_{\partial_i}\partial_k, \partial_{\bar{l}}) = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + g^{s\bar{t}}\frac{\partial g_{s\bar{j}}}{\partial z_k}\frac{\partial g_{i\bar{t}}}{\partial \bar{z}_l}.$$

The curvature tensor is an obstruction for the commuting of covariant differentiations, but when we apply covariant differentiation successively with respect to two indices without bar, we can interchange these two indices. The same principle applies when the two indices have a bar at the same time. We shall use this fact in the next section.

Let $e_1,\ldots,e_{2n}$ be a local orthonormal basis of TM such that $Je_i = e_{n+i}$ for $1 \le i \le n$ and set $u_k = \frac{1}{\sqrt{2}}(e_k - iJe_k)$, then $\{u_k\}$ is a unitary basis of $T_{\mathbb{C}}M$. Recall the Ricci curvature tensor is defined by

$$Rc(X,Y) = \sum_{i=1}^{2n} R(e_i, X, Y, e_i).$$

It is easy to see that

$$Rc(JX,JY) = \sum_{i=1}^{2n} R(e_i, JX, JY, e_i) = \sum_{i=1}^{2n} R(Je_i, X, Y, Je_i) = Rc(X,Y),$$

therefore, $Rc(X,Y)=0$ unless X and Y are of different type. Moreover,

$$
\begin{aligned}
Rc(u_k,\bar{u}_l) &= \sum_{i=1}^{2n} R(e_i,u_k,\bar{u}_l,e_i) \\
&= \sum_{i=1}^{n} R(\frac{u_i+\bar{u}_i}{\sqrt{2}},u_k,\bar{u}_l,\frac{u_i+\bar{u}_i}{\sqrt{2}}) + \sum_{i=1}^{n} R(\frac{\bar{u}_i-u_i}{\sqrt{2}i},u_k,\bar{u}_l,\frac{\bar{u}_i-u_i}{\sqrt{2}i}) \\
&= \sum_{i=1}^{n} R(\bar{u}_i,u_k,\bar{u}_l,u_i) = \sum_{i=1}^{n} R(u_i,\bar{u}_i,u_k,\bar{u}_l),
\end{aligned}
$$

where we use the first Bianchi identity in the last equality. So in local complex coordinates we have a nice expression for the Ricci tensor,

$$R_{k\bar{l}} := Rc(\partial_k,\partial_{\bar{l}}) = g^{i\bar{j}}R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 \log\det(g_{i\bar{j}})}{\partial z_k \partial \bar{z}_l}.$$

Recall that if $|X|=|Y|=1$ and X is perpendicular to Y, then $R(X,Y,Y,X)$ is the sectional curvature of the plane spanned by $\{X,Y\}$. Set now

$$U=\frac{1}{\sqrt{2}}(X-iJX),\quad V=\frac{1}{\sqrt{2}}(Y-iJY),$$

so that $|U|=|V|=1$ and $U,V\in T^{1,0}M$, then

Definition 2.1.1. The bisectional curvature is defined to be

$$B(U,V) := R(U,\bar{U},V,\bar{V}) = R(X,Y,Y,X)+R(X,JY,JY,X).$$

2.2 $C^{1,1}$ geodesics in Kähler case when bisectional curvature is non-negative

2.2.1 The space of Kähler metrics and the related geodesic problem

The Riemannian structure

Before introducing the space of volume forms, we would like to make a brief review on the space of Kähler metrics, somehow we can compare with each other. The space of Kähler metrics plays an important role in the study of Kähler geometry. There are some other interesting properties of this space not listed here, see Błocki [12] for a survey.

Let (M, g) be a compact Kähler manifold of complex dimension n with the associated Kähler form ω. We consider the space of Kähler potentials with respect to ω, that is

$$\mathcal{H} := \{\phi \in C^\infty(M, \mathbb{R}) \mid \omega_\phi := \omega + i\partial\bar{\partial}\phi > 0\}.$$

We can treat $\mathcal{H}$ as an open subset of $C^\infty(M, \mathbb{R})$ with topology of uniform convergence of all partial derivatives and differential structure defined by the relation $C^\infty(U, C^\infty(M, \mathbb{R})) = C^\infty(M \times U, \mathbb{R})$ for any region $U \subset \mathbb{R}^m$.

By "$\partial\bar{\partial}$-lemma", two Käher potentials define the same metric if (and only if) they differ by an additive constant, which means that

$$\mathcal{H}_0 := \{\omega_\phi = \omega + i\partial\bar{\partial}\phi \mid \phi \in \mathcal{H}\} = \mathcal{H}/\mathbb{R},$$

where $\mathbb{R}$ acts on $\mathcal{H}$ by addition. The set $\mathcal{H}_0$ is therefore the space of Kähler metrics in the cohomology class $\{\omega\} \in H^{1,1}(M, \mathbb{R})$.

For $\phi \in \mathcal{H}$ we can associate the tangent space $T_\phi\mathcal{H}$ with $C^\infty(M, \mathbb{R})$. Mabuchi [69] introduced a Riemannian structure on $\mathcal{H}$ as follows

$$\langle\langle \psi, \eta \rangle\rangle_\phi := \frac{1}{V} \int_M \psi\eta\omega_\phi^n, \quad \psi, \eta \in T_\phi\mathcal{H},$$

where $V := \int_M \omega^n$. For a smooth curve $\phi(t) : [0, 1] \to \mathcal{H}$ $\big($which is an element of $C^\infty(M \times [0, 1], \mathbb{R})\big)$, the length is given by

$$l(\phi) := \int_0^1 \langle\langle \dot{\phi}, \dot{\phi} \rangle\rangle_{\phi_t}^{\frac{1}{2}} dt,$$

where $\dot{\phi} = \frac{\partial\phi}{\partial t}$. One can check that the Riemannian structure on $\mathcal{H}$ gives the following Levi-Civita connection: for a smooth vector field ψ along ϕ $\big($which is also an element of $C^\infty(M \times [0, 1], \mathbb{R})\big)$, we have

$$D_{\dot{\phi}}\psi = \dot{\psi} - \frac{1}{2}\langle \nabla\psi, \nabla\dot{\phi} \rangle_{\omega_\phi},$$

where the gradient ∇, the metric $\langle \cdot, \cdot \rangle$ are taken with respect to ω_ϕ.

Now consider a 2-parameters family $\phi(s,t) \in \mathcal{H}$ and a vector field $\psi(s,t) \in C^\infty(M, \mathbb{R})$ along ϕ. It is easy to see that the connection on $\mathcal{H}$ is torsion free, so the curvature tensor is given by

$$R(\phi_s, \phi_t)\psi = (D_{\phi_s} D_{\phi_t} - D_{\phi_t} D_{\phi_s})\psi.$$

Proposition 2.2.1 (Mabuchi [69], Donaldson [32])**.** *The curvature tensor on $\mathcal{H}$ can be expressed as*

$$R(\phi_s, \phi_t)\psi = \frac{1}{4}\{\{\phi_s, \phi_t\}, \psi\}.$$

In particular,

$$\langle\langle R(\phi_s, \phi_t)\phi_s, \phi_t \rangle\rangle_\phi = \frac{1}{4}||\{\phi_s, \phi_t\}||^2_\phi,$$

which implies that $K_{\phi_s, \phi_t} \leq 0$. Moreover, the covariant derivative $DR = 0$, which implies that $\mathcal{H}$ is a locally symmetric space.

Proof. We compute directly

$$D_{\phi_s} D_{\phi_t}\psi = \psi_{st} + \frac{\partial}{\partial s}\Big(-\frac{1}{2}\langle \nabla\phi_t, \nabla\psi\rangle\Big) - \frac{1}{2}\langle \nabla\phi_s, \nabla\psi_t\rangle + \frac{1}{4}\langle \nabla\phi_s, \nabla\langle \nabla\phi_t, \nabla\psi\rangle\rangle.$$

Since

$$\frac{\partial}{\partial s}\langle \nabla\phi_t, \nabla\psi\rangle = \langle \nabla\phi_{ts}, \nabla\psi\rangle + \langle \nabla\phi_t, \nabla\psi_s\rangle - i\partial\bar{\partial}\phi_s(\nabla\phi_t, J\nabla\psi),$$

and

$$\begin{aligned} 2i\partial\bar{\partial}\phi_s(\nabla\phi_t, J\nabla\psi) =& \nabla^2\phi_s(\nabla\phi_t, \nabla\psi) + \nabla^2\phi_s(J\nabla\phi_t, J\nabla\psi) \\ =& - \nabla^2\psi(\nabla\phi_s, \nabla\phi_t) + \langle \nabla\phi_t, \nabla\langle \nabla\phi_s, \nabla\psi\rangle\rangle - \omega(\nabla_{J\phi_t} J\nabla\phi_s, J\nabla\psi). \end{aligned}$$

Therefore

$$\begin{aligned} D_{\phi_s} D_{\phi_t}\psi =& \psi_{st} - \frac{1}{2}\langle \nabla\phi_{ts}, \nabla\psi\rangle - \frac{1}{2}\langle \nabla\phi_t, \nabla\psi_s\rangle - \frac{1}{2}\langle \nabla\phi_s, \nabla\psi_t\rangle - \frac{1}{4}\nabla^2\psi(\nabla\phi_s, \nabla\phi_t) \\ &+ \frac{1}{4}\langle \nabla\phi_t, \nabla\langle \nabla\phi_s, \nabla\psi\rangle\rangle + \frac{1}{4}\langle \nabla\phi_s, \nabla\langle \nabla\phi_t, \nabla\psi\rangle\rangle - \frac{1}{4}\omega(\nabla_{J\phi_t} J\nabla\phi_s, J\nabla\psi). \end{aligned}$$

Then

$$
\begin{aligned}
R(\phi_s, \phi_t)\psi =& (D_{\phi_s} D_{\phi_t} - D_{\phi_t} D_{\phi_s})\psi \\
=& \frac{1}{4}\omega(\nabla_{J\phi_s} J\nabla\phi_t, J\nabla\psi) - \frac{1}{4}\omega(\nabla_{J\phi_t} J\nabla\phi_s, J\nabla\psi) \\
=& \frac{1}{4}\omega([grad_\omega\phi_s, grad_\omega\phi_t], grad_\omega\psi) = \frac{1}{4}\{\{\phi_s, \phi_t\}, \psi\}.
\end{aligned}
$$

At last, $DR = 0$ follows from the expression of the curvature tensor and the fact that the covariant derivative is compatible with the Poisson bracket, that is,

$$
D_\eta\{\eta_1, \eta_2\} = \{D_\eta\eta_1, \eta_2\} + \{\eta_1, D_\eta\eta_2\}, \ \eta, \eta_1, \eta_2 \in T\mathcal{H}.
$$

□

The decomposition of $\mathcal{H}$

Each tangent space $T_\phi\mathcal{H}$ admits the following orthogonal decomposition

$$
T_\phi\mathcal{H} = \{\psi \in C^\infty(M, \mathbb{R}) \mid \int_M \psi\omega_\phi^n = 0\} \oplus \mathbb{R}.
$$

This gives a 1-form α on $\mathcal{H}$ given by

$$
\alpha_\phi(\psi) = \frac{1}{V}\int_M \psi\omega_\phi^n,
$$

which is closed, indeed,

$$
d\alpha_\phi(\psi_1, \psi_2) = d(\alpha_\phi(\psi_1))_\phi(\psi_2) - d(\alpha_\phi(\psi_2))_\phi(\psi_1) = \frac{n}{V}\int_M (\psi_1 i\partial\bar{\partial}\psi_2 \wedge \omega_\phi^{n-1} - \psi_2 i\partial\bar{\partial}\psi_1 \wedge \omega_\phi^{n-1}) = 0.
$$

Since $\mathcal{H}$ is convex, $d\alpha = 0$ implies that there exists a functional I, often called *Aubin-Yau functional*, such that $dI = \alpha$ and $I(0) = 0$.

Proposition 2.2.2 (Aubin [2]). *The functional I can be expressed explicitly as*

$$
I(\phi) = \frac{1}{n+1}\sum_{p=0}^{n}\frac{1}{V}\int_M \phi\omega_\phi^p \wedge \omega^{n-p}, \ \phi \in \mathcal{H}.
$$

Obviously, we have

$$I(\phi + c) = I(\phi) + c,\ \ c \in \mathbb{R},$$

$$\frac{d^2}{dt^2} I(\phi) = \frac{d}{dt} \langle\langle \dot{\phi}, 1 \rangle\rangle_\phi = \langle\langle \nabla_{\dot{\phi}} \dot{\phi}, 1 \rangle\rangle_\phi,$$

which implies that $I^{-1}(0)$ is a totally geodesic space of $\mathcal{H}$. Since any Kähler metric in $\mathcal{H}_0$ has a unique Kähler potential in $I^{-1}(0)$, the restriction of the Mabuchi metric to $I^{-1}(0)$ induces a Riemannian structure on $\mathcal{H}_0$. Thus there is a Riemannian decomposition $\mathcal{H} = \mathcal{H}_0 \times \mathbb{R}$.

The Dirichlet problem for the geodesic equation

Donaldson [32] outlined a strategy to relate the geometry of $\mathcal{H}$ to the existence problems of special Kähler metrics and proposed several conjectures, the most fundamental one among which is the so called geodesic conjecture: any two points in $\mathcal{H}$ can be joined by a smooth geodesic.

A curve ϕ in $\mathcal{H}$ is a geodesic if $D_{\dot{\phi}} \dot{\phi} = 0$, that is

$$\ddot{\phi} - \frac{1}{2} |\nabla \dot{\phi}|^2_{\omega_\phi} = 0. \tag{2.2.1}$$

Writing locally $u = g + \phi$, since g is independent of t, we can rewrite the equation (2.2.1) as

$$\ddot{u} - u^{j\bar{k}} \dot{u}_j \dot{u}_{\bar{k}} = 0.$$

Multiplying both sides by $\det(u_{i\bar{j}})$, which is non-vanishing, we arrive at a new equation

$$\begin{vmatrix} u_{1\bar{1}} & \dots & u_{1\bar{n}} & u_{1t} \\ \vdots & \ddots & \vdots & \vdots \\ u_{n\bar{1}} & \dots & u_{n\bar{n}} & u_{nt} \\ u_{t\bar{1}} & \dots & u_{t\bar{n}} & u_{tt} \end{vmatrix} = 0.$$

This suggests to complexify the variable t by adding an imaginary variable, i.e., we extend ϕ on $M \times [0, 1]$ to $M \times S$ by $\phi(\cdot, \zeta) := \phi(\cdot, Re\, \zeta)$, where $S := \{\zeta \in \mathbb{C} : 0 \le Re\, \zeta \le 1\}$, then ϕ is a geodesic if and only if ϕ defined on $M \times S$ satisfies the following homogeneous complex Monge-Ampère equation

$$(\omega + i\partial\bar{\partial}_{M \times S} \phi)^{n+1} = 0.$$

We thus have obtained the following characterization of geodesics in $\mathcal{H}$:

Proposition 2.2.3 (Semmes [78], Donaldson [32]). *For $\phi_0, \phi_1 \in \mathcal{H}$, the existence of a geodesic connecting these two points is equivalent to solving the following Dirichlet problem:*

$$\begin{cases} \phi \in C^\infty(M \times S, \mathbb{R}), \\ \omega + i\partial\bar{\partial}\phi(\cdot, \zeta) > 0, \ 0 \le Re\ \zeta \le 1, \\ (\omega + i\partial\bar{\partial}_{M\times S}\phi)^{n+1} = 0, \\ \phi(\cdot, \zeta) = \phi_j, \ Re\ \zeta = j, \ j = 0, 1. \end{cases} \tag{2.2.2}$$

The uniqueness of (2.2.2) is a direct consequence of a general comparison principle for the generalized solutions given by the Bedford-Taylor theory [4, 5], see also Błocki [8]. In one dimension, Monge-Ampère equation is just Laplacian equation, so the regularity of solutions is clear. However, in higher dimensions, one cannot expect C^∞-regularity of solutions of homogeneous Monge-Ampère equation. This can be seen from the following simple example.

Example 2.2.1 (Gamelin and Sibony [38]). Let $\mathbb{B} \subset \mathbb{C}^2$ be the open unit ball centered at the origin. For $(z, w) \in \mathbb{B}$, define

$$u(z, w) := \big(\max\{0, |z|^2 - \frac{1}{2}, |w|^2 - \frac{1}{2}\}\big)^2.$$

Then u is a plurisubharmonic function on $\mathbb{B}$. Observe that if $(z, w) \in \mathbb{B}$ then either $|z|^2 < 1/2$ or $|w|^2 < 1/2$, in each case u depends only on one variable hence it is maximal, which means that $(i\partial\bar{\partial}u)^2 = 0$ in pluripotential sense. Note that u is smooth on $\partial\mathbb{B}$, since for $(z, w) \in \partial\mathbb{B}$,

$$u(z, w) = (|z|^2 - \frac{1}{2})^2 = (|w|^2 - \frac{1}{2})^2.$$

But $u \notin C^2(\mathbb{B})$. Since when $|z|^2 \ge 1/2$, $u(z, w) = (|z|^2 - 1/2)^2$, and $u_z = 2(|z|^2 - 1/2)\bar{z}$, it is obvious that u_z is only Lipschitz near the line $|z|^2 = 1/2$.

In [34], Donaldson studied the case when S is replaced by a unit disk in $\mathbb{C}$ using the so called Monge-Ampère foliation [3]. He gave an example showing that there exist smooth boundary data for which there does not exist a smooth solution. Also using the foliation method, Lempert-Vivas [61] constructed smooth boundary data for which the solution of (2.2.2) fails to be smooth, so that disproved Donaldson's geodesic conjecture, see also later work by Darvas-Lempert [27].

However, Chen [23] showed that any $\phi_0, \phi_1 \in \mathcal{H}$ can be joined by a *weak* geodesic whose mixed complex derivatives are bounded. This is recently improved somehow by He [51] where he assumed the boundary condition not necessarily smooth. Here "weak" means that $\omega_\phi \geq 0$ instead of $\omega_\phi > 0$. His idea is considering approximations of geodesics, sometimes called ϵ-geodesics, i.e.

$$(\ddot{\phi} - \frac{1}{2}|\nabla\dot{\phi}|^2_{\omega_\phi})\omega_\phi^n = \epsilon\omega^n,$$

where $\epsilon > 0$ is a small constant. This perturbed equation is then non-degenerate. As shown by Chen [23], smooth ϵ-geodesics always exist, which is a key step to show that $\mathcal{H}$ is a metric space. He actually established the a priori weak C^2 estimates(that is, $\Delta\phi$, $\nabla\dot{\phi}$ and $\ddot{\phi}$ are bounded while it might not be fully $C^{1,1}$) of the solutions independent of $\inf \epsilon$.

As explained before, the existence problem of ϵ-geodesic is equivalent to solving the following modification of the Dirichlet problem (2.2.2),

$$\begin{cases} \phi \in C^\infty(M \times S, \mathbb{R}), \\ \omega + i\partial\bar{\partial}\phi(\cdot, \zeta) > 0, \ 0 \leq Re\ \zeta \leq 1, \\ (\omega + i\partial\bar{\partial}_{M\times S}\phi)^{n+1} = \epsilon(\omega + i\partial\bar{\partial}|\zeta|^2)^{n+1}, \\ \phi(\cdot, \zeta) = \phi_j, \ Re\ \zeta = j, \ j = 0, 1. \end{cases} \tag{2.2.3}$$

Although ω is degenerate on $M \times S$, we can write

$$\omega + i\partial\bar{\partial}\phi = (\omega + i\partial\bar{\partial}|\zeta|^2) + i\partial\bar{\partial}(\phi - |\zeta|^2),$$

so that $\widetilde{\omega} := \omega + i\partial\bar{\partial}|\zeta|^2$ is a Kähler form on $M \times S$ and consider the related equation with solution $\widetilde{\phi} := \phi - |\zeta|^2$. For convenience, we still denote $\widetilde{\omega}$ as ω, and $\widetilde{\phi}$ as ϕ when there is no confusion.

Now we can consider a more general Dirichlet problem. We assume that M is a compact complex manifold with smooth boundary with a Kähler form ω. Take $f \in C^\infty(M, \mathbb{R})$, $f > 0$, and

$\psi \in C^\infty(\partial M, \mathbb{R})$, we look for ϕ satisfying

$$\begin{cases} \phi \in C^\infty(M, \mathbb{R}), \\ \omega + i\partial\bar{\partial}\phi > 0, \\ (\omega + i\partial\bar{\partial}\phi)^n = f\omega^n, \\ \phi = \psi, \text{ on } \partial M. \end{cases} \tag{2.2.4}$$

Theorem 2.2.1. *If (M, ω) is a compact Kähler manifold with smooth nonempty boundary. Let $0 < f \in C^\infty(M, \mathbb{R})$ and $\psi \in C^\infty(\partial M, \mathbb{R})$. Assume that there is a smooth subsolution ρ, i.e., ρ satisfies $\omega + i\partial\bar{\partial}\rho > 0$ and*

$$(\omega + i\partial\bar{\partial}\rho)^n \geq f\omega^n \text{ in } M, \;\; \rho = \psi \text{ on } \partial M.$$

Then there exists a unique solution to the Dirichlet problem (2.2.4).

The proof of this theorem is reduced to establish the a priori estimates of the solutions up to the second order. It is a combination of the results proved in several papers [1, 87, 17, 39, 23, 11]. For bounded strictly pseudoconvex domains in $\mathbb{C}^n$, this was proved in [17], and in [39] without the assumption of strict pseudoconvexity. To our situation, most estimates from these papers carry on without much change except two exceptions: interior gradient estimate and interior C^2-estimate. As for the gradient estimate, one can either use the blowing-up analysis from [23](then one has to consider the C^2-estimate first), or we can apply a direct approach following Błocki [10], see also Hanani [46]. We will discuss the interior C^2-estimate in the next subsection.

2.2.2 Interior C^2-estimate

The interior C^2-estimate for the mixed complex derivatives was shown independently by Aubin [1] and Yau [87]. The following theorem will let us apply the real Evans-Krylov theory directly without reproving its complex version.

Theorem 2.2.2 ([11]). *If ϕ satisfies the equation (2.2.4), then*

$$|\nabla^2\phi| \le C, \tag{2.2.5}$$

where C is a constant depending only on n, upper bounds for $|R|$, $|\nabla R|$, $|\phi|$, $|\nabla\phi|$, $\Delta\phi$, $\sup_{\partial M}|\nabla^2\phi|$, $||f^{1/n}||_{C^{1,1}(M)}$, $|\nabla f^{1/2n}|$ and on a lower positive bound for f. If M has a non-negative bisectional curvature, then the estimate is independent of $\inf f$.

Proof. It suffices to estimate the eigenvalues of the mapping

$$TM \ni X \longmapsto \nabla_X\nabla\phi.$$

Since their sum is bounded from below (by $-n$), it is enough to get an upper bound of the maximal eigenvalue of this mapping. We define a function on M as follows

$$\alpha := \max_{X\in TM,\ |X|=1}\langle\nabla_X\nabla\phi, X\rangle + \frac{|\nabla\phi|^2}{2} - A\phi,$$

where A is a constant to be determined later. Clearly, to prove the estimate (2.2.5), it suffices to bound α from above.

We assume that α attains maximum at an interior point $x_0 \in M\setminus\partial M$, otherwise we are done.

Our calculations will always be carried out at the point x_0, unless otherwise indicated. Let $e_1,\dots,e_{2n} \in TM$ be an orthonormal local frame of vector fields near x_0 which is normal at x_0, such that $Je_i = e_{n+i}$ for $1 \le i \le n$. Set $\zeta_k := \frac{1}{\sqrt{2}}(e_k - iJe_k)$, then $\{\zeta_k\}$ is a unitary basis of $T_{\mathbb{C}}M$. The subscripts of a function h will always denote the covariant derivatives of h with respect to ω. For simplicity, set

$$h_i := h_{\zeta_i} = \nabla_{\zeta_i}h,\ h_{i\bar{j}} := h_{\zeta_i\bar{\zeta}_j} = \nabla^2h(\bar{\zeta}_j,\zeta_i),\ h_{i\bar{j}l} := h_{\zeta_i\bar{\zeta}_j\zeta_l} = \nabla^3h(\zeta_l,\bar{\zeta}_j,\zeta_i),\ \text{etc.}$$

Let us bear in mind that covariant differentiating the metric tensor always equals zero.

Without loss of generality, we may assume that at x_0,

$$\max_{X\in TM,\ |X|=1}\langle\nabla_X\nabla\phi, X\rangle = \phi_{e_1e_1},$$

and $\phi_{e_1e_1} > 0$, otherwise we are done. Note that $\phi_{e_1e_1}$ is a well-defined function in a small neighborhood of x_0, and if we define

$$\bar{\alpha} := \phi_{e_1e_1} + \frac{|\nabla\phi|^2}{2} - A\phi,$$

near x_0, then

$$\bar{\alpha} \le \alpha \le \alpha(x_0) = \bar{\alpha}(x_0),$$

which means that $\bar{\alpha}$ also has a maximum at x_0 in a small neighborhood, in particular, $\bar{\alpha}$ is smooth. And it remains to estimate $\bar{\alpha}(x_0)$ from above.

Following Section 3 in Guan [40], we can show that $\phi_{e_1e_j}(x_0) = 0$, for $j \ge 2$. So we can also adjust the local frame so that the matrix $(\phi_{j\bar{k}})$ is diagonal at x_0.

Set $u := g + \phi$, then the equation (2.2.4) is rephrased as

$$\det(u_{p\bar{q}}) = f \det(g_{p\bar{q}}).$$

We rewrite this equation as

$$F(\nabla^2 u) := \log\det(u_{p\bar{q}}) = \log\big(f\det(g_{p\bar{q}})\big). \tag{2.2.6}$$

Denote the linearized operator at u of F by

$$L(h) = u^{p\bar{q}} h_{p\bar{q}},$$

which is elliptic since u is strictly plurisubharmonic.

Taking covariant derivatives of (2.2.6) twice, we have

$$u^{p\bar{p}}\phi_{p\bar{p}X} = (\log f)_X, \;\; X \in T_{\mathbb{C}}M, \tag{2.2.7}$$

$$u^{p\bar{p}}\phi_{p\bar{p}e_1e_1} = u^{p\bar{p}}u^{q\bar{q}}|\phi_{p\bar{q}e_1}|^2 + (\log f)_{e_1e_1} \ge (\log f)_{e_1e_1}.$$

Here we use the facts that $(u_{p\bar{p}})_X = u_{p\bar{p}X}$, and $(u_{p\bar{p}})_{XX} = u_{p\bar{p}XX}$ at x_0, see the proof of Theorem 1.0.1 or Guan [40] for the details.

Using the commutation formular (2.1.4),

$$\begin{aligned}\phi_{e_1e_1p\bar{p}}-\phi_{p\bar{p}e_1e_1}=&\frac{1}{2}(\phi_{e_1e_1e_pe_p}+\phi_{e_1e_1e_{n+p}e_{n+p}}-\phi_{e_pe_pe_1e_1}-\phi_{e_{n+p}e_{n+p}e_1e_1})\\=&R_{e_1e_pe_pe_1}(\phi_{e_1e_1}-\phi_{e_pe_p})+R_{e_1e_{n+p}e_{n+p}e_1}(\phi_{e_1e_1}-\phi_{e_{n+p}e_{n+p}})+C(|\nabla R|,|\nabla\phi|)\\\geq&B(\zeta_1,\zeta_p)\phi_{e_1e_1}-C(|R|)\phi_{p\bar{p}}-C(|\nabla R|,|\nabla\phi|).\end{aligned}$$

So we have

$$\begin{aligned}L(\phi_{e_1e_1})=&u^{p\bar{q}}(\phi_{e_1e_1})_{p\bar{q}}=u^{p\bar{p}}\phi_{e_1e_1p\bar{p}}\\\geq&(\log f)_{e_1e_1}+\sum_p\frac{B(\zeta_1,\zeta_p)\phi_{e_1e_1}}{u_{p\bar{p}}}-C(|R|)\sum_p\frac{\phi_{p\bar{p}}}{u_{p\bar{p}}}-C(|\nabla R|,|\nabla\phi|)\sum_p\frac{1}{u_{p\bar{p}}}.\end{aligned}\tag{2.2.8}$$

Since

$$\frac{1}{n}(\log f)_{e_1e_1}=(\log f^{\frac{1}{n}})_{e_1e_1}=\frac{1}{f^{\frac{1}{n}}}\left((f^{\frac{1}{n}})_{e_1e_1}-4|(f^{\frac{1}{2n}})_{e_1}|^2\right)\geq-C(||f^{1/n}||_{C^{1,1}(M)},|\nabla f^{1/2n}|)\frac{1}{f^{\frac{1}{n}}},$$

and

$$\sum_p\frac{1}{u_{p\bar{p}}}\geq\frac{n}{(\prod_p u_{p\bar{p}})^{\frac{1}{n}}}=\frac{n}{f^{\frac{1}{n}}},$$

inserting these inequalities into (2.2.8), finally we have

$$L(\phi_{e_1e_1})\geq\sum_p\frac{B(\zeta_1,\zeta_p)\phi_{e_1e_1}}{u_{p\bar{p}}}-C_1\sum_p\frac{\phi_{p\bar{p}}}{u_{p\bar{p}}}-C_2\sum_p\frac{1}{u_{p\bar{p}}},$$

where $C_1=C(|R|),\ C_2=C(|\nabla R|,|\nabla\phi|,||f^{1/n}||_{C^{1,1}(M)},|\nabla f^{1/2n}|)$ are under control.

Next we estimate $L(\frac{|\nabla\phi|^2}{2})$. First we compute directly

$$\frac{|\nabla\phi|^2}{2}=g^{j\bar{k}}\phi_j\phi_{\bar{k}},$$

$$(\frac{|\nabla\phi|^2}{2})_p=g^{j\bar{k}}\phi_{jp}\phi_{\bar{k}}+g^{j\bar{k}}\phi_j\phi_{p\bar{k}},$$

$$\begin{aligned}(\frac{|\nabla\phi|^2}{2})_{p\bar{p}}=&g^{j\bar{k}}\phi_{jp\bar{p}}\phi_{\bar{k}}+g^{j\bar{k}}\phi_{jp}\phi_{\bar{k}\bar{p}}+g^{j\bar{k}}\phi_{j\bar{p}}\phi_{p\bar{k}}+g^{j\bar{k}}\phi_j\phi_{p\bar{k}\bar{p}},\\=&\sum_j|\phi_{jp}|^2+|\phi_{p\bar{p}}|^2+\sum_j\phi_{p\bar{p}\bar{j}}\phi_j+\sum_j\phi_{p\bar{p}j}\phi_{\bar{j}}+\sum_{j,q}R_{j\bar{p}p\bar{q}}\phi_q\phi_{\bar{j}}.\end{aligned}$$

Then

$$L(\frac{|\nabla\phi|^2}{2}) = \sum_{j,p} \frac{|\phi_{jp}|^2}{u_{p\bar{p}}} + \sum_p \frac{|\phi_{p\bar{p}}|^2}{u_{p\bar{p}}} + 2Re\sum_{j,p} \frac{\phi_{p\bar{p}\bar{j}}\phi_j}{u_{p\bar{p}}} + \sum_{j,q,p} \frac{R_{j\bar{p}p\bar{q}}\phi_q\phi_{\bar{j}}}{u_{p\bar{p}}}$$
$$\geq \sum_{j,p} \frac{|\phi_{jp}|^2}{u_{p\bar{p}}} + \sum_p \frac{|\phi_{p\bar{p}}|^2}{u_{p\bar{p}}} - C_3\sum_p \frac{1}{u_{p\bar{p}}},$$

where we use the equation (2.2.7) and $C_3 = C(|R|, |\nabla\phi|, ||f^{1/n}||_{C^{1,1}(M)})$ is under control.

Therefore, we obtain at x_0,

$$0 \geq L(\bar{\alpha}) \geq \sum_{j,p} \frac{|\phi_{jp}|^2}{u_{p\bar{p}}} + \phi_{e_1e_1}\sum_p \frac{B(\zeta_1,\zeta_p)}{u_{p\bar{p}}} + \sum_p \frac{|\phi_{p\bar{p}}|^2}{u_{p\bar{p}}} - C_1\sum_p \frac{\phi_{p\bar{p}}}{u_{p\bar{p}}} - C_4\sum_p \frac{1}{u_{p\bar{p}}} + A\sum_p \frac{1}{u_{p\bar{p}}} - An$$
$$\geq \sum_{j,p} \frac{|\phi_{jp}|^2}{u_{p\bar{p}}} + \phi_{e_1e_1}\sum_p \frac{B(\zeta_1,\zeta_p)}{u_{p\bar{p}}} - C_5\sum_p \frac{1}{u_{p\bar{p}}} + A\sum_p \frac{1}{u_{p\bar{p}}} - An,$$

where $C_5 = C(|R|, |\nabla R|, |\nabla\phi|, \sup\Delta\phi, ||f^{1/n}||_{C^{1,1}(M)}, |\nabla f^{1/2n}|)$. Since

$$\sum_{j,p} \frac{|\phi_{jp}|^2}{u_{p\bar{p}}} \geq \frac{\phi_{e_1e_1}^2}{C_6} - C_7\sum_p \frac{1}{u_{p\bar{p}}},$$

where we use that $u_{p\bar{p}}$ is bounded from above. It follows that at x_0,

$$0 \geq \frac{\phi_{e_1e_1}^2}{C_6} + \phi_{e_1e_1}\sum_p \frac{B(\zeta_1,\zeta_p)}{u_{p\bar{p}}} - C_8\sum_p \frac{1}{u_{p\bar{p}}} + A\sum_p \frac{1}{u_{p\bar{p}}} - An.$$

Now we are ready to get the estimate (2.2.5). When there is a positive lower bound for f, then $\sum_p \frac{1}{u_{p\bar{p}}}$ is bounded from above (note that we only have a lower bound for $\sum_p \frac{1}{u_{p\bar{p}}}$ depending on $\sup f$), we just take $A = 0$, then we get an upper bound for $\phi_{e_1e_1}$.

In the special case when the bisectional curvature of M is non-negative, we drop the terms involved with bisectional curvature. We take $A = C_8$, then we get the estimate which is independent of the lower bound for f.

$\square$

At last, we turn to the geodesic problem in the space of Kähler metrics. As explained before, the geodesic equation is covered by the more general equation (2.2.4). Moreover, in the geodesic equation case, ∂M is *flat*, that is, near every boundary point, after a holomorphic change of coordinates, the boundary is of the form $\{Re\ z_n = 0\}$. This condition will ensure that the

C^2 boundary estimate is independent of the lower bound for f, see Theorem 3.2' in [11]. Combined with Theorem 2.2.5, this leads to the $C^{1,1}$ regularity of geodesics in Kähler case when the bisectional curvature is non-negative.

2.3 The space of volume forms

Let (M, g) be a connected compact Riemannian manifold of dimension m with the Riemannian metric g. By a *volume form* on M we mean a differential form of degree m, positive everywhere. In local coordinates $(x_1, x_2, \ldots, x_m)$, such a volume form σ takes the form:

$$\sigma = f(x)dx, \text{ where } dx = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m, \ f \text{ is positive and smooth.}$$

Also we can write the Riemannian metric g as follows:

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

There is a canonical volume form dg coming from this metric g:

$$dg := \sqrt{det(g_{ij})} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m,$$

where $det(g_{ij})$ refers to the determinant of the $m \times m$ matrix (g_{ij}).

Now let us define $\mathcal{V}_0$ as the space of volume forms, which consists of the volume forms on (M, g) with fixed total volume $Vol := \int_M dg$. Since M is closed, this space can be expressed as:

$$\mathcal{V}_0 := \{(1 + \Delta\phi)dg \mid \phi \in C^\infty(M, \mathbb{R}), 1 + \Delta\phi > 0\},$$

where Δ is the Laplacian operator with respect to g. By this expression, every element in $\mathcal{V}_0$ is determined by some "potential" function up to a constant. So we can define another space $\mathcal{V}$ of such "potential" functions for the volume forms:

$$\mathcal{V} := \{\phi \in C^\infty(M, \mathbb{R}) \mid 1 + \Delta\phi > 0\}.$$

As we just explained, we can get

$$\mathcal{V}_0 = \mathcal{V}/\mathbb{R},$$

where $\mathbb{R}$ acts on $\mathcal{V}$ by addition.

We review in this section some well known facts about the Riemannian structure of this space $\mathcal{V}$, as introduced by Donaldson in [33].

2.3.1 The Riemannian structure

We can observe from the definition of $\mathcal{V}$ that $\mathcal{V}$ is an open subset of $C^\infty(M, \mathbb{R})$ and thus $\mathcal{V}$ has a structure of an infinite dimensional differential manifold. Moreover, fix a point ϕ_0 in $\mathcal{V}$, we can identify $C^\infty(M, \mathbb{R})$ with the tangent space $T_{\phi_0}\mathcal{V}$ of $\mathcal{V}$ at ϕ_0 via the following isomorphism:

$$\begin{aligned} C^\infty(M, \mathbb{R}) \ &\cong T_{\phi_0}\mathcal{V} \\ \psi &\leftrightarrow \frac{d}{ds}|_{s=0}(\phi_0 + s\psi), \end{aligned}$$

where $s \in [-\epsilon, \epsilon] \mapsto \phi_0 + s\psi \in \mathcal{V}$ is a smooth path in $\mathcal{V}$ with a sufficiently small $\epsilon > 0$.

Definition 2.3.1. In [33], Donaldson defined a Weil-Peterson type metric on $T_{\phi_0}\mathcal{V}$ as follows:

$$\langle\langle \psi, \eta \rangle\rangle_{\phi_0} := \frac{1}{Vol}\int_M \psi\eta(1 + \Delta\phi_0)dg, \ \psi, \eta \in T_{\phi_0}\mathcal{V}.$$

In terms of this, the norm of a tangent vector $\psi \in T_{\phi_0}\mathcal{V}$ is given by:

$$\|\psi\|^2_{\phi_0} := \langle\langle \psi, \psi \rangle\rangle_{\phi_0} = \frac{1}{Vol}\int_M \psi^2(1 + \Delta\phi_0)dg.$$

Given a smooth path $\phi(t) : [0, 1] \to \mathcal{V}$, which is simply a smooth function on $M \times [0, 1]$, the "energy" of this path is:

$$E(\phi(t)) := \frac{1}{2}\int_0^1 \|\dot{\phi}\|^2_{\phi(t)}dt = \frac{1}{2Vol}\int_0^1\int_M |\dot{\phi}|^2(1 + \Delta\phi)dgdt, \tag{2.3.1}$$

where we denote $\dot{\phi} = \frac{d\phi}{dt}$, $\ddot{\phi} = \frac{d^2\phi}{dt^2}$ and so on.

Geodesics between two points ϕ_0, ϕ_1 in $\mathcal{V}$ are defined as the extremals of the energy functional:

$$\phi \mapsto E(\phi),$$

where $\phi = \phi(t)$ is a path joining ϕ_0 and ϕ_1. The geodesic equation is thus obtained by computing the Euler-Lagrange equation associated to the energy functional (2.3.1) with fixed end points.

Lemma 2.3.1. *The geodesic equation is:*

$$\ddot{\phi}(1 + \Delta\phi) - |\nabla\dot{\phi}|^2 = 0, \tag{2.3.2}$$

where Δ, ∇ are with respect to the metric g.

Proof. Let $\psi(t)$ be a small variation with end points vanished, then

$$\begin{aligned}
&\frac{d}{ds}|_{s=0}E(\phi(t) + s\psi(t)) \\
&= \frac{1}{2Vol}\int_0^1\int_M (2\dot{\phi}\dot{\psi}(1 + \Delta\phi) + |\dot{\phi}|^2\Delta\psi)dgdt \\
&= \frac{1}{2Vol}\int_0^1\int_M (\psi(-2\frac{d}{dt}(\dot{\phi}(1 + \Delta\phi)) + \Delta(|\dot{\phi}|^2)\psi)dgdt \\
&= \frac{1}{2Vol}\int_0^1\int_M \psi(-2\ddot{\phi}(1 + \Delta\phi) - 2\dot{\phi}\Delta\dot{\phi} + (2\dot{\phi}\Delta\dot{\phi} + 2|\nabla\dot{\phi}|^2))dgdt \\
&= \frac{1}{2Vol}\int_0^1\int_M \psi(-2\ddot{\phi}(1 + \Delta\phi) + 2|\nabla\dot{\phi}|^2)dgdt
\end{aligned}$$

where we use integration by parts in the second equality and $|\nabla\dot{\phi}|$ means the norm of gradient of function $\dot{\phi}$ with respect to the metric g. Since the variation can be taken arbitrary, the geodesic equation is:

$$\ddot{\phi}(1 + \Delta\phi) - |\nabla\dot{\phi}|^2 = 0.$$

□

Remark 2.3.2. When M is 2-dimensional and orientable, M is a Riemann surface and thus Kähler. Then the geodesic equation (2.3.2) coincides with a Homogeneous Complex Monge-Ampère equation. Indeed, we can extend ϕ on $M \times [0, 1]$ to $M \times \{z \in \mathbb{C} : 0 \leq Re\ z \leq 1\}$ by $\phi(\cdot, z) := \phi(\cdot, Re\ z)$ and the Kähler form ω with respect to g can also be generalized since it only depends in the Manifold direction. Simple computation shows that the equation (2.3.2) is equivalent to:

$$(\omega + i\partial\bar{\partial}\phi)^2 = 0.$$

For a vector field $\psi(t)$ along the path $\phi(t)$, which is just an element of $C^\infty(M \times [0,1], \mathbb{R})$, now we want to define the "Levi-Civita" covariant derivative $D_{\dot{\phi}}\psi$ of ψ along ϕ such that:

$$\frac{d}{dt}\langle\langle \psi, \eta \rangle\rangle_\phi = \langle\langle D_{\dot{\phi}}\psi, \eta \rangle\rangle_\phi + \langle\langle \psi, D_{\dot{\phi}}\eta \rangle\rangle_\phi, \tag{2.3.3}$$

where η is another vector field along ϕ.

We compute

$$\begin{aligned}
\frac{d}{dt}\langle\langle \psi, \eta \rangle\rangle_\phi &= \frac{d}{dt}\left(\frac{1}{Vol}\int_M \psi\eta(1+\Delta\phi)dg\right) \\
&= \frac{1}{Vol}\int_M ((\dot{\psi}\eta + \psi\dot{\eta})(1+\Delta\phi) + \psi\eta\Delta\dot{\phi})dg \\
&= \frac{1}{Vol}\int_M ((\dot{\psi}\eta + \psi\dot{\eta})(1+\Delta\phi) - \langle\nabla\psi, \nabla\dot{\phi}\rangle\eta - \langle\nabla\eta, \nabla\dot{\phi}\rangle\psi)dg \\
&= \frac{1}{Vol}\int_M ((\dot{\psi} - \frac{1}{1+\Delta\phi}\langle\nabla\psi, \nabla\dot{\phi}\rangle)\eta + (\dot{\eta} - \frac{1}{1+\Delta\phi}\langle\nabla\eta, \nabla\dot{\phi}\rangle)\psi)(1+\Delta\phi)dg.
\end{aligned}$$

The above computation shows that the right way to define a connection is given by:

$$D_{\dot{\phi}}\psi := \dot{\psi} - \frac{1}{1+\Delta\phi}\langle\nabla\psi, \nabla\dot{\phi}\rangle, \tag{2.3.4}$$

where $\langle\cdot,\cdot\rangle$ means the inner product on TM with respect to g.

In the local "coordinate chart", which represent $\mathcal{V}$ as an open subset of $C^\infty(M, \mathbb{R})$, the "Christoffel symbol" at $\phi \in \mathcal{V}$ is just:

$$\Gamma : C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$$

$$\Gamma(\psi, \eta) = \frac{-1}{1+\Delta\phi}\langle\nabla\psi, \nabla\eta\rangle$$

This shows that the symbol is symmetric in ψ, η which means that this connection is torsion free, i.e., for a smooth two-parameters family $\psi(s,t)$ in $\mathcal{V}$, we have:

$$D_{\frac{\partial\psi}{\partial s}}\frac{\partial\psi}{\partial t} = D_{\frac{\partial\psi}{\partial t}}\frac{\partial\psi}{\partial s}.$$

2.3.2 Counterpart of the Aubin-Yau functional

Since $\mathcal{V}_0 = \mathcal{V}/\mathbb{R}$, we expect to induce a Riemannian structure on $\mathcal{V}_0$ from the structure on $\mathcal{V}$, so that $\mathcal{V}$ can be written as a Riemannian decomposition. It turns out that we can do it by constructing a functional which is the counterpart of the Aubin-Yau functional, first introduced by Aubin [2] in Kähler case. Using this functional, we then make a proper normalization on $\mathcal{V}$.

Specifically, we look for a functional I which is characterized by the following properties:

$$I(0) = 0,\ dI_\phi(\psi) = \frac{1}{Vol}\int_M \psi(1+\Delta\phi)dg,\ \phi \in \mathcal{V},\ \psi \in C^\infty(M,\mathbb{R}). \tag{2.3.5}$$

We can regard dI as a 1-form defined on the tangent space $T\mathcal{V}$. The point is that since the space $\mathcal{V}$ is convex, such a functional always exists if this 1-form is closed. Indeed:

$$\begin{aligned}(d(dI_\phi))_\phi(\psi_1,\psi_2) &= d(dI_\phi(\psi_1))_\phi(\psi_2) - d((dI_\phi(\psi_2))_\phi(\psi_1) \\ &= \frac{1}{Vol}\int_X (\psi_1\Delta\psi_2 - \psi_2\Delta\psi_1)dg = 0\end{aligned}$$

It follows that there is a functional I satisfying (2.3.5). For any smooth path $\phi(t)$ in $\mathcal{V}$ joining 0 with ϕ ,we can write I formally:

$$I(\phi) = \int_0^1 \frac{1}{Vol}\int_M \dot{\phi}(1+\Delta\phi)dgdt.$$

The fact dI is closed implies that $I(\phi)$ is independent of the choice of the path $\phi(t)$. Taking $\phi(t) = t\phi$, we obtain the formula explicitly:

$$I(\phi) = \frac{1}{Vol}\int_M (\phi + \frac{1}{2}\phi\Delta\phi)dg, \tag{2.3.6}$$

Remark 2.3.3. This functional coincides with the one in the Kähler case when the manifold is of complex dimension 1.

By direct computation we have:

$$I(\phi + c) = I(\phi) + c$$

for any real constant c.

For any smooth path $\phi(t)$ in $\mathcal{V}$ joining 0 with ϕ, by (2.3.3) since the covariant derivative is compatible with the metric on $T\mathcal{V}$, we have

$$\frac{d^2}{dt^2}I(\phi) = \frac{d}{dt}\langle\langle\dot{\phi}, 1\rangle\rangle_{\phi(t)} = \langle\langle\nabla_{\dot{\phi}}\dot{\phi}, 1\rangle\rangle_{\phi(t)}$$

which yields that I is affine along geodesics. And ϕ is a geodesic implies that $\phi - I(\phi)$ is also a geodesic. We call a "volume potential" ϕ *normalized* if $I(\phi) = 0$. Then any volume form in $\mathcal{V}_0$ has a unique normalized "volume potential" in $\mathcal{V}$, and the restriction of the metric on $\mathcal{V}$ to $I^{-1}(0)$ endows $\mathcal{V}_0$ with a Riemannian structure, which is independent of the choice of dg as long as the total volume of M is fixed. And clearly the tangent space of $\mathcal{V}_0$ at a point $(1 + \Delta\phi)dg$ written as dg_ϕ can be realized as:

$$T_{dg_\phi}\mathcal{V}_0 = \{\psi \in C^\infty(M, \mathbb{R}) \mid \int_M \psi dg_\phi = 0\}.$$

To summarize, we now state as follows:

Proposition 2.3.1. *[43] There is a functional I given by*

$$I(\phi) = \frac{1}{Vol}\int_M (\phi + \frac{1}{2}\phi\Delta\phi)dg, \;\; \phi \in \mathcal{V}$$

such that $I^{-1}(0)$ is a totally geodesic subspace of $\mathcal{V}$. Moreover, the bijective mapping

$$I^{-1}(0) \ni \phi \;\mapsto (1 + \Delta\phi)dg \in \mathcal{V}_0$$

induces a Riemannian structure on $\mathcal{V}_0$ with tangent space of $\mathcal{V}_0$ at a point $d\nu$ realized as

$$T_{d\nu}\mathcal{V}_0 = \{\psi \in C^\infty(M, \mathbb{R}) \mid \int_M \psi d\nu = 0\}.$$

Thus there is a Riemannian decomposition $\mathcal{V} = \mathcal{V}_0 \times \mathbb{R}$.

2.3.3 $\mathcal{V}$ as a space of nonpositive sectional curvature

Consider a two-parameters family $\phi(s, t) \in \mathcal{V}$ and a vector field $\psi(s, t) \in C^\infty(M, \mathbb{R})$ along ϕ. For simplicity, we will write ϕ_s for $\frac{\partial\phi}{\partial s}$, ϕ_{st} for $\frac{\partial^2\phi}{\partial s\partial t}$, and so on.

Since the covariant derivative with respect to the metric on $\mathcal{V}$ is torsion free, the curvature tensor is just defined by:

$$R(\phi_s, \phi_t)\psi := (D_{\phi_s} D_{\phi_t} - D_{\phi_t} D_{\phi_s})\psi.$$

The sectional curvature corresponding to a pair of tangent vectors ϕ_s, ϕ_t is given by:

$$K_{\phi_s, \phi_t} := -\frac{\langle\langle R(\phi_s, \phi_t)\phi_s, \phi_t\rangle\rangle_\phi}{\langle\langle \phi_s, \phi_s\rangle\rangle_\phi \cdot \langle\langle \phi_t, \phi_t\rangle\rangle_\phi - \langle\langle \phi_s, \phi_t\rangle\rangle_\phi^2}.$$

We will compute the curvature tensor directly, which is a little different from the way in [33]. It is convenient to compute first $D_{\phi_s} D_{\phi_t}\psi$. By definition,

$$D_{\phi_t}\psi = \psi_t - \frac{1}{1+\Delta\phi}\langle\nabla\phi_t, \nabla\psi\rangle,$$

so we have

$$\begin{aligned}
D_{\phi_s} D_{\phi_t}\psi = \frac{\partial}{\partial s}(\psi_t - \frac{1}{1+\Delta\phi}\langle\nabla\phi_t, \nabla\psi\rangle) &- \frac{1}{1+\Delta\phi}\langle\nabla\phi_s, \nabla(\psi_t - \frac{1}{1+\Delta\phi}\langle\nabla\phi_t, \nabla\psi\rangle)\rangle \\
= \psi_{st} + \frac{\Delta\phi_s}{(1+\Delta\phi)^2}\langle\nabla\phi_t, \nabla\psi\rangle& \\
- \frac{1}{1+\Delta\phi}\langle\nabla\phi_{ts}, \nabla\psi\rangle - \frac{1}{1+\Delta\phi}\langle\nabla\phi_t, \nabla\psi_s\rangle& \\
- \frac{1}{1+\Delta\phi}\langle\nabla\phi_s, \nabla\psi_t\rangle + \frac{1}{1+\Delta\phi}\langle\nabla\phi_s, \nabla(\frac{1}{1+\Delta\phi}\langle\nabla\phi_t, \nabla\psi\rangle)\rangle.&
\end{aligned}$$

Similarly, we can compute $D_{\phi_t} D_{\phi_s}\psi$, which leads to the following explicit expression:

$$\begin{aligned}
R(\phi_s, \phi_t)\psi = D_{\phi_s} D_{\phi_t}\psi - D_{\phi_t} D_{\phi_s}\psi& \\
= \frac{\Delta\phi_s}{(1+\Delta\phi)^2}\langle\nabla\phi_t, \nabla\psi\rangle - \frac{\Delta\phi_t}{(1+\Delta\phi)^2}\langle\nabla\phi_s, \nabla\psi\rangle& \\
+ \frac{1}{1+\Delta\phi}\langle\nabla\phi_s, \nabla(\frac{1}{1+\Delta\phi}\langle\nabla\phi_t, \nabla\psi\rangle)\rangle& \\
- \frac{1}{1+\Delta\phi}\langle\nabla\phi_t, \nabla(\frac{1}{1+\Delta\phi}\langle\nabla\phi_s, \nabla\psi\rangle)\rangle.&
\end{aligned}$$

Remark 2.3.4. According to Donaldson [Nam], the curvature tensor has the following form:

$$R(\phi_s, \phi_t)\psi = \langle\frac{1}{1+\Delta\phi} curl(\frac{1}{1+\Delta\phi}\nabla\phi_s \times \nabla\phi_t), \nabla\psi\rangle,$$

where $curl(v \times w) := [v,w] + (div\ v)w - (div\ w)v$ for any pair of vectors v, w on M. Indeed, computing a bit more and arranging terms, we have

$$
\begin{aligned}
R(\phi_s,\phi_t)\psi &= \frac{1}{1+\Delta\phi}[\frac{1}{1+\Delta\phi}\Delta\phi_s\langle\nabla\phi_t,\nabla\psi\rangle - \frac{1}{1+\Delta\phi}\Delta\phi_t\langle\nabla\phi_s,\nabla\psi\rangle \\
&+ \langle\nabla\phi_s,\nabla(\frac{1}{1+\Delta\phi})\rangle\langle\nabla\phi_t,\nabla\psi\rangle + \frac{1}{1+\Delta\phi}\langle\nabla\phi_s,\nabla\langle\nabla\phi_t,\nabla\psi\rangle\rangle \\
&- \langle\nabla\phi_t,\nabla(\frac{1}{1+\Delta\phi})\rangle\langle\nabla\phi_s,\nabla\psi\rangle - \frac{1}{1+\Delta\phi}\langle\nabla\phi_t,\nabla\langle\nabla\phi_s,\nabla\psi\rangle\rangle]. \\
&= \frac{1}{1+\Delta\phi}[\frac{1}{1+\Delta\phi}(\Delta\phi_s\langle\nabla\phi_t,\nabla\psi\rangle - \Delta\phi_t\langle\nabla\phi_s,\nabla\psi\rangle \\
&+ \langle\nabla\phi_s,\nabla\langle\nabla\phi_t,\nabla\psi\rangle\rangle - \langle\nabla\phi_t,\nabla\langle\nabla\phi_s,\nabla\psi\rangle\rangle) \\
&+ \langle\nabla\phi_s,\nabla(\frac{1}{1+\Delta\phi})\rangle\langle\nabla\phi_t,\nabla\psi\rangle - \langle\nabla\phi_t,\nabla(\frac{1}{1+\Delta\phi})\rangle\langle\nabla\phi_s,\nabla\psi\rangle] \\
&= \frac{1}{1+\Delta\phi}[\frac{1}{1+\Delta\phi}\langle curl(\nabla\phi_s\times\nabla\phi_t),\nabla\psi\rangle \\
&+ \langle\nabla\phi_s,\nabla(\frac{1}{1+\Delta\phi})\rangle\langle\nabla\phi_t,\nabla\psi\rangle - \langle\nabla\phi_t,\nabla(\frac{1}{1+\Delta\phi})\rangle\langle\nabla\phi_s,\nabla\psi\rangle] \\
&= \langle\frac{1}{1+\Delta\phi}curl(\frac{1}{1+\Delta\phi}\nabla\phi_s\times\nabla\phi_t),\nabla\psi\rangle,
\end{aligned}
$$

note that in the last equality we use the formula $curl(fv\times w) = fcurl(v\times w) + \langle v,\nabla f\rangle w - \langle w,\nabla f\rangle v$ for any smooth function f on M .

Then by definition of the curvature tensor, we compute

$$
\begin{aligned}
&\langle\langle R(\phi_s,\phi_t)\phi_s,\phi_t\rangle\rangle_\phi \\
&= \frac{1}{Vol}\int_M \frac{\Delta\phi_s\cdot\phi_t}{1+\Delta\phi}\langle\nabla\phi_t,\nabla\phi_s\rangle dg \\
&\ - \frac{1}{Vol}\int_M \frac{\Delta\phi_t\cdot\phi_t}{1+\Delta\phi}\langle\nabla\phi_s,\nabla\phi_s\rangle dg \\
&\ + \frac{1}{Vol}\int_M \langle\nabla\phi_s,\nabla(\frac{1}{1+\Delta\phi}\langle\nabla\phi_t,\nabla\phi_s\rangle)\rangle\phi_t dg \\
&\ - \frac{1}{Vol}\int_M \frac{1}{1+\Delta\phi}\langle\nabla\phi_t,\nabla(\frac{1}{1+\Delta\phi}\langle\nabla\phi_s,\nabla\phi_s\rangle)\rangle\phi_t dg.
\end{aligned}
\tag{2.3.7}
$$

By integration by parts, we have

$$
\begin{aligned}
&\int_M \frac{\Delta\phi_s\cdot\phi_t}{1+\Delta\phi}\langle\nabla\phi_t,\nabla\phi_s\rangle dg \\
= -&\int_M \frac{\langle\nabla\phi_s,\nabla\phi_t\rangle^2}{1+\Delta\phi}dg - \int_M \langle\nabla\phi_s,\nabla(\frac{1}{1+\Delta\phi}\langle\nabla\phi_t,\nabla\phi_s\rangle)\rangle\phi_t dg,
\end{aligned}
$$

similarly,

$$\int_M \frac{\Delta\phi_t \cdot \phi_t}{1+\Delta\phi}\langle\nabla\phi_s, \nabla\phi_s\rangle dg$$
$$= -\int_M \frac{|\nabla\phi_s|^2 \cdot |\nabla\phi_t|^2}{1+\Delta\phi} dg - \int_M \frac{1}{1+\Delta\phi}\langle\nabla\phi_t, \nabla(\frac{1}{1+\Delta\phi}\langle\nabla\phi_s, \nabla\phi_s\rangle)\rangle\phi_t dg.$$

Inserting the above two formulas into (2.3.7), we have:

$$\langle\langle R(\phi_s, \phi_t)\phi_s, \phi_t\rangle\rangle_\phi = \frac{1}{Vol}\int_M \frac{|\nabla\phi_s|^2 \cdot |\nabla\phi_t|^2 - \langle\nabla\phi_s, \nabla\phi_t\rangle^2}{1+\Delta\phi} dg \geq 0,$$

from which the following consequence immediately follows:

Theorem 2.3.5 (Donaldson [33])**.** *The infinite dimensional Riemannian manifold $\mathcal{V}$ has nonpositive sectional curvature.*

Remark 2.3.6. In fact, there is another way to parameterize the space $\mathcal{V}$ as follows:

$$\mathcal{C} := \{u \in C^\infty(M, \mathbb{R}) \mid \int_M e^u dg = \int_M dg\}.$$

Following an idea by Calabi [19], we can define the so called Calabi metric at any $u \in \mathcal{C}$ by:

$$< v, w >_u := \int_M vwe^u dg, \; v, w \in T_u\mathcal{C},$$

where $T_u\mathcal{C} = \{v \in C^\infty(M, \mathbb{R}) \mid \int_M ve^u dg = 0\}$.

And it is proven by Calamai [21] that the space $\mathcal{C}$ endowed with the Calabi metric admits the Levi-Civita covariant derivative and has sectional curvature exactly equal to $\frac{1}{4Vol}$. Moreover, $\mathcal{V}$ is a metric space and the Dirichlet problem for the geodesic equation has explicit unique smooth solution.

2.3.4 The distance on $\mathcal{V}$

With the metric defined on $T\mathcal{V}$ already, the length of a smooth curve $[0,1] \ni t \to \phi(t) \in \mathcal{V}$ is given by:

$$l(\phi) := \int_0^1 \|\dot{\phi}\|_{\phi(t)} dt = \int_0^1 \sqrt{\frac{1}{Vol}\int_M \dot{\phi}^2(1+\Delta\phi) dg} dt.$$

The distance $d(\phi_0, \phi_1)$ between two points $\phi_0, \phi_1 \in \mathcal{V}$ is then defined to be the infimum of the length of smooth curves joining ϕ_0 and ϕ_1, that is:

$$d(\phi_0, \phi_1) := \inf\{\ l(\phi) \mid \phi \text{ is a smooth curve joining } \phi_0 \text{ and } \phi_1\}.$$

It is easy to verify that d defines a semi-distance, i.e. nonnegative, symmetric and satisfying the triangle inequality. But to show that $(\mathcal{V}, d)$ is a metric space, we have to prove that $d(\phi_0, \phi_1) = 0$ if and only if $\phi_0 = \phi_1$, which is non trivial. We will discuss this subject later after we study the next section.

2.4 The Dirichlet problem for the geodesic equation

Similar to the problem in the Kähler setting, Donaldson asked whether there is a smooth geodesic joining any two distinct points in $\mathcal{V}$. Precisely, we are interested in the boundary value problem for the geodesic equation: given u_0, u_1 in $\mathcal{V}$, we want to seek a solution u to the following equation

$$\begin{cases} u \in C^\infty(M \times [0,1], \mathbb{R}), \\ 1 + \Delta u > 0, \\ u_{tt}(1 + \Delta u) - |\nabla u_t|^2 = 0, \\ u(\cdot, 0) = u_0,\ u(\cdot, 1) = u_1 \end{cases} \tag{$*$}$$

where we denote $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$ and so on. From the PDE point of view, this equation is relevant to the nonlinear operator

$$u \to u_{tt}(1 + \Delta u) - |\nabla u_t|^2.$$

According to Remark 2.3.2, when M is of complex dimension 1, the equation $(*)$ can be reformulated as a homogeneous complex Monge-Ampère equation. In general, for homogeneous complex Monge-Ampère equation, the solution, if exists, is at most $C^{1,1}$ that we can expect. This can be seen from an example provided by Gamelin and Sibony [38]. In [23], Chen studied the

Dirichlet problem for the geodesic equation in the space of Kähler metrics, he showed the existence and uniqueness of **almost** $C^{1,1}$ solution, i.e., the mixed complex derivatives are bounded. We mention that these results affirmatively answered the question of uniqueness of constant scalar curvature metrics if the first Chern class is either strictly negative or 0. His idea is to consider a perturbed equation of the original one with right hand side replaced by a small positive ϵ. This perturbed equation is then non degenerate and elliptic. By the continuity method, he shows that there is a unique smooth solution for any $\epsilon > 0$ and these smooth solutions approximate generalized geodesics with lower regularity called **weak geodesics** as ϵ tends to 0.

It is expected that the techniques used by Chen can be extended to the equation $(*)$. To carry this out, we should derive some concavity property of the operator first.

One should be clear that we mainly follow Chen and He [24] in this section.

2.4.1 Concavity of the nonlinear operator

In the following lemma, we show some basic property of a function inspired by the structure of the geodesic equation.

Lemma 2.4.1. *Let $n \geq 3$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the function*

$$f(x) = \log(x_1 x_2 - \sum_{i=3}^{n} {x_i}^2),$$

is concave on the domain $D := \{x \in \mathbb{R}^n \mid x_1 > 0,\ x_2 > 0,\ x_1 x_2 - \sum_{i=3}^{n} {x_i}^2 > 0\}$.

Proof. Observe that f is induced from a quadratic form of Minkowski space. For simplicity, we define the Minkowski norm for any $x \in D$ as

$$\|x\|_{\mathcal{M}} := \sqrt{x_1 x_2 - \sum_{i=3}^{n} {x_i}^2}.$$

Then to prove the lemma, we just need to prove

$$\left\| \frac{x+y}{2} \right\|_{\mathcal{M}}^2 \geq \|x\|_{\mathcal{M}} \cdot \|y\|_{\mathcal{M}}, \quad x, y \in D.$$

It suffices to show the *Reversed triangle inequality*:

$$\|x+y\|_{\mathcal{M}} \geq \|x\|_{\mathcal{M}} + \|y\|_{\mathcal{M}}. \tag{2.4.1}$$

Though it is well known, for readers' convenience, we prove (2.4.1) through elementary calculus. By definition,

$$\begin{aligned}
\|x+y\|_{\mathcal{M}}^2 &= (x_1+y_1)(x_2+y_2) - \sum_{i=3}^{n}(x_i+y_i)^2 \\
&= x_1x_2 + y_1x_2 + x_1y_2 + y_1y_2 - \sum_{i=3}^{n}(x_i^2 + 2x_iy_i + y_i^2) \\
&= (\|x\|_{\mathcal{M}} + \|y\|_{\mathcal{M}})^2 + x_1y_2 + y_1x_2 - \sum_{i=3}^{n} 2x_iy_i - 2\|x\|_{\mathcal{M}} \cdot \|y\|_{\mathcal{M}}
\end{aligned} \tag{2.4.2}$$

Since

$$\begin{aligned}
x_1y_2 + y_1x_2 &= \frac{y_2}{x_2}x_1x_2 + \frac{x_2}{y_2}y_1y_2 \\
&= \frac{y_2}{x_2}(x_1x_2 - \sum_{i=3}^{n}{x_i}^2) + \frac{x_2}{y_2}(y_1y_2 - \sum_{i=3}^{n}{y_i}^2) \\
&\quad + \frac{y_2}{x_2}\sum_{i=3}^{n}{x_i}^2 + \frac{x_2}{y_2}\sum_{i=3}^{n}{y_i}^2 \\
&\geq 2\|x\|_{\mathcal{M}} \cdot \|y\|_{\mathcal{M}} + \sum_{i=3}^{n} 2x_iy_i,
\end{aligned}$$

inserted into (2.4.2), it is done. □

As a direct consequence, we get that the operator

$$u \to \log(u_{tt}(1+\Delta u) - |\nabla u_t|^2) \tag{2.4.3}$$

is concave on $\{u \in C^2(M \times [0,1], \mathbb{R}) \mid u_{tt} > 0,\ u_{tt}(1+\Delta u) - |\nabla u_t|^2 > 0\}$.

Following Donaldson [33], we define a "norm" still denoted as $\|\cdot\|_{\mathcal{M}}$ on symmetric matrices as follows

$$\|A\|_{\mathcal{M}} = A_{00}\sum_{i=1}^{n} A_{ii} - \sum_{i=1}^{n} A_{i0}^2,$$

where $A = (A_{ij})_{0 \leq i,j \leq n}$ is symmetric.

The following lemma will be useful when we derive a priori estimates.

Lemma 2.4.2. *1. Let A be a symmetric matrix, if $A > 0$, then $\|A\|_{\mathcal{M}} > 0$; if $A \geq 0$, then $\|A\|_{\mathcal{M}} \geq 0$.*

2. If A, B are two symmetric matrices with $\|A\|_{\mathcal{M}} = \|B\|_{\mathcal{M}} > 0$ and the entries A_{00}, B_{00} positive, then for any $s \in [0,1]$,

$$\|sA + (1-s)B\|_{\mathcal{M}} \geq \|A\|_{\mathcal{M}}, \ \|A - B\|_{\mathcal{M}} \leq 0.$$

Moreover, the equality holds if and only if $A_{ii} = B_{ii}, A_{i0} = B_{i0}$, $\forall i$.

Proof. The first item is obvious. For the second item, just apply Lemma 2.4.1 and the Reversed triangle inequality. $\square$

2.4.2 A priori estimates

As pointed out by Donaldson, to approach the equation $(*)$, he introduced a perturbed equation as follows

$$\begin{cases} u \in C^{\infty}(M \times [0,1], \mathbb{R}), \\ u_{tt}(1+\Delta u) - |\nabla u_t|^2 = \epsilon, \\ u(\cdot, 0) = u_0 \in \mathcal{V}, \ u(\cdot, 1) = u_1 \in \mathcal{V} \end{cases} \tag{$**$}$$

for any $\epsilon > 0$. He also explained that the equation $(**)$ is actually related to some free boundary problems.

We rewrite the equation as

$$\log(u_{tt}(1+\Delta u) - |\nabla u_t|^2) = \log \epsilon. \tag{2.4.4}$$

The linearization of the left-hand side of (2.6.2) is given by

$$\begin{aligned} L(h) :&= \frac{d}{d\gamma}\bigg|_{\gamma=0} \log\left((u_{tt} + \gamma h_{tt})(1 + \Delta u + \gamma \Delta h) - |\nabla u_t + \gamma \nabla h_t|^2\right) \\ &= \frac{1}{\epsilon}\left(u_{tt}\Delta h + (1+\Delta u)h_{tt} - 2\langle \nabla u_t, \nabla h_t\rangle\right). \end{aligned}$$

Proposition 2.4.1. *If u satisfies $(**)$, then:*

$$u_{tt} + 1 + \Delta u > 0,$$

It follows that $u_{tt} > 0$, $1 + \Delta u > 0$. In particular, L is elliptic.

Proof. At the boundary, it is obvious true by the assumption. If at some point $(x_0, t_0) \in M \times (0, 1)$,

$$u_{tt} + 1 + \Delta u = 0.$$

Then at the point (x_0, t_0),

$$u_{tt}(1 + \Delta u) - |\nabla u_t|^2 = -(1 + \Delta u)^2 - |\nabla u_t|^2 < \epsilon,$$

contradiction. $\square$

C^0-**estimate**

Lemma 2.4.3 ([24])**.** *If u satisfies $(**)$, denote $u_a(x, t) := at(1 - t) + (1 - t)u_0 + tu_1$, where a is a fixed constant, then for some a big enough depending on (M, g) and u_0, u_1, we have*

$$u_{-a} \leq u \leq (1 - t)u_0 + tu_1.$$

Proof. First note that by Proposition (2.4.1), $u_{tt} > 0$, which means that u is convex with respect to t. It follows that

$$u \leq (1 - t)u_0 + tu_1.$$

For the other side, we assume the contrary, then $u < u_{-a}$ at some point, since u and u_{-a} coincide on the boundary, then $u - u_{-a}$ obtains its minimum in the interior, say at p. Then $D^2u \geq D^2u_{-a}$ at p. Since

$$\|D^2u\|_{\mathcal{M}} = \epsilon,$$

$$\|D^2u_{-a}\|_{\mathcal{M}} = 2a(1 + (1 - t)\Delta u_0 + t\Delta u_1) - |\nabla u_1 - \nabla u_0|^2,$$

we can always choose a big enough depending only on (M, g), u_0, u_1 such that

$$\|D^2 u\|_{\mathcal{M}} < \|D^2 u_{-a}\|_{\mathcal{M}}. \tag{2.4.5}$$

Next we construct two $(n+2)\times(n+2)$ symmetric matrices A, B, such that the $(n+1)\times(n+1)$ block of A is $D^2 u_{-a}$ at p, $A_{i(n+2)} = 0$ for $1 \le i \le n+1$, $A_{(n+2)(n+2)} = 1$, and the $(n+1)\times(n+1)$ block of B is $D^2 u$ at p, $B_{i(n+2)} = 0$ for $1 \le i \le n+1$, $B_{(n+2)(n+2)} = \lambda$, where λ is a constant satisfying

$$\|B\|_{\mathcal{M}} = u_{tt}(\lambda + \Delta u) - |\nabla u_t|^2 = \|D^2 u_{-a}\|_{\mathcal{M}} = \|A\|_{\mathcal{M}}.$$

From (2.4.5) we know $\lambda > 1$. It follows from Lemma 2.4.2(2) that $\|B - A\|_{\mathcal{M}} < 0$, but $B - A$ is semi-positive since $D^2 u \ge D^2 u_{-a}$ at p, so $\|B - A\|_{\mathcal{M}} \ge 0$ by Lemma 2.4.2(1). Contradiction. $\square$

C^1-estimate

By Proposition 2.4.1, the linearized operator L is elliptic, so we can apply maximum principle that the estimates largely rely on.

Note that for a fixed point p on $M \times [0, 1]$ with local coordinates $(x_1, \ldots, x_n, t)$, we can always choose normal coordinates in the manifold direction such that the metric tensor g satisfies $g_{ij} = \delta_{ij}$, $\partial_k g_{ij} = 0$ at p.

Lemma 2.4.4 ([24]). *If u satisfies (**), then there is a uniform constant C, depending on (M, g), and u_0, u_1, such that:*

$$|u_t| + |\nabla u| \le C. \tag{2.4.6}$$

Proof. By Proposition (2.4.1), $u_{tt} > 0$, which implies that u_t is increasing. Then

$$|u_t(x, t)| \le max\{\ |u_t(x, 0)|,\ |u_t(x, 1)|\ \}.$$

By Lemma (2.4.3), we have

$$|u_t(x, 0)| = \left|\lim_{t\to 0} \frac{u(x, t) - u(x, 0)}{t}\right| \le \lim_{t\to 0} \left|\frac{u(x, t) - u(x, 0)}{t}\right| \le a + |u_0 - u_1|.$$

Similarly, we can bound $|u_t(x,1)|$. Therefore, $u_t(x,t)$ is uniformly bounded.

To bound ∇u, we take

$$h := \frac{1}{2}(|\nabla u|^2 + bu^2),$$

where b is a constant to be determined later.

Since h is uniformly bounded on the boundary, we assume that u attains maximum at $(p, t_0) \in M \times (0,1)$.

Take covariant derivatives under normal coordinates, we have:

$$\begin{aligned}
h_t &= \sum_k u_{tk}u_k + bu_t u, \quad h_k = \sum_i u_{ik}u_i + bu_k u, \\
h_{tt} &= \sum_k u_{ttk}u_k + \sum_k u_{tk}^2 + b(u_t^2 + u_{tt}u), \\
h_{tk} &= \sum_i u_{tik}u_i + \sum_i u_{ik}u_{ti} + b(u_{tk}u + u_t u_k), \\
\Delta h &= \sum_{i,k} u_{ikk}u_i + \sum_{i,k} u_{ik}^2 + b(u\Delta u + \sum_k u_k^2), \\
&= \sum_i (\Delta u)_i u_i + Ric(\nabla u, \nabla u) + \sum_{i,k} u_{ik}^2 + b(u\Delta u + \sum_k u_k^2).
\end{aligned}$$

It follows that

$$\begin{aligned}
\epsilon L(h) &= u_{tt}\Delta h + (1+\Delta u)h_{tt} - 2\sum_k u_{tk}h_{tk} \\
&= u_{tt}[\sum_i (\Delta u)_i u_i + Ric(\nabla u, \nabla u) + \sum_{i,k} u_{ik}^2 + b(u\Delta u + \sum_k u_k^2)] \\
&+ (1+\Delta u)[\sum_k u_{ttk}u_k + \sum_k u_{tk}^2 + b(u_t^2 + u_{tt}u)] \\
&- 2\sum_k u_{tk}[\sum_i u_{tik}u_i + \sum_i u_{ik}u_{ti} + b(u_{tk}u + u_t u_k)].
\end{aligned}$$

Take derivative of $\|u\|_{\mathcal{M}} = \epsilon$, we have

$$u_{tt}(\Delta u)_i + u_{tti}(1+\Delta u) - 2\sum_k u_{tki}u_{tk} = 0.$$

We can arrange that

$$\begin{aligned}\epsilon L(h) &= u_{tt} Ric(\nabla u, \nabla u) + u_{tt} \sum_{i,k} u_{ik}^2 + (1+\Delta u) \sum_k u_{tk}^2 - 2\sum_{i,k} u_{tk} u_{ti} u_{ik} \\ &+ bu[u_{tt}\Delta u + u_{tt}(1+\Delta u) - 2\sum_k u_{tk}^2] \\ &+ b[u_{tt}|\nabla u|^2 + u_t^2(1+\Delta u) - 2\sum_k u_t u_k u_{tk}]\end{aligned}$$

Since at (p, t_0), $h_t = 0$, we have

$$\sum_k u_{tk} u_k = -bu_t u.$$

Furthermore,

$$\begin{aligned}u_{tt}\sum_{i,k} u_{ik}^2 + (1+\Delta u)\sum_k u_{tk}^2 &\geq 2\sqrt{u_{tt}\sum_{i,k} u_{ik}^2 (1+\Delta u)\sum_k u_{tk}^2} \\ &\geq 2\sqrt{\sum_{i,k} u_{ik}^2 \sum_k u_{tk}^2 (\epsilon + \sum_i u_{ti}^2)} \\ &\geq 2\sum_{i,k} u_{tk} u_{ti} u_{ik}.\end{aligned}$$

At (p, t_0) it follows that

$$\begin{aligned}\epsilon L(h) &\geq -R_0 u_{tt}|\nabla u|^2 + bu(2\epsilon - u_{tt}) + b[u_{tt}|\nabla u|^2 + u_t^2(1+\Delta u) + 2buu_t{}^2] \\ &= u_{tt}(\frac{b}{2}|\nabla u|^2 - bu - R_0|\nabla u|^2) + \frac{b}{2}u_{tt}|\nabla u|^2 + bu_t{}^2(1+\Delta u) + 2\epsilon bu + 2b^2 u_t{}^2 u \\ &\geq u_{tt}(\frac{b}{2}|\nabla u|^2 - bu - R_0|\nabla u|^2) + b|u_t| \cdot |\nabla u|\sqrt{2\epsilon} + 2\epsilon bu + 2b^2 u_t{}^2 u,\end{aligned}$$

where $R_0 = 1 + max|R_{ij}|$.

Notice that if u satisfies $(**)$, then $\tilde{u} = u + C_1 t + C_0$ also solves the equation but with boundary condition

$$\tilde{u}(\cdot, 0) = u_0 + C_0, \quad \tilde{u}(\cdot, 1) = u_1 + C_1 + C_0,$$

where C_0, C_1 are any real constants. Since $|u|, |u_t|$ is uniformly bounded, we can choose C_0, C_1 big enough, such that $\tilde{u} \geq 0$, $\tilde{u}_t \geq 1$ everywhere. Thus we can assume that $u \geq 0$, $u_t \geq 1$. Then we have

$$\epsilon L(h) \geq u_{tt}(\frac{b}{2}|\nabla u|^2 - bu - R_0|\nabla u|^2) + \sqrt{2\epsilon}b|\nabla u|.$$

Since at the point (p, t_0), $\epsilon L(h) \le 0$, and now we choose $b = 4R_0$, then we have

$$0 \ge u_{tt}(R_0|\nabla u|^2 - bu),$$

which follows that

$$|\nabla u|^2(p, t_0) \le 4u.$$

$\square$

C^2 interior estimate

Lemma 2.4.5 ([24])**.** *If u satisfies (∗∗), then there is a uniform constant C, depending on (M, g), and u_0, u_1, such that:*

$$0 < u_{tt} + 1 + \Delta u \le C(1 + \max_{\partial(M \times [0,1])} |u_{tt}|). \tag{2.4.7}$$

Proof. By Proposition (2.4.1), it is clear that

$$u_{tt} + 1 + \Delta u > 0.$$

Take

$$F := \frac{bt^2}{2} - bu,$$

$$h := u_{tt} + 1 + \Delta u,$$

$$\tilde{h} := e^F h,$$

where b is a constant to be determined later. Without loss of generality, assume $\tilde{h}$ attains maximum at $(p, t_0) \in M \times (0, 1)$.

Take covariant derivatives under normal coordinates, we have:

$$\tilde{h}_t = e^F(F_t h + h_t), \quad \tilde{h}_k = e^F(F_k h + h_k),$$

$$\tilde{h}_{tt} = e^F(F_{tt} h + 2F_t h_t + h{F_t}^2 + h_{tt}),$$

$$\tilde{h}_{tk} = e^F(F_{tk} h + F_t h_k + F_k h_t + F_k F_t h + h_{tk}),$$

$$\tilde{h}_{kk} = e^F(F_{kk} h + 2F_k h_k + h{F_k}^2 + h_{kk}),$$

$$\Delta\tilde{h} = e^F(\Delta F h + 2\sum_k F_k h_k + h\sum_k {F_k}^2 + \Delta h).$$

Since at (p, t_0), $\tilde{h}_t = \tilde{h}_k = 0$, that is

$$F_t h = -h_t, \quad F_k h = -h_k.$$

Then at (p, t_0), we have

$$\tilde{h}_{tt} = e^F(F_{tt} h + h_{tt} - hF_t^2),$$

$$\tilde{h}_{tk} = e^F(F_{tk} h + h_{tk} - hF_t F_k),$$

$$\Delta\tilde{h} = e^F(\Delta F h + \Delta h - h\sum_k F_k^2).$$

At (p, t_0),

$$\begin{aligned}
\epsilon L(\tilde{h}) &= u_{tt}\Delta\tilde{h} + (1+\Delta u)\tilde{h}_{tt} - 2\sum_k u_{tk}\tilde{h}_{tk}, \\
&= e^F[u_{tt}(\Delta F h + \Delta h - h\sum_k F_k^2) + (1+\Delta u)(F_{tt} h + h_{tt} - hF_t^2) \\
&\quad - 2\sum_k u_{tk}(F_{tk} h + h_{tk} - hF_t F_k)], \\
&= e^F[h(u_{tt}\Delta F + (1+\Delta u)F_{tt} - 2\sum_k u_{tk}F_{tk}) \\
&\quad + u_{tt}\Delta h + (1+\Delta u)h_{tt} - 2\sum_k u_{tk}h_{tk} \\
&\quad - h(u_{tt}\sum_k F_k^2 + (1+\Delta u)F_t^2 - 2\sum_k u_{tk}F_t F_k)] \\
&= e^F\left(\epsilon h L(F) + \epsilon L(h) - h(u_{tt}\sum_k F_k^2 + (1+\Delta u)F_t^2 - 2\sum_k u_{tk}F_t F_k)\right).
\end{aligned}$$

Since F_t, F_k only involve first order derivative, by C^1 estimates, we know they are uniformly bounded, so

$$u_{tt}\sum_k F_k^2 + (1+\Delta u)F_t^2 - 2\sum_k u_{tk}F_tF_k \le C_1 h.$$

It follows that

$$\epsilon L(\tilde{h}) \ge e^F(\epsilon h L(F) + \epsilon L(h) - C_1 h^2).$$

Also we calculate

$$\begin{aligned}\epsilon L(F) &= u_{tt}(-b\Delta u) + (1+\Delta u)(b - bu_{tt}) + 2b\sum_k u_{tk}^2\\ &= b(1+\Delta u + u_{tt} - 2\epsilon)\\ &= b(h-2\epsilon),\end{aligned}$$

where we use the equation in the second equality.

To calculate $L(h)$, let us take derivatives of h first:

$$h_t = u_{ttt} + \Delta u_t, \quad h_{tt} = u_{tttt} + \Delta u_{tt},$$

$$h_k = u_{ttk} + (\Delta u)_k, \quad \Delta h = \Delta u_{tt} + \Delta^2 u, \quad h_{tk} = u_{tttk} + (\Delta u_t)_k,$$

We calculate

$$\begin{aligned}\epsilon L(h) &= u_{tt}(\Delta u_{tt} + \Delta^2 u) + (1+\Delta u)(u_{tttt} + \Delta u_{tt})\\ &\quad - 2\sum_k u_{tk}(u_{tttk} + (\Delta u_t)_k).\end{aligned} \tag{2.4.8}$$

We take first order derivative of the equation $(**)$,

$$u_{tt}\Delta u_t + u_{ttt}(1+\Delta u) - 2\sum_k u_{ttk}u_{tk} = 0, \tag{2.4.9}$$

$$u_{tt}(\Delta u)_i + u_{tti}(1+\Delta u) - 2\sum_k u_{tki}u_{tk} = 0. \tag{2.4.10}$$

Continue taking second order derivative we have

$$u_{tt}\Delta u_{tt} + 2u_{ttt}\Delta u_t + u_{tttt}(1+\Delta u) - 2\sum_k u_{ttk}^2 - 2\sum_k u_{tttk}u_{tk} = 0,$$

$$u_{tt}\Delta^2 u + 2\sum_i u_{tti}(\Delta u)_i + \Delta u_{tt}(1+\Delta u) - 2\sum_{k,i} u_{tki}^2 - 2\sum_k u_{tk}\Delta u_{tk} = 0.$$

Combine the above two equations into (2.4.8), we have

$$\begin{aligned}
\epsilon L(h) &= 2\sum_k u_{ttk}^2 + 2\sum_k u_{tttk}u_{tk} + 2\sum_{k,i} u_{tki}^2 + 2\sum_k u_{tk}\Delta u_{tk} \\
&\quad - 2u_{ttt}\Delta u_t - 2\sum_i u_{tti}(\Delta u)_i - 2\sum_k u_{tk}(u_{tttk} + (\Delta u_t)_k) \\
&= 2\sum_k u_{ttk}^2 + 2\sum_{k,i} u_{tki}^2 + 2\sum_{k,i} R_{ki}u_{tk}u_{ti} - 2u_{ttt}\Delta u_t - 2\sum_i u_{tti}(\Delta u)_i,
\end{aligned}$$

where we apply commutation formula in the second equality.

Note that

$$u_{tt}(\sum_k u_{ttk}^2 - u_{ttt}\Delta u_t) = u_{tt}\sum_k u_{ttk}^2 + (1+\Delta u)u_{ttt}^2 - 2\sum_k u_{tk}u_{ttk}u_{ttt} \geq 0, \tag{2.4.11}$$

where we used the equation (2.4.9).

Similarly

$$u_{tt}(\sum_{k,i} u_{tki}^2 - \sum_i u_{tti}(\Delta u)_i) = u_{tt}\sum_{k,i} u_{tki}^2 + (1+\Delta u)\sum_k u_{ttk}^2 - 2\sum_{k,i} u_{tk}u_{tki}u_{tti} \geq 0, \tag{2.4.12}$$

where we used the equation (2.4.10).

It follows from (2.4.11), (2.4.12) that

$$\epsilon L(h) \geq 2\sum_{k,i} R_{ki}u_{tk}u_{ti} \geq -C_2 h^2,$$

where C_2 depends on the Ricci curvature.

In sum, at (p, t_0), we have

$$\epsilon L(\tilde{h}) \geq \left((b - C_1 - C_2)h^2 - 2\epsilon b h\right).$$

Now take $b = C_1 + C_2 + 1$, and apply maximum principle at (p, t_0), we are done. $\square$

C^2 boundary estimate

Lemma 2.4.6 ([24])**.** *If u satisfies (**), then there is a uniform constant C, depending on (M, g), and u_0, u_1, such that*

$$\max_{\partial(M\times[0,1])} |\nabla u_t| \leq C(1 + \max_{M\times[0,1]} |\nabla u|).$$

Proof. We only consider $|\nabla u_t|$ on $M \times \{0\}$, since the case on $M \times \{1\}$ is similar.

For any point $(p, 0)$ on $M \times \{0\}$, we choose local coordinates around p such that

$$p = (0, \dots, 0,\ 0), \quad g_{ij}(0) = \delta_{ij},\ \partial g_{ij}(0) = 0.$$

Then for any $x \in B_0(\rho)$, which is a small coordinate ball with radius ρ centered at 0, we have

$$g_{ij}(x) = (1 + o(|x|))\,\delta_{ij}, \quad \partial g_{ij}(x) = o(|x|).$$

Now for any $k = 1, \dots, n$, we consider $u_k - u_{0,k}$, which is a local defined function on $B_0(\rho) \times [0, \tau]$ with τ small enough.

Take

$$h := A(u - u_0 + At - At^2) + B\sum_i x_i^2 + u_k - u_{0,k},$$

By C^1 estimates, it is easy to see that $h \geq 0$ on $\partial(B_0(\rho) \times [0, \tau])$, where we choose $A \gg B$ big enough.

We caculate

$$\begin{aligned}
\epsilon L(u - u_0 + At - At^2) &= u_{tt}(\Delta u - \Delta u_0) + (1 + \Delta u)u_{tt} - 2\sum_i u_{ti}^2 - 2A(1 + \Delta u) \\
&= 2\epsilon - u_{tt} - u_{tt}\Delta u_0 - 2A(1 + \Delta u) \\
&\leq -u_{tt}(1 + \Delta u_0),
\end{aligned}$$

since we can choose τ small enough such that on $M \times [0, \tau]$,

$$A(1 + \Delta u) > \epsilon.$$

Here we should mention that τ might depend on ϵ, but A is uniformly bounded.

And

$$\begin{aligned}
\epsilon L(u_k - u_{0,k}) &= u_{tt}(\Delta u_k - \Delta(u_{0,k})) + (1+\Delta u)u_{ttk} - 2\sum_i u_{ti}u_{tki} \\
&= u_{tt}(\Delta u)_k + u_{tt}\sum_i R_{ki}u_i + (1+\Delta u)u_{ttk} - 2\sum_i u_{ti}u_{tki} - u_{tt}\Delta(u_{0,k}) \\
&= u_{tt}\left(\sum_i R_{ki}u_i - \Delta(u_{0,k})\right) \\
&\leq Cu_{tt},
\end{aligned}$$

where we use (2.4.10) in the third equality and C depends on C^1 estimates and boundary value.

Moreover,

$$\epsilon L(\sum_i x_i^2) = u_{tt}\Delta(\sum_i x_i^2) < (2n+1)u_{tt},$$

So in $B_0(\rho)\times[0,\tau]$, we have

$$\epsilon L(h) < u_{tt}\left(-A(1+\Delta u_0) + (2n+1)B + C\right) < 0.$$

By maximum principle, $h \geq 0$ in $B_0(\rho)\times[0,\tau]$. Since $h(0)=0$,

$$\frac{\partial h}{\partial t}|_{t=0} \geq 0,$$

which means that

$$u_{tk} \geq -Au_t - A^2.$$

If we take

$$\tilde{h} := A(u - u_0 + At - At^2) + B\sum_i x_i^2 + u_{0,k} - u_k,$$

similarly we get $u_{tk} \leq Au_t + A^2$. □

Observe from the equation $u_{tt} = \frac{\epsilon + |\nabla u_t|^2}{1+\Delta u}$ that u_{tt} is also uniformly bounded on the boundary.

Summing up, we have

Proposition 2.4.2 ([24])**.** *If u satisfies (**), then there is a uniform bound on $|u|_{C^0}$, $|u|_{C^1}$, Δu, u_{tt} and $|\nabla u_t|$, independent of ϵ.*

This Proposition implies that the equation $(**)$ is uniformly elliptic. Moreover, if we rewrite the equation as

$$\log\left(u_{tt}(1+\Delta u) - |\nabla u_t|^2\right) = \log \epsilon,$$

then from Lemma 2.4.1, we know this new equation is also concave. Thus, the Hölder estimate of $|\nabla^2 u|$ which may depend on ϵ just follows from the Evans-Krylov theory. Then we can get estimates for all higher order derivatives of u using the boot-strapping argument.

2.4.3 Weak geodesics

Solving the perturbed equation

We will use the method of continuity to solve the equation $(**)$. We consider the following continuity family for $s \in [0,1]$,

$$\|D^2 u\|_{\mathcal{M}} = (1-s)\|D^2 u_{-a}\|_{\mathcal{M}} + s\epsilon, \tag{2.4.13}$$

with the boundary condition

$$u(\cdot,0,s) = u_0,\ u(\cdot,1,s) = u_1,$$

where u_{-a} is defined as in Lemma 2.4.3. Note that we can choose a big enough, so that $\|D^2 u_{-a}\|_{\mathcal{M}}$ is positive. Now consider the set

$$S := \{s \in [0,1]:\ \text{the equation (2.4.13) has a unique smooth solution.}\}.$$

Obviously $0 \in S$. We explain that S is both open and closed. Firstly, since the linearized operator L is uniformly elliptic, S is open by the inverse function theorem in the Banach space; secondly, by the a priori estimates derived in the previous section, S is also closed. Therefore, $S = [0,1]$. In particular, $1 \in S$, which means that

Theorem 2.4.1. *For any two points $u_0, u_1 \in \mathcal{V}$ and any $\epsilon > 0$, there exists a unique smooth solution to the equation $(**)$, $u(t) : [0,1] \to \mathcal{V}$ connecting u_0 and u_1.*

Existence and uniqueness of the weak solution

Similar as in the Kähler case, we can introduce the **weak C^2 geodesics**.

Here the weak C^2 means that only Δu, u_{tt}, $\nabla u_t \in L^\infty(M)$, while other second order derivatives might not be bounded. We call a segment $U(t)$ connecting two points in $\mathcal{V}$ is a weak C^2 geodesic if it satisfies the geodesic equation $(*)$ in the weak sense.

Using Proposition 2.4.2 and Theorem 2.4.1, by an approximation process, we know that for any two points in $\mathcal{V}$, there exists a weak C^2 geodesic segment connecting these two points.

To show the weak solution of the geodesic equation $(*)$ is unique with the fixed boundary data, we prove the following lemma.

Lemma 2.4.7. *Suppose U and V are the weak solutions of the geodesic equation $(*)$ with prescribed boundary data (u_0, u_1) and (v_0, v_1) respectively. Then*

$$\max_{M\times[0,1]} |U - V| \leq \max_{\partial(M\times[0,1])} (|u_0 - v_0|, |u_1 - v_1|).$$

Proof. For any $\epsilon > 0$, let u and v be the unique smooth solutions of the perturbed equation $(**)$ with the boundary data (u_0, u_1) and (v_0, v_1) respectively.

It suffices to show that

$$\max_{M\times[0,1]} |u - v| \leq \max_{\partial(M\times[0,1])} (|u_0 - v_0|, |u_1 - v_1|).$$

Suppose on the contrary, that $\max(u-v) > \max_{\partial(M\times[0,1])}(|u_0-v_0|, |u_1-v_1|)$, then $u-v-at(1-t)$ attains its maximum in the interior, say at p, when a is a positive constant small enough. So at the point p, we have $D^2u \leq D^2(v + at(1-t)) \leq D^2v$, in particular,

$$u_{tt}(p) \leq v_{tt}(p) - 2a < v_{tt}(p). \tag{2.4.14}$$

Since $\|D^2u\|_{\mathcal{M}} = \|D^2v\|_{\mathcal{M}} = \epsilon$, using Lemma 2.4.2(2) and (2.4.14), we have

$$\|D^2v - D^2u\|_{\mathcal{M}} < 0 \text{ at the point } p.$$

This is a contradiction since at this point $D^2u \leq D^2v$ implies that $\|D^2v - D^2u\|_{\mathcal{M}} \geq 0$ by lemma 2.4.2(1). Switch u and v, then the lemma follows.

□

As a consequence, we have the following

Theorem 2.4.2 ([24])**.** *There exists a unique weak solution to the geodesic equation (∗). In other words, for any two points u_0, u_1 in $\mathcal{V}$, there exist a weak C^2 geodesic segment $U(t)$ connecting these two points and a uniform constant C such that*

$$0 \leq 1 + \Delta U \leq C, \quad |U_{tt}| + |\nabla U_t| \leq C.$$

2.5 $\mathcal{V}$ as a metric space

As explained in Section 2.3.4, the problem to show that d defines a distance on $\mathcal{V}$ is whether $d(u_0, u_1) > 0$ for any two distinct points $u_0, u_1 \in \mathcal{V}$. We will prove the following quantitative version of this statement:

Theorem 2.5.1 ([24])**.** *For any $u_0, u_1 \in \mathcal{V}$, we have*

$$d(u_0, u_1)$$
$$\geq \sqrt{\frac{1}{Vol}\max\left(\int_{\{u_0>u_1\}}(u_0-u_1)^2(1+\Delta u_0)dg, \int_{\{u_1>u_0\}}(u_1-u_0)^2(1+\Delta u_1)dg\right)}.$$

We will prove this theorem following two steps as in [11]: establishing an estimate for the geodesic distance; showing that d is equal to the geodesic distance.

For simplicity, we call a segment $u(t) : [0,1] \to \mathcal{V}$ connecting u_0 and u_1 an ϵ-geodesic if u satisfies the perturbed equation (∗∗). As in [24], we have the following estimate for the ϵ-geodesic distance.

Lemma 2.5.1. *Let u be an ϵ-geodesic connecting $u_0, u_1 \in \mathcal{V}$, and denote the energy element as*

$$e(t) := \frac{1}{2}\|\dot{u}\|^2_{u(t)} = \frac{1}{2Vol}\int_M u_t^2\,(1+\Delta u)\,dg.$$

Then we have

$$e(t) \geq \mathcal{E}(u_0, u_1) - \epsilon|u_t|_{C^0},$$

where

$$\mathcal{E}(u_0, u_1) := \frac{1}{2Vol}\max\{\int_{\{u_0>u_1\}}(u_0-u_1)^2(1+\Delta u_0)dg, \int_{\{u_1>u_0\}}(u_1-u_0)^2(1+\Delta u_1)dg\}.$$

In particular,

$$l(u)^2 \geq 2\mathcal{E}(u_0, u_1) - 2\epsilon|u_t|_{C^0}.$$

Proof. Since

$$\begin{aligned}
\frac{d}{dt}e(t) &= \frac{1}{2Vol}\int_M (2u_t u_{tt}(1+\Delta u) + u_t^2\Delta u_t)dg \\
&= \frac{1}{Vol}\int_M u_t(u_{tt}(1+\Delta u) - |\nabla u_t|^2)dg \\
&= \frac{\epsilon}{Vol}\int_M u_t dg,
\end{aligned}$$

we have $|\frac{d}{dt}e(t)| \leq \epsilon|u_t|_{C^0}$, which implies

$$e(t) \geq \max\{e(0), e(1)\} - \epsilon|u_t|_{C^0}.$$

Since $u_{tt} > 0$, it follows that

$$u_t(\cdot, 0) \leq u(\cdot, 1) - u(\cdot, 0) \leq u_t(\cdot, 1).$$

Therefore,

$$e(0) \geq \frac{1}{2Vol}\int_{\{u_0>u_1\}}(u_0-u_1)^2(1+\Delta u_0)dg,$$

similarly,

$$e(1) \geq \frac{1}{2Vol}\int_{\{u_1>u_0\}}(u_1-u_0)^2(1+\Delta u_1)dg.$$

So the desired estimate follows. $\square$

We will also need the following lemma.

Lemma 2.5.2 ([24]). *Suppose that $u_i(s) : [0,1] \to \mathcal{V}(i = 1,2)$ are two smooth curves in $\mathcal{V}$. For any $\epsilon > 0$ and fixed s, set $u(t,s)$ as the unique ϵ-geodesic connecting $u_1(s)$ and $u_2(s)$. Then $u(t,s) \in C^\infty([0,1] \times [0,1], \mathcal{V})$ and there exists a uniform constant $C = C(M, u_1, u_2)$ such that*

$$|u| + |\frac{\partial u}{\partial t}| + |\frac{\partial u}{\partial s}| \le C;\ 0 < \frac{\partial^2 u}{\partial t^2} \le C;\ \frac{\partial^2 u}{\partial s^2} \le C.$$

Proof. According to Section 2.4.2, the estimates are clear except

$$|\frac{\partial u}{\partial s}| \le C,\ \frac{\partial^2 u}{\partial s^2} \le C. \tag{2.5.1}$$

Since u satisfies the equation $\log(u_{tt}(1 + \Delta u) - |\nabla u_t|^2) = \log \epsilon$, taking derivative with respect to s, we have

$$L(\frac{\partial u}{\partial s}) = 0,$$

taking derivative with respect to s twice, we have

$$L(\frac{\partial^2 u}{\partial s^2} \ge 0,$$

where we use the concavity of the operator (2.4.3).

Then the inequalities (2.5.1) follow from the maximal principle since the operator L is elliptic.

□

Theorem 2.5.2. *Suppose that $v(s) : [0,1] \to \mathcal{V}$ is a smooth curve and $u \in \mathcal{V} \setminus v([0,1])$. For any $\epsilon > 0$, set $u(t,s) \in C^\infty([0,1] \times [0,1], \mathcal{V})$ as the unique ϵ-geodesic connecting u and $v(s)$. Then for ϵ small enough, we have*

$$l\big(u(\cdot,0)\big) \le l(v) + l\big(u(\cdot,1)\big) + C\epsilon,$$

where $C > 0$ is independent of ϵ.

Proof. Denote

$$l_1(s) := l(v|_{[0,s]}) = \int_0^s \sqrt{\frac{1}{Vol} \int_M v_{\tilde{s}}^2 (1 + \Delta v) dg d\tilde{s}},$$

$$l_2(s) := l\left(u(\cdot,s)\right) = \int_0^1 \sqrt{\frac{1}{Vol}\int_M u_t^2(1+\Delta u)dgdt} = \int_0^1 \sqrt{2e(t,s)}dt,$$

where we use the notation e as in Lemma 2.5.1.

It suffices to show that $l_1' + l_2' \geq -C\epsilon$ on $[0,1]$.

Compute directly, we have

$$e_t = \frac{1}{Vol}\int_M (u_t u_{tt}(1+\Delta u) - u_t|\nabla u_t|^2)dg = \frac{\epsilon}{Vol}\int_M u_t dg,$$

$$\begin{aligned} e_s &= \frac{1}{Vol}\int_M (u_t u_{ts}(1+\Delta u) - u_t\langle \nabla u_t, \nabla u_s\rangle)dg \\ &= \frac{1}{Vol}\left(\frac{\partial}{\partial t}\int_M u_t u_s(1+\Delta u)dg - \int_M (u_s u_{tt}(1+\Delta u) + u_s u_t \Delta u_t + u_t\langle \nabla u_t, \nabla u_s\rangle)dg\right) \\ &= \frac{1}{Vol}\left(\frac{\partial}{\partial t}\int_M u_t u_s(1+\Delta u)dg - \epsilon\int_M u_s dg\right). \end{aligned}$$

Therefore

$$\begin{aligned} l_2' &= \int_0^1 (2e)^{-1/2} e_s dt \\ &= \frac{1}{Vol}\left[(2e)^{-1/2}\int_M u_t u_s(1+\Delta u)dg\right]_{t=0}^{t=1} \\ &\quad + \frac{1}{Vol}\left(\int_0^1 \left((2e)^{-3/2} e_t \int_M u_t u_s(1+\Delta u)dg\right)dt - \epsilon\int_0^1\left((2e)^{-1/2}\int_M u_s dg\right)dt\right). \end{aligned}$$

Since $u(0,s) = u$, $u(1,s) = v(s)$, we have $u_s(0,s) = 0$, $u_s(1,s) = v_s$. Then by Lemma 2.5.2, we have

$$l_2' \geq \frac{1}{Vol}\frac{1}{\sqrt{2e(1,s)}}\int_M u_t(1,s)v_s(1+\Delta v)dg - C\epsilon\int_0^1\left((2e)^{-3/2} + (2e)^{-1/2}\right)dt.$$

By the Schwartz inequality,

$$\begin{aligned} \left|\int_M u_t(1,s)v_s(1+\Delta v)dg\right| &\leq \sqrt{\int_M u_t(1,s)^2\big(1+\Delta u(1,s)\big)dg}\sqrt{\int_M v_s^2(1+\Delta v)dg} \\ &= Vol \cdot \sqrt{2e(1,s)} \cdot l_1', \end{aligned}$$

So we have

$$l_2' \geq -l_1' - C\epsilon \int_0^1 \Big((2e)^{-3/2} + (2e)^{-1/2} \Big) dt.$$

According to Lemma 2.5.1,

$$e(t,s) \geq \mathcal{E}\big(u, v(s)\big) - \epsilon |u_t|_{C^0},$$

Since we assume that $u \in \mathcal{V} \setminus v([0,1])$, which is crucial for this theorem, $\mathcal{E}\big(u, v(s)\big)$ is always positive for any s by definition, and clearly $\mathcal{E}\big(u, v(s)\big)$ is continuous with respect to s, so $\mathcal{E}\big(u, v(s)\big)$ has a uniformly positive lower bound. Then for ϵ small enough, we have $e(t,s) \geq c > 0$. Therefore,

$$l_2' \geq -l_1' - C\epsilon,$$

and the theorem follows.

$\square$

The following theorem shows that d is the same as the geodesic distance.

Theorem 2.5.3. *Let u^ϵ be the ϵ-geodesic connecting $u_0, u_1 \in \mathcal{V}$, then*

$$d(u_0, u_1) = \lim_{\epsilon \to 0^+} l(u^\epsilon).$$

Proof. Let $v(s) : [0,1] \to \mathcal{V}$ be an arbitrary smooth curve connecting u_0 and u_1, we just need to show that

$$\overline{\lim_{\epsilon \to 0^+}}\, l(u^\epsilon) \leq l(v).$$

Without loss of generality, we assume that $u_1 \notin v\big([0,1)\big)$. We extend $u^\epsilon(t) : [0,1] \to \mathcal{V}$ to $u^\epsilon(t,s) : [0,1] \times [0,1) \to \mathcal{V}$ such that for any $s \in [0,1)$, $u^\epsilon(\cdot, s)$ is the ϵ-geodesic connecting u_1 and $v(s)$. By Theorem 2.5.2, we have

$$l\big(u^\epsilon(\cdot, 0)\big) \leq l(v|_{[0,s]}) + l\big(u^\epsilon(\cdot, s)\big) + C(s)\epsilon,\ \ s \in [0,1).$$

Since clearly $\lim_{s \to 1^-} l(v|_{[0,s]}) = l(v)$, it remains to show that

$$\overline{\lim_{s \to 1^-}}\ \overline{\lim_{\epsilon \to 0^+}}\, l\big(u^\epsilon(\cdot, s)\big) = 0.$$

□

But this follows immediately from the next lemma.

Lemma 2.5.3. *Let u be the ϵ-geodesic connecting $u_0, u_1 \in \mathcal{V}$, then*

$$l(u) \leq C\Big(|u_0 - u_1|_{C^0} + \frac{2\epsilon + |\nabla u_1 - \nabla u_0|^2}{2\min\big(1+\Delta u_0, 1+\Delta u_1\big)}\Big),$$

where C is a uniform constant independent of ϵ.

Proof. By Proposition 2.4.2,

$$l(u) = \int_0^1 \sqrt{\frac{1}{Vol}\int_M u_t^2(1+\Delta u)dgdt} \leq C|u_t|_{C^0},$$

where C is a uniform constant independent of ϵ.

By Lemma 2.4.4, we have

$$l(u) \leq C\big(a + |u_0 - u_1|_{C^0}\big),$$

where a is a constant the same as in Lemma 2.4.3. But in the proof of Lemma 2.4.3, to make the inequality (2.4.5) hold, we can choose

$$a = \frac{2\epsilon + |\nabla u_1 - \nabla u_0|^2}{2\min\big(1+\Delta u_0, 1+\Delta u_1\big)},$$

then the result follows.

□

Corollary 2.5.1 ([24])**.** *The space $(\mathcal{V}, d)$ is a metric space. Moreover, the distance function is at least C^1.*

Proof. We only need to show the differentiability of the distance function. Following from the proof of Theorem 2.5.2, we have

$$\frac{dl\big(u^\epsilon(\cdot,s)\big)}{ds} = \frac{1}{Vol}\frac{1}{\sqrt{2e(1,s)}}\int_M u_t(1,s)v_s(1+\Delta v)dg + O(\epsilon),$$

by Theorem 2.5.3,

$$d\big(u, v(s)\big) = \lim_{\epsilon\to 0^+} l\big(u^\epsilon(\cdot,s)\big),$$

we have

$$\lim_{s \to s_0} \frac{d\big(u, v(s)\big) - d\big(u, v(s_0)\big)}{s - s_0} = \frac{1}{Vol} \frac{1}{\sqrt{2e(1, s_0)}} \int_M u_t(1, s_0) v_s(s_0)(1 + \Delta v) dg,$$

which means that d is a differential function.

$\square$

Lastly, we show that $(\mathcal{V}, d)$ is non-positively curved in the sense of Alexandrov following Calabi and Chen [20]. We will need the following lemma, which says that the Jacobi vector field along any geodesic grows super-linearly.

Lemma 2.5.4 ([24])**.** *Let $u(t, s)$ be the two-parameter families of ϵ-geodesics defined as in Lemma 2.5.2. Let $X(t, s) = \frac{\partial u}{\partial t}$ be the tangential vector fields and $Y(t, s) = \frac{\partial u}{\partial s}$ the deformation vector fields along the ϵ-geodesics. Then we have*

$$\frac{\partial^2 \|Y\|}{\partial t^2} \geq 0.$$

Proof. By definition,

$$\|Y\|^2 = \frac{1}{Vol} \langle\langle Y, Y \rangle\rangle = \int_M |u_s|^2 (1 + \Delta u) dg.$$

Then

$$\frac{1}{2} \frac{\partial \|Y\|^2}{\partial t} = \langle\langle D_X Y, Y \rangle\rangle = \langle\langle D_Y X, Y \rangle\rangle,$$

since the covariant derivative D is compatible with the metric and torsion free.

It follows that

$$\begin{aligned}
\frac{1}{2} \frac{\partial^2 \|Y\|^2}{\partial t^2} &= \langle\langle D_Y X, D_X Y \rangle\rangle + \langle\langle D_X D_Y X, Y \rangle\rangle \\
&= \|D_X Y\|^2 + \langle\langle R(X, Y)X, Y \rangle\rangle + \langle\langle D_Y D_X X, Y \rangle\rangle \\
&\geq \|D_X Y\|^2 + \langle\langle D_Y D_X X, Y \rangle\rangle,
\end{aligned}$$

where the last inequality follows from Theorem 2.3.5 saying that the sectional curvature of $\mathcal{V}$ is non-negative.

Since for any fixed s, $u(t,s)$ is an ϵ-geodesic, we have

$$D_X X = u_{tt} - \frac{1}{1+\Delta u}\langle \nabla u_t, \nabla u_t \rangle = \frac{\epsilon}{1+\Delta u}.$$

Then by definition

$$\begin{aligned}
\langle\langle D_Y D_X X, Y \rangle\rangle &= \frac{1}{Vol}\int_M u_s D_Y \left(\frac{\epsilon}{1+\Delta u}\right)(1+\Delta u)dg \\
&= \frac{\epsilon}{Vol}\int_M \left(-\frac{u_s \Delta u_s}{1+\Delta u} - u_s \langle \nabla u_s, \nabla (\frac{1}{1+\Delta u}) \rangle \right) dg \\
&= \frac{\epsilon}{Vol}\int_M \frac{|\nabla u_s|^2}{1+\Delta u} dg \geq 0.
\end{aligned}$$

It follows that

$$\frac{1}{2}\frac{\partial^2 \|Y\|^2}{\partial t^2} \geq \|D_X Y\|^2.$$

Since

$$\frac{1}{2}\frac{\partial^2 \|Y\|^2}{\partial t^2} = \|Y\| \frac{\partial^2 \|Y\|}{\partial t^2} + \left(\frac{\partial \|Y\|}{\partial t} \right)^2,$$

and

$$\frac{\partial \|Y\|}{\partial t} = \frac{1}{2\|Y\|}\frac{\partial \|Y\|^2}{\partial t} = \frac{\langle\langle Y, D_X Y \rangle\rangle}{\|Y\|} \leq \|D_X Y\|,$$

we have

$$\frac{\partial^2 \|Y\|}{\partial t^2} \geq 0.$$

$\square$

Theorem 2.5.4 ([24]). *The space $(\mathcal{V}, d)$ is non-positive curved in the following sense. Given any three points $A, B, C \in \mathcal{V}$, for any $\lambda \in [0,1]$, let P be the point on the weak geodesic segment connecting B and C such that $d(B,P) = \lambda d(B,C)$ and $d(P,C) = (1-\lambda)d(B,C)$. Then the following inequality holds*

$$d^2(A,P) \leq (1-\lambda)d^2(A,B) + \lambda d^2(A,C) - \lambda(1-\lambda)d^2(B,C).$$

Proof. For any small $\epsilon > 0$, let $u(s) \equiv A$, and $v(s)$ be the ϵ-geodesic connecting B and C. Set $u(t,s)$ as the ϵ-geodesic connecting A and $v(s)$. The energy of the path $u(\cdot, s)$ connecting A and

$v(s)$ is by definition given by

$$E_\epsilon(u(\cdot, s)) = \int_0^1 e_\epsilon(t, s)dt,$$

where

$$e_\epsilon(t, s) := \frac{1}{2}\|\dot{u}\|^2_{u(t,s)} = \frac{1}{2Vol}\int_M u_t^2 \left(1 + \Delta u\right) dg.$$

Since

$$\frac{\partial e_\epsilon(u(t, s))}{\partial t} = \frac{\epsilon}{Vol}\int_M u_t dg,$$

when $\epsilon \to 0$, $e_\epsilon(t, s)$ is a constant for any fixed s, therefore

$$d(A, v(s)) = \lim_{\epsilon\to 0} l(u(\cdot, s)) = \lim_{\epsilon\to 0}\sqrt{2e_\epsilon(t, s)} = lim_{\epsilon\to 0}\sqrt{2E_\epsilon(u(t, s))}. \tag{2.5.2}$$

As in Theorem 2.5.2, we have

$$\begin{aligned}
\frac{\partial E_\epsilon}{\partial s} &= \int_0^1 \frac{\partial e_\epsilon}{\partial s}dt \\
&= \int_0^1 \frac{1}{Vol}\left(\frac{\partial}{\partial t}\int_M u_t u_s(1 + \Delta u)dg - \epsilon\int_M u_s dg\right) dt \\
&= \frac{1}{Vol}\int_M u_t(1, s)v_s(1 + \Delta v)dg - \frac{\epsilon}{Vol}\int_0^1\int_M u_s dgdt.
\end{aligned}$$

We follow the notations in Lemma 2.5.4, thus

$$\frac{\partial E_\epsilon}{\partial s} = \langle\langle X, Y\rangle\rangle_{t=1} - \frac{\epsilon}{Vol}\int_0^1\int_M u_s dgdt.$$

Then

$$\frac{\partial^2 E_\epsilon}{\partial s^2} = \langle\langle D_Y X, Y\rangle\rangle_{t=1} + \langle\langle X, D_Y Y\rangle\rangle_{t=1} - \frac{\epsilon}{Vol}\int_0^1\int_M u_{ss} dgdt.$$

According to Lemma 2.5.4, $\|Y\|$ is a convex function of t. And here we have $Y(0) = 0$, it follows that

$$\frac{\partial}{\partial t}\|Y(t)\|_{t=1} \geq \|Y(1)\|,$$

therefore,

$$\langle\langle D_X Y, Y\rangle\rangle_{t=1} \geq \langle\langle Y, Y\rangle\rangle_{t=1}.$$

So

$$\frac{\partial^2 E_\epsilon}{\partial s^2} \geq \langle\langle Y, Y \rangle\rangle_{t=1} + \langle\langle X, D_Y Y \rangle\rangle_{t=1} - \frac{\epsilon}{Vol} \int_0^1 \int_M u_{ss} dg dt,$$

moreover, at $t = 1$, $u(1, s) = v(s)$ is also an ϵ-geodesic, we have

$$D_Y Y = \frac{\epsilon}{1 + \Delta u},$$

and

$$\langle\langle Y, Y \rangle\rangle_{t=1} = \frac{1}{Vol} \int_M v_s^2 (1 + \Delta v) dg \geq 2E_\epsilon(v(\cdot)) - C\epsilon,$$

finally we obtain

$$\frac{\partial^2 E_\epsilon}{\partial s^2} \geq 2E_\epsilon(v(\cdot)) - C\epsilon.$$

Therefore

$$E_\epsilon(u(\cdot, \lambda)) \leq (1 - \lambda)E_\epsilon(u(\cdot, 0)) + \lambda E_\epsilon(u(\cdot, 1)) - \lambda(1 - \lambda)E_\epsilon(v(\cdot)) - C\epsilon.$$

Now we fix λ and let $\epsilon \to 0$, by the equality (2.5.2), the theorem follows. □

We know that every geodesic segment achieves the minimum length of all paths connecting its end points. The following statement shows that any length minimizing sequence of paths joining any two points in $\mathcal{V}$ must contain a subsequence converging to the unique weak geodesic in the distance topology.

Corollary 2.5.2. *Let $u_0, u_1 \in \mathcal{V}$, and let $\{u_i\}$ be any sequence of paths in $\mathcal{V}$ from u_0 to u_1 and whose length converges to the least possible limit. Then $\{u_i\}$ converges, in the distance topology, to the unique weak geodesic joining u_0 and u_1.*

Proof. Let $u(t) : [0, 1] \to \mathcal{V}$ be the unique weak geodesic joining u_0 and u_1. Then $l(u) = d(u_0, u_1)$. By assumption, we have

$$\lim_{i \to +\infty} l(u_i) = l(u).$$

For simplicity, we assume that each curve involved is parameterized proportionally to the arclength. For any $t \in [0,1]$, using Theorem 2.5.4, we have

$$d(u_i(t), u(t))^2 \leq td(u_i(t), u_1)^2 + (1-t)d(u_i(t), u_0)^2 - t(1-t)d(u_0, u_1)^2,$$

thus

$$\begin{aligned} d(u_i(t), u(t))^2 &\leq tl(u_i|_{[t,1]})^2 + (1-t)l(u_i|_{[0,t]})^2 - t(1-t)l(u)^2 \\ &= t(1-t)^2 l(u_i)^2 + (1-t)t^2 l(u_i)^2 - t(1-t)l(u)^2 \\ &= t(1-t)(l(u_i)^2 - l(u)^2), \end{aligned}$$

as a result,

$$d(u_i(t), u(t)) \to 0,$$

uniformly as $i \to +\infty$. The theorem follows. □

2.6 $C^{1,1}$ geodesics in real case when sectional curvature is non-negative

To prove Theorem 1.0.1, it suffices to prove the following a priori estimates:

Proposition 2.6.1. *Assume that the sectional curvature of M is nonnegative. If u satisfies $(**)$, then there is a uniform constant C which is independent of ϵ such that*

$$|\nabla^2 u| \leq C. \tag{2.6.1}$$

Proof. According to Chen and He [24], we know that there is a uniform bound for u, ∇u, Δu, u_{tt} and ∇u_t, independent of ϵ. On $M \times [0,1]$, we define a function

$$\alpha := \max_{X \in TM,\ |X|=1} \langle \nabla_X \nabla u, X \rangle + |\nabla u|^2 + A(-u + \frac{t^2}{2}),$$

where A is a constant to be determined later. Thus, to prove the estimate (2.6.1), it suffices to bound α.

We assume that α attains maximum at an interior point $(x_0, t_0) \in M \times (0, 1)$, otherwise we are done.

Let $e_1, ..., e_m$ be an orthonormal local frame of vector fields near x_0 which is normal at x_0. This means that $g_{ij} = \langle e_i, e_j \rangle = \delta_{ij}$ near x_0 and $\Gamma_{ij}^k = 0$ at x_0. We may also assume that at (x_0, t_0),

$$\max_{X \in TM,\ |X|=1} \langle \nabla_X \nabla u, X \rangle = u_{11}.$$

Note that u_{11} is a well-defined function near (x_0, t_0), and if we define

$$\tilde{\alpha} := u_{11} + |\nabla u|^2 + A(-u + \frac{t^2}{2}),$$

near (x_0, t_0), then $\tilde{\alpha}$ also has a maximum at (x_0, t_0), since

$$\tilde{\alpha} \leq \alpha \leq \alpha(x_0, t_0) = \tilde{\alpha}(x_0, t_0).$$

Similarly as in Guan [40], one can show that $u_{1j}(x_0, t_0) = 0$ for $j \geq 2$, and therefore the $m \times m$ matrix (u_{ij}) can be diagonalized at (x_0, t_0). To see this let $e_\theta = e_1 \cos\theta + e_j \sin\theta$, then

$$u_{e_\theta e_\theta}(x_0, t_0) = u_{11} \cos^2\theta + 2u_{1j} \sin\theta\cos\theta + u_{jj} \sin^2\theta$$

has a maximum at $\theta = 0$. So

$$\left.\frac{d}{d\theta}\right|_{\theta=0} u_{e_\theta e_\theta}(x_0) = 0,$$

which gives $u_{1j}(x_0, t_0) = 0$.

Near (x_0, t_0) we define the $(m+1) \times (m+1)$ matrix

$$\nabla^2 u = \begin{pmatrix} u_{11} & \dots & u_{1m} & u_{1t} \\ \vdots & \ddots & \vdots & \vdots \\ u_{m1} & \dots & u_{mm} & u_{mt} \\ u_{t1} & \dots & u_{tm} & u_{tt} \end{pmatrix},$$

that is, the Hessian of the function u defined on $M \times [0,1]$, and we rewrite the equation as

$$F(\nabla^2 u) := \log(u_{tt}(1+\Delta u) - |\nabla u_t|^2) = \log \epsilon. \tag{2.6.2}$$

We mention again the linearization of the left-hand side of (2.6.2) given by

$$L(h) = \frac{1}{\epsilon}\left(u_{tt}\Delta h + (1+\Delta u)h_{tt} - 2\langle \nabla u_t, \nabla h_t\rangle\right),$$

is an elliptic and concave operator.

We compute

$$L(u_{11}) = \frac{1}{\epsilon}\left(\sum_i u_{tt}(u_{11})_{ii} + (1+\Delta u)u_{tt11} - 2\sum_i u_{ti}(u_{t11})_i\right). \tag{2.6.3}$$

At (x_0, t_0), in fact we have

$$(u_{ii})_j = u_{iij}, \tag{2.6.4}$$

and

$$(u_{kk})_{ij} = u_{kkij}. \tag{2.6.5}$$

Following Guan [40], here we also provide some explanations. First, the equality (2.6.4) is obvious when the local frame is normal at x_0. In general, since $e_1, ..., e_m$ are orthonormal, we have

$$\langle \nabla_{e_k} e_i, e_j\rangle + \langle e_i, \nabla_{e_k} e_j\rangle = 0,$$

$$\langle \nabla_{e_i} e_k, \nabla_{e_j} e_k\rangle + \langle e_k, \nabla_{e_i}\nabla_{e_j} e_k\rangle = 0.$$

Thus

$$\Gamma^j_{ki} + \Gamma^i_{kj} = 0,$$

$$\Gamma^l_{ik}\Gamma^l_{jk} + \nabla_{e_i}(\Gamma^k_{jk}) + \Gamma^l_{jk}\Gamma^k_{il} = 0,$$

which means $\Gamma^i_{ki} = 0$ and $\nabla_{e_i}(\Gamma^k_{jk}) = 0$.

By definition, at (x_0, t_0),

$$(u_{ii})_j = u_{iij} + 2\Gamma^k_{ji} u_{ki} = u_{iij} + 2\Gamma^i_{ji} u_{ii},$$

since (u_{ij}) is diagonal at (x_0, t_0). By $\Gamma^i_{ji} = 0$, we have (2.6.4).

Next we compute directly

$$\begin{aligned}
(u_{kk})_{ij} &= \nabla_{e_j} \nabla_{e_i} (u_{kk}) - \Gamma^l_{ji} \nabla_{e_l} (u_{kk}) \\
&= \nabla_{e_j} (u_{kki} + 2\Gamma^l_{ik} u_{lk}) - \Gamma^l_{ji} u_{kkl} \\
&= u_{kkij} + 2\Gamma^l_{jk} u_{lki} + \Gamma^l_{ji} u_{kkl} + 2\nabla_{e_j}(\Gamma^l_{ik}) u_{lk} \\
&\quad + 2\Gamma^l_{ik} u_{lkj} + 2\Gamma^l_{ik} \Gamma^p_{jl} u_{pk} + 2\Gamma^l_{ik} \Gamma^p_{jk} u_{lp} - \Gamma^l_{ji} u_{kkl} \\
&= u_{kkij} + 2\Gamma^l_{jk} u_{lki} + 2\nabla_{e_j}(\Gamma^k_{ik}) u_{kk} + 2\Gamma^l_{ik} u_{lkj} - 2\Gamma^l_{ik} \Gamma^l_{jk} u_{kk} + 2\Gamma^l_{ik} \Gamma^l_{jk} u_{ll},
\end{aligned}$$

by $\Gamma^k_{ij} = 0$ at x_0 and $\nabla_{e_j}(\Gamma^k_{ik}) = 0$, we have (2.6.5).

From now on, we emphasize that all the calculation will be carried out at the point (x_0, t_0).

Taking covariant derivative of (2.6.2) twice, we have

$$F^{pq}(u_{pq})_k = 0, \quad \text{for any } k, \tag{2.6.6}$$

$$F^{pq}(u_{pq})_{11} + F^{pq,rs}(u_{pq})_1 (u_{rs})_1 = 0,$$

where we use the notation $F^{pq} := \frac{\partial F}{\partial u_{pq}}$, $F^{pq,rs} := \frac{\partial^2 F}{\partial u_{pq} \partial u_{rs}}$.

By concavity of the operator F, we obtain

$$F^{pq}(u_{pq})_{11} \geq 0,$$

which means that

$$u_{tt} \sum_i u_{ii11} + (1 + \Delta u) u_{tt11} - 2 \sum_i u_{ti} (u_{ti})_{11} \geq 0. \tag{2.6.7}$$

By the commutation formula (2.1.4),

$$
\begin{aligned}
u_{11ii} - u_{ii11} &= (\nabla_{e_i} R)^m_{1i1} u_m + R^m_{1i1} u_{mi} + R^m_{1ii} u_{1m} \\
&\quad + R^m_{1i1} u_{mi} + (\nabla_{e_1} R)^m_{1ii} u_m + R^m_{1ii} u_{m1} \\
&= 2R_{1ii1}(u_{11} - u_{ii}) + C(|\nabla R|, |\nabla u|),
\end{aligned}
$$

where C is a constant under control which may be different from line to line in the following.

Since we assume the sectional curvature of M is nonnegative, which implies $R_{1ii1} \geq 0$, therefore,

$$
u_{11ii} \geq u_{ii11} + C(|\nabla R|, |\nabla u|). \tag{2.6.8}
$$

In fact here is the only time we use this assumption.

Inserting the inequalities (2.6.7) and (2.6.8) into (2.6.3), we get

$$
L(u_{11}) \geq \frac{1}{\epsilon} \left[C(|\nabla R|, |\nabla u|) u_{tt} + 2 \sum_i u_{ti} \left((u_{ti})_{11} - u_{t11i} \right) \right].
$$

By definition,

$$
\begin{aligned}
(u_{ti})_{11} &= \nabla_{e_1} \nabla_{e_1} (u_{ti}) - \Gamma^k_{11} \nabla_k (u_{ti}) \\
&= u_{ti11} + C(|R|, |\nabla u_t|) \\
&= u_{t11i} + C(|R|, |\nabla u_t|),
\end{aligned}
$$

where we use the commutation formula again, finally we have

$$
L(u_{11}) \geq -C \frac{u_{tt} + \Delta u + 1}{\epsilon}. \tag{2.6.9}
$$

Next we estimate $L(|\nabla u|^2)$. First, we compute directly

$$
\begin{aligned}
(|\nabla u|^2)_i &= 2 \sum_k u_k u_{ki} = 2 u_i u_{ii}, \quad (|\nabla u|^2)_t = 2 \sum_k u_k u_{tk}, \\
(|\nabla u|^2)_{ii} &= 2 \sum_k (u_{ki}^2 + u_k u_{kii}) = 2 u_{ii}^2 + 2 \sum_k u_k (u_{iik} + R^m_{iik} u_m), \\
(|\nabla u|^2)_{tt} &= 2 \sum_k (u_{tk}^2 + u_k u_{ttk}), \\
(|\nabla u|^2)_{ti} &= 2 \sum_k u_{ki} u_{tk} + 2 \sum_k u_k u_{tki} = 2 u_{ii} u_{ti} + 2 \sum_k u_k u_{tki},
\end{aligned}
$$

then

$$L(|\nabla u|^2) = \frac{1}{\epsilon}\left(\sum_i u_{tt}(|\nabla u|^2)_{ii} + (1+\Delta u)(|\nabla u|^2)_{tt} - 2\sum_i u_{ti}(|\nabla u|^2)_{ti}\right)$$
$$= \frac{2}{\epsilon}(\sum_i u_{tt}u_{ii}^2 + \sum_k u_{tt}u_k(\Delta u)_k + u_{tt}Ric(\nabla u, \nabla u)$$
$$+ (1+\Delta u)\sum_k u_{tk}^2 + (1+\Delta u)\sum_k u_k u_{ttk} - 2\sum_i u_{ti}^2 u_{ii} - 2\sum_{k,i} u_k u_{ti} u_{tki}).$$

Using the equation (2.6.6), we have

$$L(|\nabla u|^2) = \frac{2}{\epsilon}\left(\sum_i u_{tt}u_{ii}^2 + u_{tt}Ric(\nabla u, \nabla u) + (1+\Delta u)\sum_k u_{tk}^2 - 2\sum_i u_{ti}^2 u_{ii}\right)$$
$$\geq \frac{2}{\epsilon}\left(\sum_i u_{tt}u_{ii}^2 + (1+\Delta u - 2u_{11})\left(u_{tt}(1+\Delta u) - \epsilon\right) - C(|R|, |\nabla u|)u_{tt}\right)$$
$$\geq 4u_{11} - C\frac{u_{tt} + \Delta u + 1}{\epsilon}.$$

Summing up, by maximal principle, we will obtain at (x_0, t_0),

$$0 \geq L(\tilde{\alpha}) \geq 4u_{11} + (A - C)\frac{u_{tt} + \Delta u + 1}{\epsilon} - 2A,$$

now we may take $A >> 0$, it follows that $u_{11} \leq C$ at (x_0, t_0), and

$$\alpha \leq \alpha(x_0, t_0) \leq C,$$

and the theorem follows.

□

Chapter 3

Complex Hessian equations on compact Hermitian manifolds with boundary

Organisation. In Section 3.1 we give definitions for generalised $m-$subharmonic functions together with some basic properties. Assuming Theorem 1.0.2, in Section 3.2 we develop "pluripotential theory" for generalised $m-$subharmonic functions in a Euclidean ball. Section 3.3 is devoted to study weak solutions to the Dirichlet problem. Finally, in Sections 3.4, we prove Theorem 1.0.2 independent of other sections. The appendix is given in Section 3.5.

3.1 Generalised m-subharmonic functions

Fix a Hermitian metric $\alpha = \sqrt{-1}\alpha_{i\bar{j}}dz^i \wedge d\bar{z}^j$ on a bounded open set Ω in $\mathbb{C}^n$. A C^2 function u on Ω is called α-subharmonic if

$$\Delta_\alpha u(z) = \sum \alpha^{\bar{j}i}(z)\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}(z) \geq 0, \tag{3.1.1}$$

where $\alpha^{\bar{j}i}$ is the inverse of $\alpha_{i\bar{j}}$. We can rewrite it simply in term of (n, n)-positive forms

$$dd^c u \wedge \alpha^{n-1} \geq 0, \quad \text{where } d = \partial + \bar{\partial},\ d^c = \frac{\sqrt{-1}}{2\pi}(\bar{\partial} - \partial).$$

This form has the advantage that one can generalise to non-smooth functions and with possibility define higher power of the wedge product of $dd^c u$ (see Remark 3.1.5). We start with the following definition which is adapted from subharmonic functions.

Definition 3.1.1. A function $u : \Omega \to [-\infty, +\infty[$ is called $\alpha-$subharmonic if

(a) u is upper semicontinuous and $u \in L^1_{loc}(\Omega)$;

(b) for every relatively compact open set $D \subset\subset \Omega$ and every $h \in C^0(\bar{D})$ satisfying $\Delta_\alpha h = 0$ in D, if $h \geq u$ on ∂D, then $h \geq u$ on $\bar{D}$.

Remark 3.1.2. Comparing with subharmonic functions we have that

1. If an upper semicontinuous u satisfies (b), then by Harvey-Lawson [47, Theorem 9.3(A)] it follows that either $u \equiv -\infty$ or $u \in L^1_{loc}(\Omega)$.
2. The $\alpha-$subharmonicity for continuous function u is equivalent to the inequality $\Delta_\alpha u \geq 0$ in the distributional sense, a detailed statement of this fact will be given in Lemma 3.5.3 (Appendix).

Given another real $(1, 1)$-form $\chi = \sqrt{-1}\chi_{i\bar{j}} dz^i \wedge d\bar{z}^j$. We shall define $(\chi, m) - \alpha$-subharmonicity for non-smooth functions. We denote by

$$\Gamma_m = \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n : S_1(\lambda) > 0, ..., S_m(\lambda) > 0\}$$

the symmetric positive cone associated with k-th elementary symmetric functions

$$S_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

It is well known that Γ_m is an open convex cone in $\mathbb{R}^n$. The positive cone $\Gamma_m(\alpha)$ associated with the metric α is defined as

$$\Gamma_m(\alpha) = \{\gamma \text{ real } (1, 1) - \text{form: } \gamma^k \wedge \alpha^{n-k} > 0 \text{ for every } k = 1, ..., m\}.$$

In other words, in the orthonormal coordinate such that $\alpha = \sum_i \sqrt{-1}dz^i \wedge d\bar{z}^i$ at a given point in Ω, also diagonalising $\gamma = \sum_i \lambda_i \sqrt{-1}dz^i \wedge d\bar{z}^i$, then $\gamma \in \Gamma_m(\alpha)$ if $(\lambda_1, ..., \lambda_n) \in \Gamma_m$.

Definition 3.1.3. A function $u : \Omega \to [-\infty, +\infty[$ is called $m - \alpha$-subharmonic if u is $\tilde{\alpha}$-subharmonic for any $\tilde{\alpha}$ of the form $\tilde{\alpha}^{n-1} = \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \alpha^{n-m}$, where $\gamma_1, ..., \gamma_{m-1} \in \Gamma_m(\alpha)$.

Here, the metric $\tilde{\alpha}$ is uniquely defined thanks to a result of Michelsohn [70]. By a simple consideration we have a generalisation

Definition 3.1.4. A function $u : \Omega \to [-\infty, +\infty[$ is called $(\chi, m) - \alpha$-subharmonic if $u + \rho$ is $\tilde{\alpha}$-subharmonic for any $\tilde{\alpha}$ of the form $\tilde{\alpha}^{n-1} = \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \alpha^{n-m}$, where $\gamma_1, ..., \gamma_{m-1} \in \Gamma_m(\alpha)$, and the smooth function ρ is defined, up to a constant, by the equation $dd^c\rho \wedge \tilde{\alpha}^{n-1} = \chi \wedge \tilde{\alpha}^{n-1}$.

Notice that when $\chi \equiv 0$, Definition 3.1.4 coincides with Definition 3.1.3. Thanks to Lemma 3.5.3 in Appendix, we get that for a $(\chi, m) - \alpha$-subharmonic function u,

$$(\chi + dd^c u) \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \alpha^{n-m} \geq 0 \tag{3.1.2}$$

for any collection $\gamma_i \in \Gamma_m(\alpha)$, in the weak sense of currents. We denote the set of all $(\chi, m) - \alpha$-subharmonic functions in Ω by

$$SH_{\chi,m}(\alpha, \Omega) \quad \text{or} \quad SH_{\chi,m}(\alpha)$$

(for short) if the considered set is clear from the context.

Remark 3.1.5. (1) For a C^2 function u the inequality (3.1.2) is equivalent to the inequalities

$$(\chi + dd^c u)^k \wedge \alpha^{n-k} \geq 0 \quad \text{for } k = 1, ..., m. \tag{3.1.3}$$

In other words, $u \in SH_{\chi,m}(\alpha, \Omega)$ if and only if $\chi + dd^c u \in \overline{\Gamma_m(\alpha)}$ at any given point in Ω.

This fact can be seen as follows: for any real $(1,1)$-form $\tau \in \Gamma_m(\alpha)$ and $1 \leq k \leq m$,

$$\frac{\tau^k \wedge \alpha^{n-k}}{\alpha^n} = \left(\inf_{\gamma} \left\{ \frac{\tau \wedge \gamma^{k-1} \wedge \alpha^{n-k}}{\alpha^n} \right\} \right)^k, \tag{3.1.4}$$

where γ is taken such that $\gamma \in \Gamma_m(\alpha)$ and $\gamma^k \wedge \alpha^{n-k}/\alpha^n = 1$.

Indeed, By Gårding inequality,

$$\frac{\tau \wedge \gamma^{k-1} \wedge \alpha^{n-k}}{\alpha^n} \geq \left(\frac{\tau^k \wedge \alpha^{n-k}}{\alpha^n}\right)^{\frac{1}{k}} \cdot \left(\frac{\gamma^k \wedge \alpha^{n-k}}{\alpha^n}\right)^{\frac{k-1}{k}},$$

thus $\leq$ follows. To show $\geq$, consider the orthonormal coordinate with respect to α at a given point, diagonalising $\tau = \sum_i \tau_i \sqrt{-1} dz^i \wedge d\bar{z}^i$, we let $\gamma = \sum_i \gamma_i \sqrt{-1} dz^i \wedge d\bar{z}^i$, where $\gamma_i = \tau_i \Big/ \left(\frac{\tau^k \wedge \alpha^{n-k}}{\alpha^n}\right)^{\frac{1}{k}}$, then the infimum is attained. Also, the equality (3.1.4) implies that the operator $S_k(\lambda)^{\frac{1}{k}}$ is concave.

(2) There are other definitions for $m-\alpha$-subharmonic functions. The first one is suggested by Błocki [9] and the second one is given by Lu [66, Definition 2.3] in a more general setting. All definitions are equivalent in the case of $m-$subharmonic functions, i.e. $\alpha = dd^c|z|^2$. Later on, by Lemma 3.5.8, we will find that our definition is equivalent to the one in [66].

We list here some basic properties of $(\chi, m) - \alpha$-subharmonic functions.

Proposition 3.1.1. *Let Ω be a bounded open set in $\mathbb{C}^n$.*

(a) If $u_1 \geq u_2 \geq \cdots$ is a decreasing sequence of $(\chi, m) - \alpha$-subharmonic functions, then $u := \lim_{j\to\infty} u_j$ is either $(\chi, m) - \alpha$-subharmonic or $\equiv -\infty$.

(b) If u, v belong to $SH_{\chi,m}(\alpha)$, then so does $\max\{u, v\}$.

(c) Let a family of functions $\{u_\alpha\}_{\alpha\in I} \subset SH_{\chi,m}(\alpha)$ be locally uniformly bounded above. Then, the upper semicontinuous regularisation u^* of $u(z) := \sup_\alpha u_\alpha(z)$ is $(\chi, m) - \alpha$-subharmonic.

Proof. It is enought to verify $\tilde{\alpha}-$subharmonicity for every $\tilde{\alpha}^{n-1} = \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \alpha^{n-m}$ with $\gamma_i \in \Gamma_m(\alpha)$. Once $\tilde{\alpha}$ is fixed the proof follows from Appendix (Proposition 3.5.1, Corollary 3.5.5). □

3.2 Pluripotential estimates in a ball

In this section we develop pluripotential theory for $(\chi, m)-\alpha$-subharmonic functions in a Euclidean ball, where α is conformal to a Kähler metric on this ball. To do this we fix a ball $B := B(z, r) \subset\subset$

Ω with small radius, where Ω is a bounded open set in $\mathbb{C}^n$. We also fix a smooth function $G : \bar{B} \to \mathbb{R}$ such that $\omega := e^G \alpha$ is Kähler, i.e.,

$$d(e^G \alpha) = 0 \quad \text{on } \bar{B}. \tag{3.2.1}$$

Notice that by Definition 3.1.4 we have $SH_{\chi,m}(\alpha) \equiv SH_{\chi,m}(\omega)$ as $\Gamma_m(\alpha) \equiv \Gamma_m(\omega)$.

For $1 \leq m \leq n$ and a general Hermitian metric α, it is not known yet whether any $(\chi, m) - \alpha$-subharmonic function can be approximated by a decreasing sequence of smooth $(\chi, m) - \alpha$-subharmonic functions. So we make the following definition.

Definition 3.2.1. Let v be a $(\chi, m) - \alpha$-subharmonic function in a neighborhood of $\bar{B}$. v is said to belong to $\mathcal{A}$ if there exists a sequence of smooth $(\chi, m) - \alpha$-subharmonic functions $v_j \in C^\infty(\bar{B})$ decreasing point-wise to v in B as j goes to $+\infty$.

For simplicity we also assume in this section that for every $z \in \bar{\Omega}$,

$$\chi(z) \in \Gamma_m(\alpha), \tag{3.2.2}$$

(otherwise we replace χ by $\tilde{\chi} := \chi + Cdd^c\rho$ where ρ is a strictly plurisubharmonic function in $\bar{\Omega}$ and $C > 0$ large.) Since $\bar{B}$ is compact, there exist $0 < c_0 \leq 1$, depending on $\chi, \alpha, \bar{B}$, such that

$$\chi - c_0\alpha \in \Gamma_m(\alpha).$$

Throughout this chapter we often write

$$\chi_u := \chi + dd^c u \quad \text{for } u \in SH_{\chi,m}(\alpha).$$

3.2.1 Hessian operator

According to the results in [60] for any $v_1, ..., v_m \in \mathcal{A} \cap C^0(\bar{B})$, the wedge product

$$\chi_{v_1} \wedge \cdots \wedge \chi_{v_m} \wedge \alpha^{n-m}$$

is a well-defined positive Radon measure for a general Hermitian metric α. However, to define the wedge product for $v_i \in \mathcal{A} \cap L^\infty(\bar{B})$ we will need the Kähler property of $\omega = e^G\alpha$ in (3.2.1).

Following Bedford-Taylor [5], by a simple modification, we can define the wedge product for $v_i \in \mathcal{A} \cap L^\infty(\bar{B})$ as follows. Fix a strictly plurisubharmonic function φ in a neighborhood of $\bar{B}$ such that

$$\tau := dd^c\varphi - \chi > 0.$$

Let us denote $w_i := v_i + \varphi$. Then w_i is $m - \omega$-subharmonic and bounded. Since ω is Kähler, we define inductively for $1 \leq k \leq m$,

$$dd^c w_k \wedge \cdots \wedge dd^c w_1 \wedge \omega^{n-m} := dd^c(w_k dd^c w_{k-1} \wedge \cdots \wedge dd^c w_1 \wedge \omega^{n-m}). \tag{3.2.3}$$

The resulted wedge product is a positive $(n-m+k, n-m+k)-$current. Then, one puts

$$dd^c w_k \wedge \cdots \wedge dd^c w_1 \wedge \alpha^{n-m} := e^{(m-n)G} dd^c w_k \wedge \cdots \wedge dd^c w_1 \wedge \omega^{n-m}. \tag{3.2.4}$$

We see that local properties that hold for a positive current on the right hand side will be preserved to the positive currents on the left hand side. Finally, using a formal expansion, we set

$$\begin{aligned} &\chi_{v_1} \wedge \cdots \wedge \chi_{v_m} \wedge \alpha^{n-m} \\ &:= \sum_{1 \leq i_1 < \cdots < i_k \leq m} (-1)^{m-k} dd^c w_{i_1} \wedge \cdots \wedge dd^c w_{i_k} \wedge \alpha^{n-m} \wedge \tau^{m-k}. \end{aligned} \tag{3.2.5}$$

This is an honest equality in the case $v_i's$ are smooth functions. The right hand side still makes sense, when $v_i's$ are only bounded, by (3.2.3) and (3.2.4). Thus, the wedge product on the left hand side is a well-defined $(n,n)-$positive current.

We also observe that the equality (3.2.5) does not depend on the choice of φ. Moreover, let $T = \chi_{v_1} \wedge \cdots \wedge \chi_{v_k} \wedge \alpha^{n-m}$ for $v_i \in \mathcal{A} \cap L^\infty(\bar{B})$ and $w \in \mathcal{A} \cap L^\infty(\bar{B})$. Then, we have

$$(\chi + dd^c w) \wedge T = \chi \wedge T + dd^c w \wedge T.$$

In other words, the definition of the wedge product obeys the linearity as in the smooth case.

Remark 3.2.2. If we do not assume $d\alpha = 0$ (or $d(e^G\alpha) = 0$ for some function G), then in the inductive process we cannot get rid of the extra terms, e.g.,

$$dd^c v_k \wedge \cdots \wedge dd^c v_1 \wedge dd^c \alpha^{n-m}.$$

As $dd^c v_i$ is not positive, we do not know how to define the wedge product for bounded functions v_i in $\mathcal{A}$ once the power of the base α is less than $n-m$. It is worth mentioning that if $v_i's$ are continuous and belong to $\mathcal{A}$, then we can use the uniform convergence of potentials to define wedge product as in [60].

As in [60] the Chern-Levine-Nirenberg (CLN) inequalities are proved quickly in the present setting.

Lemma 3.2.1. *Let $u_1, ..., u_m \in \mathcal{A} \cap L^\infty(\bar{B})$. Let $K \subset\subset B$ be a compact set. Then,*

$$\int_K \chi_{u_1} \wedge \cdots \wedge \chi_{u_m} \wedge \alpha^{n-m} \leq C,$$

where C depends on $\alpha, K, B, \|u_1\|_{L^\infty(B)}, ..., \|u_m\|_{L^\infty(B)}$.

Proof. Since $\omega = e^G \alpha$ is Kähler and G is bounded on $\bar{B}$, using formulas (3.2.4), (3.2.5),

$$\int_K \chi_{u_1} \wedge \cdots \wedge \chi_{u_m} \wedge \alpha^{n-m} \leq e^{(n-m)\sup_{\bar{B}}|G|} \int_K \chi_{u_1} \wedge \cdots \wedge \chi_{u_m} \wedge \omega^{n-m}.$$

Then, the lemma follows from integration by parts (see [60, Proposition 2.9]). $\square$

The following Bedford-Taylor convergence theorems are crucial in our approach.

Theorem 3.2.3. *Let $\{u_1^j\}_{j\geq 1}, ..., \{u_m^j\}_{j\geq 1} \subset \mathcal{A} \cap L^\infty(\bar{B})$ be decreasing (or increasing) sequences which converge point-wise to $u_1, ..., u_m \in \mathcal{A} \cap L^\infty(\bar{B})$, respectively. Then, the sequence of positive measures*

$$(\chi + dd^c u_1^j) \wedge \cdots \wedge (\chi + dd^c u_m^j) \wedge \alpha^{n-m}$$

converges weakly to the positive measure

$$(\chi + dd^c u_1) \wedge \cdots \wedge (\chi + dd^c u_m) \wedge \alpha^{n-m}$$

as $j \to +\infty$.

Proof. Recall that $\omega := e^G \alpha$ is a Kähler form on $\bar{B}$. By definitions (3.2.4) and (3.2.5) it is enough to show that if decreasing sequences of bounded $m-\omega$-suharmonic functions $\{v_1^j\}_{j\geq 1}, ..., \{v_m^j\}_{j\geq 1}$

converge to bounded $m-\omega$-subharmonic functions $v_1, ..., v_m$, respectively, then the sequence of $(n,n)-$positive currents $dd^c v_1^j \wedge \cdots \wedge dd^c v_m^j \wedge \omega^{n-m}$ weakly converges to $dd^c v_1 \wedge \cdots \wedge dd^c v_m \wedge \omega^{n-m}$. Therefore, the theorem follows from an easy adaption of arguments in Bedford-Taylor [5]. □

Let us define the notion of capacity associated with Hessian operators which plays an important role in the study of bounded $(\chi, m)-\alpha$-subharmonic functions. For a Borel set $E \subset B$,

$$cap(E) := \sup \left\{ \int_E (\chi + dd^c v)^m \wedge \alpha^{n-m} : v \in \mathcal{A}, 0 \le v \le 1 \right\}. \tag{3.2.6}$$

We first observe that this capacity is equivalent to another capacity.

Lemma 3.2.2. *For a Borel set $E \subset B$,*

$$\mathbf{c}_m(E) := \sup \left\{ \int_E (dd^c w)^m \wedge \alpha^{n-m} : w \in \mathcal{A}_0, 0 \le w \le 1 \right\}, \tag{3.2.7}$$

where $\mathcal{A}_0$ is the class $\mathcal{A}$ with $\chi \equiv 0$. Then, there exists a constant C depending on χ, α such that

$$\frac{1}{C} cap(E) \le \mathbf{c}_m(E) \le C cap(E)$$

for any Borel set $E \subset B$.

Proof. Since $\chi \le dd^c \varphi$ for some smooth plurisubharmonic function on $\bar{B}$, the first inequality follows. To show the second one, we need to use the positivity of α. By (3.2.2) there is a constant $C > 0$ such that

$$\chi - \frac{1}{C} dd^c \rho \in \Gamma_m(\alpha),$$

where $\rho = |z|^2 - r^2 \le 0$. We can choose C such that $|\rho/C| \le 1/2$. Take a function $0 \le w \le 1/2$ in $\mathcal{A}_0$, then it is easy to see that

$$\int_E (dd^c w)^m \wedge \alpha^{n-m} \le \int_E \left(\chi + dd^c(w - \frac{\rho}{C})\right)^m \wedge \alpha^{n-m} \le cap(E).$$

Hence, $\mathbf{c}_m(E) \le 2^m cap(E)$. □

Corollary 3.2.1. *Let $u \in \mathcal{A} \cap L^\infty(\bar{B})$. Then, u is quasi-continuous with respect to the capacity $cap(\cdot)$.*

Proof. Observe that $v := u + \varphi$ is $m - \alpha$ subharmonic for some smooth plurisubharmonic function φ on $\bar{B}$. Therefore, v is also approximated by a decreasing sequence of smooth $m - \alpha$ subharmonic functions. By the arguments in Bedford-Taylor [5] adapted to the case $\omega = e^G \alpha$ (see similar arguments in Lemma 3.5.10), we get that v is quasi-continuous with respect to $\mathbf{c}_m(\cdot)$. By Lemma 3.2.2 the proof is completed. □

The next consequence is an inequality between volume and capacity.

Lemma 3.2.3. *Fix $1 < \tau < n/(n-m)$. There exists a constant $C(\tau)$ such that for any Borel set $E \subset B$,*

$$V_\alpha(E) \leq C(\tau)\big[cap(E)\big]^\tau, \tag{3.2.8}$$

where $V_\alpha(E) := \int_E \alpha^n$.

The exponent here is optimal because if we take $\alpha = dd^c|z|^2$, then the explicit formula for $\mathbf{c}_m(B(0,s))$ in $B = B(0,r)$ with $0 < s < r$, provides an example.

Proof. From [29, Proposition 2.1] we knew that $V_\alpha(E) \leq C[\mathbf{c}_m(E)]^\tau$ with $\mathbf{c}_m(E)$ defined in (3.2.7). Note that the argument in [29] remains valid for non-Kähler α since the mixed form type inequality used there still holds by stability estimates for the Monge-Ampère equation. Thanks to Lemma 3.2.2 the proof follows. □

3.2.2 Comparison principle in $\mathcal{A} \cap L^\infty(\bar{B})$

For simplicity if $u, v \in \mathcal{A} \cap L^\infty(\bar{B})$ we write

$$u \geq v \quad \text{on } \partial B \quad \text{meaning that} \quad \liminf_{z \to \partial B}(u - v) \geq 0. \tag{3.2.9}$$

Lemma 3.2.4. *Let $u, v \in SH_{\chi,m}(\alpha) \cap L^\infty(\bar{B})$ be such that $u \geq v$ on ∂B. Let $T = \chi_{v_1} \wedge \cdots \wedge \chi_{v_{m-1}} \wedge \alpha^{n-m}$ with $v_i \in SH_{\chi,m} \cap L^\infty(\bar{B})$. Then,*

$$\int_{\{u<v\}} dd^c v \wedge T \leq \int_{\{u<v\}} dd^c u \wedge T + \int_{\{u<v\}} (v-u) dd^c T.$$

Notice that by the equations (3.2.4) and (3.2.5)

$$\begin{aligned} dd^c T &= dd^c \left(e^{(m-n)G} \chi_{v_1} \wedge \cdots \wedge \chi_{v_{m-1}} \wedge \omega^{n-m} \right) \\ &= dd^c \left(e^{(m-n)G} \chi_{v_1} \wedge \cdots \wedge \chi_{v_{m-1}} \right) \wedge \omega^{n-m}, \end{aligned}$$

where $\omega = e^G \alpha$ is a fixed Kähler form as in (3.2.1).

Proof. By replacing u by $u+\delta$ for $\delta > 0$ and then letting $\delta \searrow 0$ we will work with $\{u < v\} \subset\subset K \subset\subset B$, where K is an open set. By the CLN inequality (Lemma 3.2.1)

$$\int_K \|dd^c T\| < +\infty.$$

By Theorem 3.2.3, Corollary 3.2.1, and arguments in [6] we get that

$$\mathbf{1}_{\{u<v\}} dd^c \max\{u, v\} \wedge T = \mathbf{1}_{\{u<v\}} dd^c v \wedge T \tag{3.2.10}$$

as two measures. Since $\{u + \varepsilon < v\} \subset\subset K$ for $\varepsilon > 0$, Stokes' theorem gives

$$\begin{aligned} &\int_K dd^c \max\{u+\varepsilon, v\} \wedge T \\ &= \int_{\partial K} d^c u \wedge T + \int_K d^c \max\{u+\varepsilon, v\} \wedge dT \\ &= \int_{\partial K} d^c u \wedge T + \int_{\partial K} u \wedge dT + \int_K \max\{u+\varepsilon, v\} dd^c T \\ &= \int_K dd^c u \wedge T - \int_K u dd^c T + \int_K \max\{u+\varepsilon, v\} dd^c T \\ &= \int_K dd^c u \wedge T + \int_{\{u+\varepsilon<v\}\cap K} (v-u) dd^c T + \varepsilon \int_{\{u+\varepsilon \geq v\}\cap K} dd^c T. \end{aligned}$$

Moreover, by the identity (3.2.10)

$$\begin{aligned} &\int_{\{u+\varepsilon<v\}} dd^c v \wedge T \\ &= \int_{\{u+\varepsilon<v\}} dd^c \max\{u+\varepsilon, v\} \wedge T \\ &= \int_K dd^c \max\{u+\varepsilon, v\} \wedge T - \int_{\{u+\varepsilon \geq v\}\cap K} dd^c \max\{u+\varepsilon, v\} \wedge T \\ &\leq \int_K dd^c \max\{u+\varepsilon, v\} \wedge T - \int_{\{u+\varepsilon>v\}\cap K} dd^c u \wedge T. \end{aligned}$$

Thus, it follows that

$$\int_{\{u+\varepsilon<v\}} dd^c v \wedge T \leq \int_{\{u+\varepsilon\leq v\}} dd^c u \wedge T + \int_{\{u+\varepsilon<v\}} (v-u) dd^c T$$
$$+ \varepsilon \int_K \|dd^c T\|.$$

Letting $\varepsilon \searrow 0$ we get the desired inequality. □

In the Hermitian setting due to the torsion of α and χ, the classical comparison principle no longer holds. However, its weak versions in [28] and [58] are enough for several applications. We state the local counterparts of those.

Let D_1, D_2 be two constants such that on $\bar{B}$,

$$\begin{aligned} &-D_1\alpha^2 \leq dd^c\alpha \leq D_1\alpha^2, \quad -D_1\alpha^3 \leq d\alpha \wedge d^c\alpha \leq D_1\alpha^3; \\ &-D_2\alpha^2 \leq dd^c\chi \leq D_2\alpha^2, \quad -D_2\alpha^3 \leq d\chi \wedge d^c\chi \leq D_2\alpha^3. \end{aligned} \tag{3.2.11}$$

Lemma 3.2.5. *Let $u, v \in \mathcal{A} \cap L^\infty(\bar{B})$ be such that $u \geq v$ on ∂B. Assume that $d = \sup_{\bar{B}}(v-u) > 0$. and $D_1 D_2 \sup_{\{u<v\}}(v-u) \leq 1$. Then,*

$$\int_{\{u<v\}} (\chi + dd^c v)^m \wedge \alpha^{n-m} \leq \int_{\{u<v\}} (\chi + dd^c u)^m \wedge \alpha^{n-m} +$$
$$+ C D_1 D_2 \sup_{\{u<v\}} (v-u) \sum_{k=0}^{m-1} \int_{\{u<v\}} (\chi + dd^c u)^k \wedge \alpha^{n-k}.$$

The constant C depends only on n, m.

Proof. We use repeatedly Lemma 3.2.4 (for $T = \chi_u^k \wedge \chi_v^l \wedge \alpha^{n-k-l}$, $k + l \leq m - 1$), and bounds in (3.2.11) to replace v by u. Thanks to results in [60, Section 2] the arguments go through for general Hessian operators with respect to the Hermitian metric α. □

Recall from (3.2.2) that there exists $0 < c_0 \leq 1$, depending on $\chi, \alpha, \bar{B}$, such that

$$\chi - c_0\alpha \in \Gamma_m(\alpha). \tag{3.2.12}$$

The weak comparison principle is a crucial tool in pluripotential theory approach to study weak solutions of Hessian type equations [58, 59, 60]. We state a local version.

Lemma 3.2.6. *Let $u, v \in \mathcal{A} \cap L^\infty(\bar{B})$ be such that $u \geq v$ on ∂B. Assume that $d = \sup_B(v-u) > 0$. Fix $0 < \varepsilon < \min\{1/2, d/(1+2\|v\|_\infty)\}$. Denote $S(\varepsilon) = \inf_B[u - (1-\varepsilon)v]$, and for $s > 0$,*

$$U(\varepsilon, s) := \{u < (1-\varepsilon)v + S(\varepsilon) + s\}.$$

Then, for $0 < s < (c_0\varepsilon)^3/(16 D_1 D_2)$,

$$\int_{U(\varepsilon,s)} \big(\chi + (1-\varepsilon)dd^c v\big)^m \wedge \alpha^{n-m} \leq \left(1 + \frac{Cs}{(c_0\varepsilon)^m}\right) \int_{U(\varepsilon,s)} (\chi + dd^c u)^m \wedge \alpha^{n-m}.$$

The constant depends on n, m, D_1, D_2.

Proof. We only give here a brief argument as it is very similar to the one of [58, Theorem 2.3]. Set for $0 \leq k \leq m$,

$$a_k := \int_{U(\varepsilon,s)} \chi_u^k \wedge \alpha^{n-k}.$$

Then,

$$(c_0\varepsilon) a_k \leq \varepsilon \int_{U(\varepsilon,s)} \chi_u^k \wedge \chi \wedge \alpha^{n-k-1} \leq \int_{U(\varepsilon,s)} \chi_u^k \wedge \chi_{(1-\varepsilon)v} \wedge \alpha^{n-k-1}.$$

By Lemma 3.2.4

$$\int_{U(\varepsilon,s)} \chi_u^k \wedge \chi_{(1-\varepsilon)v} \wedge \alpha^{n-k-1} \leq \int_{U(\varepsilon,s)} \chi_u^{k+1} \wedge \alpha^{n-k-1} + R,$$

where $R = \int_{U(\varepsilon,s)} [(1-\varepsilon)v + S(\varepsilon) + s - u] dd^c \left(\chi_u^k \wedge \alpha^{n-k-1}\right)$. It is bounded by

$$R \leq s D_1 D_2 (a_k + a_{k-1} + a_{k-2}),$$

where we simply understand $a_k \equiv 0$ for $k < 0$. To be honest, here we used [60, Lemma 2.3], hence we should multiply the right hand side with a constant $C_{m,n} > 0$ depending only on m, n. This is no harm as we could adjust the definitions of D_1, D_2.

Thus, for $0 < s < \delta := \frac{(c_0\varepsilon)^3}{D_1 D_2}$, $(c_0\varepsilon) a_k \leq \delta (D_1 D_2)(a_k + a_{k-1} + a_{k-2}) + a_{k+1}$. The rest goes in the same way as in [58, Theorem 2.3]. $\square$

The following result is obvious if potential functions are smooth.

Corollary 3.2.2. *Let $u, v \in \mathcal{A} \cap L^\infty(\bar{B})$ be such that $u \geq v$ on ∂B. Suppose that $\chi_u^m \wedge \alpha^{n-m} \leq \chi_v^m \wedge \alpha^{n-m}$ in B. Then, $u \geq v$ on $\bar{B}$.*

Proof. It follows from the proof of [58, Corollary 3.4.] with obvious modifications. The reason is that there exists a C^2 strictly plurisubharmonic function on $\bar{B}$. $\square$

We have proved the comparison principle (Lemma 3.2.6) and volume-capacity inequality (Lemma 3.2.3). The following uniform a priori estimate is proved in the identically way as [60, Theorem 3.10].

Theorem 3.2.4. *Let $u, v \in \mathcal{A} \cap L^\infty(\bar{B})$ be such that*

$$\liminf_{z \to \partial B}(u - v) \geq 0, \quad d := \sup_B (v - u) > 0.$$

Let us fix the following constants:

$$\begin{aligned} &p > n/m, \quad 0 < \tau < \frac{p - \frac{n}{m}}{p(n-m)}, \quad \tau^* = \frac{(1 + m\tau)p}{p-1}; \\ &0 < \varepsilon < \min\{1/2, d/3(1 + \|v\|_\infty)\}; \\ &\varepsilon_0 := \frac{1}{3}\min\{(c_0\varepsilon)^m, \frac{(c_0\varepsilon)^3}{16 D_1 D_2}\}. \end{aligned}$$

Suppose that $(\chi + dd^c u)^m \wedge \alpha^{n-m} = f\alpha^n$ on B with $f \in L^p(B, \alpha^n)$. Assume that v is continuous and put

$$U(\varepsilon, s) = \{u < (1-\varepsilon)v + \inf_B[u - (1-\varepsilon)v] + s\}.$$

Then, there exists a constant $C = C(\tau, \alpha, B)$ such that for every $0 < s < \varepsilon_0$,

$$s \leq C(1 + \|v\|_{L^\infty(B)})\|f\|_{L^p(B)}^{\frac{1}{m}} \left[V_\alpha(U(\varepsilon, s))\right]^{\frac{\tau}{\tau^*}},$$

where $V_\alpha(E) = \int_E \alpha^n$ for a Borel set E.

Notice that from assumptions, the sub-level sets near the infimum point will be non-empty and relatively compact in the ball B. The restriction on the class $\mathcal{A}$ will be relaxed later (see Remark 3.2.7).

3.2.3 The Dirchlet problem on $\bar{B}$

Consider the Dirichlet problem with the right hand side in $L^p(B)$, $p > n/m$. Notice that n/m is the optimal exponent.

$$\begin{aligned} &u \in \mathcal{A} \cap C^0(\bar{B}), \\ &(\chi + dd^c u)^m \wedge \alpha^{n-m} = f\alpha^n, \\ &u = \varphi \in C^0(\partial B). \end{aligned} \tag{3.2.13}$$

Lemma 3.2.7. *Let f, g be non-negative functions in $L^p(B), p > n/m$. Let $\varphi, \psi \in C^0(\partial B)$. Suppose that u, v are solutions to the corresponding Dirichlet problems* (3.2.13) *with the datum (f, φ) and (g, ψ). Then,*

$$\|u - v\|_{L^\infty(B)} \le \sup_{\partial B} |\varphi - \psi| + C\|f - g\|_{L^p(B)}^{\frac{1}{m}},$$

where C depends only on p and the diameter of B.

Proof. We use an idea in [29], which used the uniform *a priori* estimate for Monge-Ampère equation due to Kołodziej [54]. The proof here is similar to [72, Theorem 3.11]. Put $h = |f - g|^{\frac{n}{m}}$ in B. It follows that $h \in L^{\frac{pm}{n}}(B)$, where $\frac{pm}{n} > 1$. Moreover,

$$\|h\|_{L^{\frac{pm}{n}}(B)}^{\frac{1}{n}} = \|f - g\|_{L^p(B)}^{\frac{1}{m}}.$$

By a theorem in [54], there exists $\rho \in PSH(B) \cap C^0(\bar{B})$ solving

$$(dd^c\rho)^n = h\alpha^n, \quad \rho_{|\partial B} = 0.$$

We also have

$$\|\rho\|_{L^\infty} \le C\|h\|_{L^{\frac{pm}{n}}(B)}^{\frac{1}{n}} = C\|f - g\|_{L^p(B)}^{\frac{1}{m}},$$

where $C = C(m, n, p, B, \alpha)$ a uniform constant. Furthermore, by the mixed-form inequality,

$$(dd^c\rho)^m \wedge \alpha^{n-m} \ge h^{\frac{m}{n}}\alpha^n = |f - g|\alpha^n.$$

Therefore,

$$\begin{aligned}[\chi_u + dd^c\rho]^m \wedge \alpha^{n-m} &\geq \chi_u^m \wedge \alpha^{n-m} + (dd^c\rho)^m \wedge \alpha^{n-m} \\ &\geq f\alpha^n + |f-g|\alpha^n \\ &\geq g\alpha^n.\end{aligned}$$

Since $\rho \leq 0$ in $\bar{B}$, it follows from the domination principle (Corollary 3.2.2) that $u + \rho \leq v + \sup_{\partial B} |u - v|$. Hence,

$$u - v \leq -\rho + \sup_{\partial B} |u-v| \leq \sup_{\partial B} |u-v| + C\|f-g\|_{L^p(B)}^{\frac{1}{m}}.$$

Similarly, $v - u \leq \sup_{\partial B} |u-v| + C\|f-g\|_{L^p(B)}^{\frac{1}{m}}$. Thus, the theorem follows. □

We also need another stability estimate for solutions whose Hessian operators are in L^p, $p > n/m$.

Lemma 3.2.8. *Under the assumptions of Lemma 3.2.7 there exist a uniform constant $C = C(p, m, n, \|f\|_p, \|g\|_p)$ and a constant $a = a(p, m, n) > 0$ such that*

$$\|u - v\|_{L^\infty(B)} \leq \sup_{\partial B} |\varphi - \psi| + C\|u - v\|_{L^1(B)}^a.$$

Proof. Having Theorem 3.2.4 we can repeat the proof of [60, Theorem 3.11] two times, one for the pair $u + \sup_{\partial B} |\varphi - \psi|$ and v, another for the pair $v + \sup_{\partial B} |\varphi - \psi|$ and u. □

From the existence of smooth solutions (Theorem 1.0.2) and stability estimates (Lemma 3.2.7), we obtain the existence and uniqueness of weak solutions on $\bar{B}$.

Theorem 3.2.5. *Let $0 \leq f \in L^p(B)$ with $p > n/m$. Then, there exists a unique solution to the Dirichlet problem* (3.2.13).

The last ingredient to prove the approximation property for $(\chi, m) - \alpha$-subharmonic functions is the existence of smooth solutions for a Hessian type equation.

Lemma 3.2.9. *Let H be a smooth function on $\bar{B}$ and $\varphi \in C^\infty(\partial B)$. Then, there exists a unique $u \in SH_{\chi,m}(\alpha) \cap C^\infty(\bar{B})$ solving the Hessian equation*

$$(\chi + dd^c u)^m \wedge \alpha^{n-m} = e^{u+H}\alpha^n,$$

$$u = \varphi \quad \text{on } \partial B.$$

Proof. The right hand side depends also on u but with the right sign. We solve the equation by the continuity method as in the proof of Theorem 1.0.3, provided a priori estimates up to second order. The C^0-estimate easily follows by considering the maximum point and the minimum point of the solution. So does the C^1-estimate on the boundary. The proof of C^1-estimate at an interior point will be affected at equations (3.4.12) and (3.4.13) in Section 3.4.2. The extra terms appear in these equations are $O(|\nabla u|^2)$. So this will not affect the conclusion of the inequality (3.4.14). Therefore, we will get C^1-estimate. The C^2-estimate at an interior point goes through as in Section 3.4.3, as it is explained in [60, Lemma 3.18]. For the C^2-estimates at a boundary point, the equation (3.4.32) contains a bounded term $O(|\nabla u|)$ under control by the C^1-estimate. Therefore, the equality (3.4.33) still holds and we get the desired estimates. □

Lemma 3.2.10. *Let $0 \leq f \in L^p(B)$, $p > n/m$, and $\varphi \in C^0(\partial B)$. Let $\{f_j\}_{j\geq 1}$ be smooth and positive functions on $\bar{B}$, converging in $L^p(B)$ to f, and $\{\varphi_j\}_{j\geq 1} \in C^\infty(\partial B)$, converging uniformly to φ, as $j \to +\infty$. Assume that*

$$\chi_{u_j}^m \wedge \alpha^{n-m} = e^{u_j} f_j \alpha^n,$$

$$u_j = \varphi_j \quad \text{on } \partial B.$$

Then, u_j converges uniformly to $u \in \mathcal{A} \cap C^0(\bar{B})$, which is the unique solution in $\mathcal{A} \cap C^0(\bar{B})$ of

$$\chi_u^m \wedge \alpha^{n-m} = e^u f \alpha^n,$$

$$u = \varphi \quad \text{on } \partial B.$$

Proof. Observe that u_j is uniformly bounded above. It follows that the right hand side of equations are uniformly bounded in L^p. Applying Lemma 3.2.7 for $\psi = 0$ and $g = 0$, this gives the uniform bound for u_j. Then, by compactness of the sequence u_j in L^1 and Lemma 3.2.8 we get a continuous solution by passing to a limit. The uniqueness follows as in [73, Lemma 2.3]. □

3.2.4 Approximation property on $\bar{B}$

We have all ingredients ready to prove the main theorem of this section. By using results of Pliś [77], Harvey - Lawson - Pliś [50, Theorem 6.1] also proved this theorem in the case when $\chi \equiv 0$ and α is Kähler.

Theorem 3.2.6. *Let u be $(\chi, m) - \alpha$-subharmonic in a neighborhood of $\bar{B}$. Then, there exists a sequence of smooth functions $u_j \in SH_{\chi,m}(\alpha) \cap C^\infty(\bar{B})$ such that u_j decreases to u point-wise in B as $j \to +\infty$.*

Proof. We follow closely the proof of [60, Lemma 3.20], which in turn uses the scheme introduced by Berman [7] and Eyssidieux-Guedj-Zeriahi [36] (see also Lu-Nguyen [68]).

By positivity assumption on $\chi \in \Gamma_m(\alpha)$ for every $z \in \bar{B}$ we have that $j \in SH_{\chi,m}(\alpha)$ for any constant j. As $\max\{u, -j\}$ belongs to $SH_{\chi,m}(\alpha)$, we may assume that u is bounded. Since u is upper semicontinuous on $\bar{B}$, there exists a sequence of smooth functions ϕ_j decreasing to u on $\bar{B}$. Fix such an $h := \phi_j$. Consider the envelope

$$\tilde{h} := \sup\{v \in SH_{\chi,m}(\alpha) \cap L^\infty(B) : v \leq h\}. \tag{3.2.14}$$

Then, $\tilde{h} \in SH_{\chi,m}(\alpha)$ and $u \leq \tilde{h} \leq h$. Therefore, if $\tilde{h} \in \mathcal{A}$, i.e. it has the approximation property, then so does u by letting $h = \phi_j \searrow u$. We shall prove that $\tilde{h}$ can be approximated uniformly and then the lemma will follow.

Since $h \in C^\infty(\bar{B})$, we can write $\chi_h^m \wedge \alpha^{n-m} = F\alpha^n$ with F being a smooth function on $\bar{B}$. Let us denote $F_* = \max\{F, 0\}$. We choose a sequence of smoothly non-negative functions F_j decreasing uniformly to F_* as $j \to \infty$. Fix such a $\tilde{F} := F_j \geq F_*$. By Lemma 3.2.9 we solve for $0 < \varepsilon \leq 1$,

$$\chi_{\tilde{w}_\varepsilon}^m \wedge \alpha^{n-m} = e^{\frac{1}{\varepsilon}(\tilde{w}_\varepsilon - h)}[\tilde{F} + \varepsilon]\alpha^n,$$

$$\tilde{w}_\varepsilon = h \quad \text{on } \partial B.$$

By maximum principle, $\tilde{w}_\varepsilon \leq h$ and $\tilde{w}_\varepsilon$ is increasing as ε decreases to 0. Keep ε fixed, and take limit on both sides for $\tilde{F} = F_j \to F_*$, i.e. letting $j \to \infty$, we get from Lemma 3.2.10,

$$\chi_{w_\varepsilon}^m \wedge \alpha^{n-m} = e^{\frac{1}{\varepsilon}(w_\varepsilon - h)}[F_* + \varepsilon]\alpha^n,$$

$$w_\varepsilon = h \quad \text{on } \partial B.$$

Here $\tilde{w}_\varepsilon$ uniformly increases to w_ε. Thus, $w_\varepsilon \in \mathcal{A} \cap C^0(\bar{B})$ and w_ε is increasing as ε decreases to 0. Since $w_\varepsilon \leq h$, the right hand side is uniformly bounded in $L^\infty(\bar{B})$. The monotone sequence w_ε, bounded above by h, is a Cauchy sequence in $L^1(B)$. By Lemma 3.2.8, this sequence is also Cauchy in the uniform norm in $\bar{B}$. So, w_ε uniformly increases to w which satisfies

$$\chi_w^m \wedge \alpha^{n-m} \leq \mathbf{1}_{\{w=h\}} F_* \alpha^n,$$

$$w = h \quad \text{on } \partial B.$$

In particular, $w \in \mathcal{A} \cap C^0(\bar{B})$. Now, we claim that $w = \tilde{h}$. The inequality $w \leq \tilde{h}$ is clear. One needs to verify that $w \geq \tilde{h}$ on $\{w < h\}$. Take a candidate v in the envelope (3.2.14), i.e, $v \leq h$. Observe that $\chi_w^m \wedge \alpha^{n-m} = 0$ on $\{w < v\} \subset \{w < h\}$. By Corollary 3.2.2 it follows that w is maximal on $\{w < h\}$. Thus, the set $\{w < v\}$ is empty, i.e., $w \geq v$. Since v is arbitrary, so $w \geq \tilde{h}$. The claim follows and so does the theorem. □

Remark 3.2.7. **(a)** In the proof we only used the wedge product for continuous potentials, so Theorem 3.2.6 holds for a general Hermitian metric α. In this case one should use a counterpart of [60, Theorem 2.16] instead of Corollary 3.2.2 in the last argument.

(b) An immediate consequence is that the class $\mathcal{A}$ coincides with $SH_{\chi,m}(\alpha)$.

Thanks to the quasi-continuity and approximation property of $(\chi, m)-\alpha$-subharnonic functions we get an inequality similar to the one for plurisubharmonic functions in Cegrell-Kołoldziej [16].

Proposition 3.2.1. *Let $u, v \in SH_{\chi,m}(\alpha) \cap L^\infty(B)$. Let μ be a positive measure such that $\chi_u^m \wedge \alpha^{n-m} \geq \mu$ and $\chi_v^m \wedge \alpha^{n-m} \geq \mu$. Then*

$$(\chi + dd^c \max\{u, v\})^m \wedge \alpha^{n-m} \geq \mu.$$

Proof. It is readily adaptable from [16, Theorem 1] with an obvious change of notations. □

3.3 Weak solutions to the Dirichlet problem on Hermitian manifolds

On the complex manifold $M = \bar{M} \setminus \partial M$ we define the class $SH_{\chi,m}(\alpha, M)$ in local coordinates. One main difference is that for an arbitrary real $(1,1)$-form χ on M, there are plenty of local $(\chi, m) - \alpha$-subharmonic functions on each local chart. However, the global class $SH_{\chi,m}(\alpha, M)$ may be empty, e.g. for negative χ. Thus, the existence of a subsolution will guarantee that $SH_{\chi,m}(\alpha)$ is non empty.

In this section we shall study weak solution to the Dirichlet problem for the complex Hessian type equation. As we pointed out in Section 3.2.1 the assumption that α is locally conformal to a Kähler metric on M is needed to develop potential theory for bounded functions.

Fix a continuous right hand side density $0 \le f \in C^0(\bar{M})$ and a continuous boundary data $\varphi \in C^0(\partial M)$. Let us denote

$$\mu := f\alpha^n.$$

We wish to solve the Dirichlet problem:

$$\begin{aligned} &w \in SH_{\chi,m}(\alpha) \cap C^0(\bar{M}), \\ &(\chi + dd^c w)^m \wedge \alpha^{n-m} = \mu, \\ &w = \varphi \quad \text{on } \partial M. \end{aligned} \tag{3.3.1}$$

The C^2 subsolution ρ to the equation (3.3.1) satisfies:

$$\chi_\rho := \chi + dd^c\rho \in \Gamma_m(\alpha),$$

and

$$(\chi + dd^c\rho)^m \wedge \alpha^{n-m} \ge \mu, \quad \rho = \varphi \quad \text{on } \partial M. \tag{3.3.2}$$

By replacing χ by χ_ρ and u by $u - \rho$ we can reduce the problem to the case of zero boundary data and $\chi \in \Gamma_m(\alpha)$ as follows:

$$
\begin{aligned}
& w \in SH_{\chi,m}(\alpha) \cap C^0(\bar{M}), \\
& (\chi + dd^c w)^m \wedge \alpha^{n-m} = \mu, \\
& w = 0 \quad \text{on } \partial M.
\end{aligned}
\tag{3.3.3}
$$

Then 0 is a subsolution to the equation (3.3.3), and there exists $0 < c_0 \leq 1$ such that

$$
\chi - c_0 \alpha \in \Gamma_m(\alpha).
$$

3.3.1 Envelope of continuous subsolutions

By assumption (3.3.2) the set

$$
\mathcal{S} = \{v \in SH_{\chi,m}(\alpha) \cap C^0(\bar{M}) : \chi_v^m \wedge \alpha^{n-m} \geq \mu, \ v_{|\partial M} \leq 0\}
$$

is not empty. Hence, we define the envelope

$$
u_0(z) := \sup_{v \in \mathcal{S}} v(z). \tag{3.3.4}
$$

One expects that it will be a solution to the continuous Dirichlet problem.

Theorem 3.3.1. *If u_0 is continuous, then it solves the Dirichlet problem* (3.3.1).

Proof. We first have $u_0 \in S$ by Proposition 3.1.1-(b) and Proposition 3.2.1. In particular,

$$
(\chi + dd^c u_0)^m \wedge \alpha^{n-m} \geq \mu.
$$

It remains to show that $\chi_{u_0}^m \wedge \alpha^{n-m} = \mu$. Fix a small ball $B \subset M$ and find $w \in SH_{\chi,m}(\alpha) \cap C^0(\bar{B})$ solving $w = u_0$ on ∂B and

$$
(\chi + dd^c w)^m \wedge \alpha^{n-m} = \mu \quad \text{in } B.
$$

Hence, $w \geq u_0$ in $\bar{B}$. Consider the lift $\tilde{u} \in \mathcal{S}$ of u_0 with respect to this ball defined by

$$
\tilde{u} = \begin{cases} \max\{w, u_0\} & \text{on } B, \\ u_0 & \text{on } \bar{M} \setminus B. \end{cases}
$$

Thus, we have $\tilde{u} \in S$ and $u_0 \leq \tilde{u}$ in B. On the other hand by the definition of u_0 we have $\tilde{u} \leq u_0$. Thus, $u_0 = \tilde{u}$ in B, which means $\chi_{u_0}^m \wedge \alpha^{n-m} = \mu$. This holds for any ball, so the theorem follows. $\square$

Remark 3.3.2. For continuous $(\chi, m) - \alpha$-subharmonic functions the wedge product is always well-defined. Theorem 3.3.1 is valid for a general Hermitian metric α. The remaining issue is to verify the continuity of the envelope u_0. So far we could not do this for a general Hermitian metric α.

Remark 3.3.3. Let us consider $m = n$ and $f \equiv 0$ in connection with the geodesic equation studied notably by Semmes [78], Donaldson [32], Chen [23] and Blocki [11]. It follows from the comparison principle (an extension of Lemma 3.2.6 for M in the place of B), that there exists at most one continuous solution to the equation. Guan and Li [41] have extended the gradient estimate in [10] to this case. Hence, we can get a continuous solution to the homogeneous equation by a compactness argument. This solution is maximal on M, thus equal to u_0. Thus, we get the unique solution even in the case when the background metric is only Hermitian.

3.3.2 Envelope of bounded subsolutions

In this section we shall prove Theorem 1.0.3, where α is locally conformal Kähler. First we enlarge the class $\mathcal{S}$ above,

$$\hat{\mathcal{S}} := \{v \in SH_{\chi,m}(\alpha) \cap L^\infty(\bar{M}) : \chi_v^m \wedge \alpha^{n-m} \geq \mu,\ v^*_{|\partial M} \leq 0\}.$$

The locally conformal Kähler assumption of α allows us to use potential theory which has been developed in Section 3.2 for bounded $(\chi, m) - \alpha$-subharmonic functions. Set

$$u(z) := \sup_{v \in \hat{\mathcal{S}}} v(z)$$

It follows from Proposition 3.1.1-(b) and Proposition 3.2.1 that $u^* \in \hat{\mathcal{S}}$. Hence, $u = u^*$. Let us solve the linear PDE

$$(\chi + dd^c \rho_1) \wedge \alpha^{n-1} = 0,$$

$$\rho_1 = 0 \quad \text{on } \partial M.$$

Therefore, $0 \leq u \leq \rho_1$. It implies that $u = 0$ and it is continuous on ∂M.

Remark 3.3.4. It is obvious that $u_0 \leq u$. If we can show that u is continuous on M, then $u \in \mathcal{S}$ automatically. Then, $u_0 = u$ is indeed continuous.

In what follows, we shall prove that u is a solution to the (bounded) Dirichlet problem, and then we will prove its regularity by using the *a priori* estimate (Theorem 3.2.4).

Lemma 3.3.1 (lift)**.** *Let $v \in \hat{\mathcal{S}}$. Let $B \subset M$ be a small ball. There exists $\tilde{v} \in \hat{\mathcal{S}}$ such that $v \leq \tilde{v}$ and $\chi_{\tilde{v}}^m \wedge \alpha^{n-m} = \mu$ in B.*

Proof. Choose $C^0(\partial B) \ni \phi_j \searrow v$ on ∂B and solve the Dirichlet problem

$$\begin{cases} v_j \in SH_{\chi,m}(\alpha) \cap C^0(\bar{B}), \\ (\chi + dd^c v_j)^m \wedge \alpha^{n-m} = \mu, \\ v_j = \phi_j \text{ on } \partial B. \end{cases}$$

It follows from Corollary 3.2.2 that $v_j \searrow w \in SH_{\chi,m}(\alpha, B)$ satisfying

$$(\chi + dd^c w)^m \wedge \alpha^{n-m} = \mu.$$

Furthermore, $\limsup_{z \to \zeta \in \partial B} w(z) \leq v(\zeta)$. By the domination principle (Corollary 3.2.2) we have $v_j \geq v$ on B. Thus, $w \geq v$ on B. Define

$$\tilde{v} = \begin{cases} \max\{w, v\} \quad \text{on } B, \\ v \quad \text{on } \bar{M} \setminus B. \end{cases}$$

Then, $\tilde{v}$ is the function we are looking for. $\square$

Lemma 3.3.2. *$u \in SH_{\chi,m}(\alpha) \cap L^\infty(M) \cap C^0(\partial M)$ and $\chi_u^m \wedge \alpha^{n-m} = \mu$.*

Proof. It only remains to show that $\chi_u^m \wedge \alpha^{n-m} = \mu$. Fix a small ball $B \subset M$ and consider the lift $\tilde{u} \in \hat{\mathcal{S}}$ of u with respect to this ball. Then, $u \leq \tilde{u}$ in B. On the other hand by the definition of u we have $\tilde{u} \leq u$. Thus, $u = \tilde{u}$ in B. Since B is arbitrary, $\chi_u^m \wedge \alpha^{n-m} = \mu$ on M. □

We shall prove the most technical part.

Lemma 3.3.3. *u is continuous on $\bar{M}$.*

By Lemma 3.3.2, the function u satisfies the (bounded) Dirichlet problem:

$$w \in SH_{\chi,m}(\alpha) \cap L^\infty(M),$$

$$(\chi + dd^c w) \wedge \alpha^n = \mu,$$

$$\lim_{z \to \zeta} w(z) = 0 \quad \text{for every } \zeta \in \partial M.$$

Proof of Lemma 3.3.3. We follow closely [55, Section 2.4]. We argue by contradiction. Suppose u is not continuous, then the discontinuity of u occurs at an interior point of M. Hence

$$d = \sup_{\bar{M}} (u - u_*) > 0,$$

where $u_*(z) = \lim_{\epsilon \to 0} \inf_{w \in B(z,\epsilon)} u(w)$ is lower regularisation of u. Consider the closed nonempty set

$$F = \{u - u_* = d\} \subset\subset M.$$

We remark that F is closed and $u_{|F}$ is continuous on F. Therefore, we may choose a point $x_0 \in F$ such that

$$u(x_0) = \min_F u.$$

Choose a local coordinate chart about x_0, relatively compact in M, which is isomorphic to a small ball $B := B(0,r) \subset \mathbb{C}^n$ with origin at $z(x_0) = 0$ and of small radius. Since $\chi \in \Gamma_m(\alpha)$, there exists $\delta > 0$ such that

$$\gamma(z) := \chi(z) - \delta dd^c |z|^2 \in \Gamma_m(\alpha) \tag{3.3.5}$$

for every $z \in \bar{B}$. Set

$$v := u + \delta |z|^2 \in SH_{\gamma,m}(\alpha).$$

Since $u \geq 0$ on M, $v_*(0) = u_*(0) \geq 0$. Hence, we have $v \in L^\infty(\bar{B})$, which solves

$$(\gamma + dd^c v)^m \wedge \alpha^{n-m} = \mu. \tag{3.3.6}$$

We also find that

$$\sup_{\bar{B}}(v - v_*) = \sup_{\bar{B}}(u - u_*) = u(0) - u_*(0) = d.$$

Let us consider the sublevel sets, for $0 < s < d$,

$$E(s) = \{u_* \leq u - d + s\} \cap \bar{B}. \tag{3.3.7}$$

It's clear that $E(s)$ is closed and by our assumption $0 \in E(s)$. Furthermore,

$$E(s) \searrow E(0) = \{u_* = u - d\} \cap \overline{B(0,r)} \ni 0.$$

Let us denote

$$\tau(s) = u(0) - \inf_{E(s)} u(z).$$

Since $E(s)$ is decreasing, it follows that $\tau(s)$ decreasing as $s \searrow 0$. Moreover, $\tau(s)$ is bounded for $0 \leq s \leq d$. We also need the following fact.

Claim 3.3.5. $\lim_{s\to 0} \tau(s) = 0$.

Proof of Claim 3.3.5. It is easy to see that $\liminf_{s\to 0} \tau(s) \geq \tau(0) = 0$. It is enough to show that $\limsup_{s\to 0} \tau(s) \leq 0$. Suppose that it is not true, i.e.,

$$\limsup_{s\to 0} \tau(s) = 2\epsilon > 0$$

for some $\epsilon > 0$. Then, there exists a sequence $s_j \to 0$ such that $\tau(s_j) > \epsilon$ for every integer $j > 0$. It means that

$$\inf_{E(s_j)} u < u(0) - \epsilon.$$

Therefore, there is a sequence $\{z_j\}_{j\geq 1} \subset E(s_j)$ satisfying $u(z_j) < u(0) - \epsilon$. Since any limit point z of $\{z_j\}_{j\geq 1}$ belongs to $E(0)$, $u(z) \geq u(0)$. Hence,

$$\limsup_{j\to\infty} u(z_j) \leq u(0) - \epsilon \leq u(z) - \epsilon.$$

The upper semicontinuity of $-u_*$ gives

$$\limsup_{j\to+\infty}[-u_*(z_j)] \le -u_*(z).$$

Hence, $d = \limsup_{j\to+\infty}[u(z_j) - u_*(z_j)] \le u(z) - \epsilon - u_*(z) = d - \epsilon$. This is not possible and the claim follows. $\square$

Take $B' = B(0, r')$ with a bit larger $r' > r$. By the approximation property in a small ball (Theorem 3.2.6), one can find a sequence

$$SH_{\gamma,m}(\alpha) \cap C^\infty(B') \ni v_j \searrow v = u + \delta|z|^2 \quad \text{in } B'. \tag{3.3.8}$$

Let us fix this sequence from now on. If there is no otherwise indication then v and v_j's are these functions. The following result is a variation of the Hartogs lemma (Lemma 3.5.7).

Lemma 3.3.4. *Let $K \subset \bar{B}$ be a compact set and $c \ge 1$ a constant. Assume that for some $t > 0$,*

$$v < c\, v_* + t \quad \textit{on } K.$$

Then

$$v_j < c\, v + t \quad \textit{on } K$$

for $j > j_0$ with a fixed $j_0 > 0$ depending only on K, t.

Proof of Lemma 3.3.4. Let $z_0 \in K$. It follows from the assumption that $z_0 \in \{v - c\, v_* < t\}$ which is an open set by the upper semicontinuity of $v - c\, v_*$. Thus, $z_0 \in \{v - c\, v_* < t'\}$ for some $0 < t' < t$. Hence, $v(z_0) - c\, v_*(z_0) < t'$, i.e., by definition

$$\lim_{\epsilon'\to 0}\left(\sup_{B(z_0,2\epsilon')} v - c \inf_{B(z_0,2\epsilon')} v\right) < t'.$$

Therefore, for $0 < t_1 = \frac{t-t'}{2}$, there exists $\epsilon' = \epsilon'(t_1, z_0) > 0$ such that

$$B(z_0, 2\epsilon') \subset \{v < v_* + t\},$$

and $\sup_{B(z_0,2\epsilon')} v - c\inf_{B(z_0,2\epsilon')} v \le t' + t_1$. It implies that

$$\sup_{\bar{B}(z_0,\epsilon')} v \le c\,v + t' + t_1 \quad \text{on } \bar{B}(z_0,\epsilon').$$

By Hartogs' lemma for $(\gamma,1)-\alpha$-subharmonic functions (Corollary 3.5.4),

$$v_j \le \sup_{\bar{B}(z_0,\epsilon')} v + t_1 < c\,v + t' + 2t_1 = c\,v + t.$$

for $j \ge j(t_1, z_0, \epsilon')$. Because K is compact it is covered by a finite many balls $B(z_j,\epsilon'_j)$. Thus, the proof follows. □

We wish to apply Theorem 3.2.4 for the function v and its approximants $v_j's$ defined in (3.3.8) to get a contradiction. Therefore, we need to study the value of v and v_j's on the boundary ∂B. More precisely, we are going to show that there exists $c > 1$, $a > 0$ and s_0, which are independent of j, such that

$$\{c\,v + d - a + s < v_j\} \tag{3.3.9}$$

is non-empty and relatively compact in $B = B(0,r)$ for every $0 < s < s_0$. For this purpose we need to analyse the value of the function $c\,v - v_j$ on the boundary $S(0,r)$ of $B(0,r)$, with the help of Lemma 3.3.4.

Take two parameters $c > 1$ and $0 < a < d$ which are determined later. We need to estimate

$$c\,v + d - a - v_j$$

on $S(0,r)$. Recall that $v = u + \delta|z|^2$ and

$$\begin{aligned} E(s) &= \{u_* \le u - d + s\} \cap \overline{B(0,r)} \\ &= \{v_* \le v - d + s\} \cap \overline{B(0,r)}. \end{aligned}$$

We consider two cases:

Case 1: $z \in S(0,r) \cap E(a)$. We have

$$\begin{aligned} v_*(z) &= u_*(z) + \delta r^2 \\ &\ge u(z) - d + \delta r^2 \\ &= (u(z) - u(0)) + (u(0) - d) + \delta r^2. \end{aligned}$$

As $0 \in E(a)$, we have $\tau(a) \geq u(0) - u(z)$. Combining with $u(0) - u_*(0) = d$, we get that

$$v_*(z) \geq v_*(0) - \tau(a) + \delta r^2.$$

Note that $r > 0$ (small) is already fixed. It implies that, for $c > 1$,

$$v(z) \leq v_*(z) + d < c\, v_*(z) + d - (c-1)\left[v_*(0) + \delta r^2 - \tau(a)\right].$$

Since $v - cv_*$ is upper semicontinuous,

$$v < c\, v_* + d - (c-1)\left[v_*(0) + \delta r^2 - \tau(a)\right]$$

on the closure of a neighbourhood V of $S(0,r) \cap E(a)$. Applying Lemma 3.3.4 for the compact set $\bar{V} \cap \bar{B}$ and

$$t := d - (c-1)\left[v_*(0) + \delta r^2 - \tau(a)\right] > 0, \tag{3.3.10}$$

we get

$$v_j < c\, v + d - (c-1)\left[v_*(0) + \delta r^2 - \tau(a)\right] \quad \text{on } \bar{V} \cap \bar{B}, \tag{3.3.11}$$

if $j > j_1(V)$.

Case 2: $z \in S(0,r) \setminus V$. Since $E(a) \cap (S(0,r) \setminus V) = \emptyset$, the inequality

$$v < v_* + d - a$$

holds on $S(0,r) \setminus V$. Applying Lemma 3.3.4 again, we get

$$v_j < v + d - a < c\, v + d - a \quad \text{on } S(0,r) \setminus V \tag{3.3.12}$$

for $j > j_2(V)$. Thus, it follows from (3.3.11) and (3.3.12) that

$$v_j < c\, v + d - \min\left\{a, (c-1)\left[v_*(0) + \delta r^2 - \tau(a)\right]\right\} \tag{3.3.13}$$

on $S(0,r)$ for $j > \max\{j_1, j_2\}$.

Next, if there exists $c > 1$ such that for $0 < s_0 < a$,

$$(c-1)v_*(0) < a - s_0 \tag{3.3.14}$$

then $c\,v_*(0)+d-(a-s_0) < v(0) \leq v_j(0)$. It follows that the set $\{c\,v+d-a+s < v_j\}$ is non-empty for $0 < s < s_0$.

According to Claim 3.3.5, (3.3.10), (3.3.13) and (3.3.14) we need to choose $0 < a < d$, $c > 1$ and $0 < s_0 < a$, in this order, such that

$$\begin{aligned}
&\tau(a) \leq \frac{\delta r^2}{2};\\
&d-(c-1)\left[v_*(0)+\delta r^2-\tau(a)\right] > 0;\\
&(c-1)v_*(0) < a < (c-1)\left(v_*(0)+\frac{\delta r^2}{2}\right);\\
&s_0 = \frac{a-(c-1)v_*(0)}{2} > 0.
\end{aligned}$$

This is always possible. Thus, we get relatively compact subsets that satisfy (3.3.9).

Now we can apply Theorem 3.2.4 to get that a contradiction. In fact, we have for $w_j := v_j/c$ and $0 < s < s_0$,

$$\{c\,v+d-a+s < v_j\} = \{v+(d-a+s)/c < w_j\} \subset\subset B.$$

It follows that

$$d_j := \sup_B (w_j - v) \geq \frac{d-a+s_0}{c} > 0.$$

We denote for $0 < s < \varepsilon_0 < \varepsilon$ (as in Theorem 3.2.4),

$$U_j(\varepsilon, s) := \{v < (1-\varepsilon)w_j + \inf_{\Omega}[v-(1-\varepsilon)w_j] + s\}.$$

Notice that ε_0 depends only on d, a, s_0. Hence, applying Theorem 3.2.4 for v in (3.3.6) and γ in (3.3.5), we get that for $0 < s < \varepsilon_0$,

$$s \leq C(1+\|v\|_\infty)\|f\|_p^{\frac{1}{m}}[V_\alpha(U_j(\varepsilon,s))],$$

where $V_\alpha(U_j(\varepsilon,s)) = \int_{U_j(\varepsilon,s)} \alpha^n$. Furthermore, for such a fixed $s > 0$,

$$U_j(\varepsilon,s) \subset \{v < w_j - d_j + \varepsilon\|w_j\|_\infty + s\} \subset \{v < v_j\}.$$

Since $V_\alpha(\{v < v_j\}) \to 0$ as $j \to +\infty$, we get the contradiction. The proof of Lemma 3.3.3 is finished. $\square$

3.3.3 Some applications

The first application is the mixed type inequality for Hessian operators with the Hermitian form. When both χ and ω are Kähler metrics the inequality is proved by Dinew and Lu [31]. Since the inequality is local, we state it for a small Euclidean ball B in $\mathbb{C}^n$.

Proposition 3.3.1. *Let $f, g \in L^p(B)$, $p > n/m$. Suppose that $u, v \in SH_{\chi,m}(\alpha) \cap C^0(\bar{B})$ satisfy*

$$\chi_u^m \wedge \alpha^{n-m} = f\alpha^n, \quad \chi_v^m \wedge \omega^{n-m} = g\alpha^n. \tag{3.3.15}$$

Then, for any $0 \leq k \leq m$,

$$\chi_u^k \wedge \chi_v^{m-k} \wedge \alpha^{n-m} \geq f^{\frac{k}{m}} g^{\frac{m-k}{m}} \alpha^n. \tag{3.3.16}$$

Proof. It is a simple consequence of the mixed type inequality in the smooth case, and then for continuous functions we use Theorem 1.0.2 and Lemma 3.2.7. □

Thanks to this type of inequality with $\chi = \alpha = \omega$ we are able to relax the smoothness assumption on potentials in the statement of [60, Proposition 3.16]. In particular, the uniqueness of continuous solutions to the complex Hessian equation on compact Hermitian manifolds with strictly positive right hand side in L^p, $p > n/m$ follows.

Corollary 3.3.1. *Let (X, ω) be a compact Hermitian manifold. Suppose that $u, v \in SH_m(\omega) \cap C^0(X)$, $\sup_X u = \sup_X v = 0$, satisfy*

$$\omega_u^m \wedge \omega^{n-m} = f\omega^n, \quad \omega_v^m \wedge \omega^{n-m} = g\omega^n, \tag{3.3.17}$$

where $f, g \in L^p(X, \omega^n)$, $p > n/m$. Assume that

$$f \geq c_0 > 0 \tag{3.3.18}$$

for some constant c_0. Fix $0 < a < \frac{1}{m+1}$. Then,

$$\|u - v\|_{L^\infty} \leq C\|f - g\|_{L^p}^a, \tag{3.3.19}$$

where the constant C depends on $c_0, a, p, \|f\|_{L^p}, \|g\|_{L^p}, \omega, X$.

We can also show that continuous solutions obtained in [60] are also the continuous solutions in the viscosity sense and vice versa (Lu [65] proved the existence and uniqueness of viscosity solutions to the complex Hessian equation on some special compact Hermitian manifolds). The viscosity approach for the Monge-Ampère equation on Kähler manifolds was used by Eyssidieux, Guedj, Zeriahi [35], Wang [86]. It seems to be interesting to investigate the viscosity method for the complex Hessian equation on compact Hermitian manifolds with or without boundary. We refer the readers to [48, Example 18.1], [49, Example 3.2.7] for some partial results in this direction.

3.4 Proof of Theorem 1.0.2

In this section we proceed to prove Theorem 1.0.2, which we used in Sections 3.2, 3.3. The proof is independent of results in those sections.

Let us rewrite the equation in the PDE form as in the paper by Székelyhidi [80]. Without loss of generality we fix $\Omega := B(0,\delta) \subset B(0,1) \subset \mathbb{C}^n$ for $0 < \delta << 1$. Let α be a Hermitian metric in $B(0,1)$. Fix a smooth real $(1,1)$-form χ on $B(0,1)$. For a C^2 function u we consider the real $(1,1)$-form $g = \chi + \sqrt{-1}\partial\bar{\partial}u$, i.e., $g_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$. We can define $A^i_j := \alpha^{\bar{p}i} g_{j\bar{p}}$, where $\alpha^{\bar{j}i}$ is the inverse of $\alpha_{i\bar{j}}$. Then, the matrix A^i_j is Hermitian with respect to the metric α, i.e., $A \times [\alpha_{i\bar{j}}]$ is a Hermitian matrix. Denote $\lambda(A) = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ the n-tuple of eigenvalues of A. In other words, λ is the eigenvector of $g_{i\bar{j}}$ with respect to the metric α. The complex Hessian equation (1.0.1) is

$$F(A) = h,$$

where

$$F(A) := f(\lambda(A)) = [(S_m(\lambda)]^{1/m},$$

and f is a symmetric increasing concave function defined on the cone Γ_m. Recall that the m-th elementary symmetric cone is

$$\Gamma_m = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, ..., S_m(\lambda) > 0\}.$$

Fix $0 < h \in C^\infty(\bar{\Omega})$ and a smooth boundary data $\varphi \in C^\infty(\partial\Omega)$. We wish to study the Dirichlet problem, seeking $u \in C^\infty(\bar{\Omega})$ and $u = \varphi$ smooth on $\partial\Omega$ such that

$$\begin{cases} \lambda(A) & \in \Gamma_m, \\ F(A) & = h, \end{cases} \tag{3.4.1}$$

where $A^i_j = \alpha^{\bar{p}i}(\chi_{j\bar{p}} + u_{j\bar{p}})$. To simplify notation, first we extend $\varphi \in C^\infty(\partial\Omega)$ smoothly to $\overline{B(0,1)}$. Upon replacing

$$\tilde{u} := u - C(|z|^2 - \delta^2) - \varphi,$$

$$\tilde{\chi} := \chi + \sqrt{-1}\partial\overline{\partial}[C(|z|^2 - \delta^2) + \varphi],$$

with $C > 0$ large enough, which does not change $g_{i\bar{j}}$, we may assume that

$$u = 0 \quad \text{on } \partial\Omega, \quad \chi \geq \alpha \quad \text{on } \bar{\Omega} \tag{3.4.2}$$

and 0 is the subsolution, i.e., $\chi^m \wedge \alpha^{n-m} \geq h$.

Let $F^{ij}(A) := \partial F/\partial a_{ij}$ be the partial derivative of F at A with respect to entry a_{ij}. We also denote

$$\mathcal{F} := \sum_{1 \leq i \leq n} f_i,$$

where $f_i = \partial f/\partial \lambda_i > 0$ are precisely eigenvalues of F^{ij} with respect to metric α. If we choose coordinates in which α is orthonormal and A being diagonal, then

$$F^{ij} = \delta_{ij} f_i,$$

and thus $\mathcal{F} = \sum_{i=1}^n F^{ii}$.

We will proceed in Sections 3.4.1, 3.4.2, 3.4.3 to get *a priori* estimates, up to second order, and then, using the results in Tosatti-Weinkove-Wang-Yang [82], to get $C^{2,\alpha}$ interior estimates. This combined with the C^2 estimates on the boundary thus gives full C^2 estimates up to the boundary.

Then we can apply Krylov's boundary estimate to get the desired $C^{2,\alpha}(\bar{\Omega})$ estimate. The higher order estimates are obtained by the bootstrapping argument, and then using the continuity method we obtain a solution to the equation (3.4.1). The uniqueness follows from the maximum principle.

3.4.1 C^0–estimate

Denote $B^i_j = \alpha^{\bar{p}i}\chi_{j\bar{p}}$. Then, $F(A) = h \leq F(B)$ and $u \geq 0$ on $\partial\Omega$. Solve the linear PDE

$$\begin{cases} n(\chi + \sqrt{-1}\partial\overline{\partial}u_1) \wedge \alpha^{n-1}/\alpha^n & = 0, \\ u_1 & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By the maximum principle we get that for some $C_0 > 0$,

$$0 \leq u \leq u_1 \leq C_0. \tag{3.4.3}$$

As $u = u_1 = 0$ on $\partial\Omega$, it also follows that for some $C_0' > 0$,

$$|\nabla u| \leq C_0' \quad \text{on } \partial\Omega. \tag{3.4.4}$$

3.4.2 C^1–estimate

In this section we prove the gradient estimate. Here the assumption of small radius is important. (Notice that Pliś [77] has claimed this estimate in the case $\chi \equiv 0$ and α Kähler for any ball but no proof was given there.)

By (3.4.2) we may suppose that for some $C_1 > 0$,

$$\frac{\delta_{ij}}{C_1} \leq \alpha_{i\bar{j}} \leq \chi_{i\bar{j}} \leq C_1\delta_{ij}. \tag{3.4.5}$$

$$L := \sup_{\Omega} |u| + 1.$$

Let ∇ denote the Chern connection with respect to α. Note that $\|z\|^2_\alpha$ is strictly plurisubharmonic as long as δ small. More precisely, we choose δ so that

$$\nabla_{\bar{p}}\nabla_p\|z\|^2_\alpha = \partial_{\bar{p}}\partial_p(\alpha_{i\bar{j}}z^i\bar{z}^j) = \alpha_{p\bar{p}} + O(|z|) \geq \alpha_{p\bar{p}}/2. \tag{3.4.6}$$

Denote $v = N(\sup_{z\in\Omega} \|z\|^2_\alpha - \|z\|^2_\alpha)$, where $N > 0$ is a constant to be determined later. We see that

$$0 \le v \le NC_1\delta^2 \quad \text{and} \quad - v_{p\bar{p}} = -\partial_p\partial_{\bar{p}} v \ge N/2C_1. \tag{3.4.7}$$

Consider

$$G = \log \|\nabla u\|^2_\alpha + \psi(u+v),$$

with

$$\psi(t) = -\frac{1}{2}\log(1 + \frac{t}{L + NC_1\delta^2}).$$

Note that a similar function was considered by Hou-Ma-Wu [53] and it satisfies

$$\psi' < 0, \quad \psi'' = 2\psi'^2. \tag{3.4.8}$$

If G attains its maximum at a boundary point, then $\sup_\Omega |\nabla u|$ is uniformly bounded by $\sup_{\partial\Omega} |\nabla u|$, up to a uniform constant. By (3.4.4), the latter one is uniformly bounded. Then, we will get the C^1- estimate. Therefore, we may assume that the maximum point belongs to Ω. We shall derive the desired estimate by using maximum principle at this point.

We choose the orthonormal coordinates for α such that at this point $\alpha_{i\bar{j}}$ is the identity matrix and A^i_j is diagonal. All computations bellow are performed at this point and here the subscripts stand for usual derivatives if there is no otherwise indication.

Differentiating G twice and evaluating the equations at the maximum point we have:

$$G_p = \frac{(\nabla_p\nabla_i u)u_{\bar{i}} + u_i\nabla_p\nabla_{\bar{i}}u}{|\nabla u|^2} + \psi'(u_p + v_p) = 0; \tag{3.4.9}$$

$$\begin{aligned} G_{p\bar{p}} = {} & \frac{(\nabla_{\bar{p}}\nabla_p\nabla_i u)u_{\bar{i}} + u_i\nabla_{\bar{p}}\nabla_p\nabla_{\bar{i}}u + |\nabla_p\nabla_i u|^2 + |\nabla_{\bar{p}}\nabla_i u|^2}{|\nabla u|^2} \\ & - \frac{1}{|\nabla u|^4}\left|u_i\nabla_p\nabla_{\bar{i}}u + u_{\bar{i}}\nabla_p\nabla_i u\right|^2 \\ & + \psi''|u_p + v_p|^2 + \psi'(u_{p\bar{p}} + v_{p\bar{p}}). \end{aligned} \tag{3.4.10}$$

Next, we have

$$\begin{aligned} \nabla_{\bar{p}}\nabla_p\nabla_i u &= u_{p\bar{p}i} - (\partial_{\bar{p}}\Gamma^q_{pi})u_q - \Gamma^q_{pi}u_{q\bar{p}} \\ &= g_{p\bar{p}i} - \chi_{p\bar{p}i} - (\partial_{\bar{p}}\Gamma^q_{pi})u_q - \Gamma^p_{pi}\lambda_p + \Gamma^q_{pi}\chi_{q\bar{p}}, \end{aligned} \tag{3.4.11}$$

where we used that $g_{i\bar{j}}$ is diagonal. Similarly,

$$\begin{aligned}\nabla_{\bar{p}}\nabla_p\nabla_{\bar{i}}u &= u_{p\bar{p}\bar{i}} - \overline{\Gamma^q_{pi}}u_{p\bar{q}} \\ &= g_{p\bar{p}\bar{i}} - \chi_{p\bar{p}\bar{i}} - \overline{\Gamma^p_{pi}}\lambda_p + \overline{\Gamma^q_{pi}}\chi_{p\bar{q}}.\end{aligned}$$

Moreover, by applying the covariant derivatives to the equation we get

$$F^{pp}\nabla_i g_{p\bar{p}} = h_i.$$

As $\nabla_i g_{p\bar{p}} = g_{p\bar{p}i} - \Gamma^m_{ip}g_{m\bar{p}}$, we have $F^{pp}g_{p\bar{p}i} = h_i + F^{pp}\Gamma^p_{ip}\lambda_p$. Combining with (3.4.11) we get that

$$\begin{aligned}F^{pp}(\nabla_{\bar{p}}\nabla_p\nabla_i u)u_{\bar{i}} = {} & h_i u_{\bar{i}} + F^{pp}(\Gamma^p_{ip} - \Gamma^p_{pi})\lambda_p u_{\bar{i}} - F^{pp}\chi_{p\bar{p}i}u_{\bar{i}} \\ & - F^{pp}(\partial_{\bar{p}}\Gamma^q_{pi})u_q u_{\bar{i}} + F^{pp}\Gamma^q_{pi}\chi_{q\bar{p}}u_{\bar{i}}.\end{aligned} \tag{3.4.12}$$

Similarly,

$$\begin{aligned}F^{pp}(\nabla_{\bar{p}}\nabla_p\nabla_{\bar{i}}u)u_i = {} & h_{\bar{i}}u_i + F^{pp}(\overline{\Gamma^p_{ip}} - \overline{\Gamma^p_{pi}})\lambda_p u_i \\ & - F^{pp}\chi_{p\bar{p}\bar{i}}u_i - F^{pp}\overline{\Gamma^q_{pi}}\chi_{p\bar{q}}u_i\end{aligned} \tag{3.4.13}$$

Let's denote

$$R := \sup_{p,q,i}|\partial_{\bar{p}}\Gamma^q_{pi}|, \quad T := \sup_{i,p}|\Gamma^p_{ip} - \Gamma^p_{pi}|,$$

which are bounds for the curvature and torsion of metric α on $\bar{B}(0,1)$.

It follows from (3.4.12) and (3.4.13) that, for $K := |\nabla u|^2$ large enough,

$$\begin{aligned}&\frac{1}{K}F^{pp}[(\nabla_{\bar{p}}\nabla_p\nabla_i u)u_{\bar{i}} + (\nabla_{\bar{p}}\nabla_p\nabla_{\bar{i}}u)u_i] \\ &\geq -C/K^{1/2} - F^{pp}|\lambda_p|T/K^{1/2} - C\mathcal{F}/K^{1/2} - R\mathcal{F} \\ &\geq -C - \frac{1}{2K}F^{pp}\lambda_p^2 - (R + T^2 + 1)\mathcal{F},\end{aligned} \tag{3.4.14}$$

where in the last inequality we used

$$\frac{|\lambda_p|T}{K^{1/2}} \leq \frac{1}{2}(\frac{\lambda_p^2}{K} + T^2).$$

By the equation (3.4.9)

$$-\frac{1}{K^2}|u_i\nabla_p\nabla_{\bar{i}}u + u_{\bar{i}}\nabla_p\nabla_i u|^2 = -\psi'^2|u_p + v_p|^2. \tag{3.4.15}$$

By $\sum_{p=1}^{n} f_p\lambda_p = h$ and $\chi_{p\bar{p}} \geq 1$ we have

$$\begin{aligned} \psi' F^{pp}(u_{p\bar{p}} + v_{p\bar{p}}) &= \psi' F^{pp}\lambda_p + |\psi'|F^{pp}[\chi_{p\bar{p}} + (-v_{p\bar{p}})] \\ &\geq -C + |\psi'|[1 + N/2C_1]\mathcal{F}. \end{aligned} \tag{3.4.16}$$

We also note that

$$\begin{aligned} \frac{1}{K}F^{pp}|\nabla_{\bar{p}}\nabla_i u|^2 &= \frac{1}{K}F^{pp}|g_{i\bar{p}} - \chi_{i\bar{p}}|^2 \\ &\geq \frac{1}{2K}F^{pp}|\lambda_p|^2 - \frac{1}{K}F^{pp}|\chi_{i\bar{p}}|^2 \\ &\geq \frac{1}{2K}F^{pp}|\lambda_p|^2 - \frac{C\mathcal{F}}{K}. \end{aligned} \tag{3.4.17}$$

Therefore, combining (3.4.10), (3.4.14), (3.4.15), (3.4.16) and (3.4.17), we get that

$$\begin{aligned} 0 \geq F^{pp}G_{p\bar{p}} \geq -C &- \frac{1}{2K}F^{pp}|\lambda_p|^2 - (R + T^2 + 1)\mathcal{F} \\ &+ \frac{1}{2K}F^{pp}|\lambda_p|^2 - \frac{C\mathcal{F}}{K} \\ &+ (\psi'' - \psi'^2)F^{pp}|u_p + v_p|^2 \\ &+ |\psi'|[1 + N/2C_1]\mathcal{F}. \end{aligned}$$

We may assume that $K > C$. As $\psi'' = 2\psi'^2$, we simplify the inequality:

$$0 \geq \psi'^2 F^{pp}|u_p + v_p|^2 + |\psi'|(1 + N/2C_1)\mathcal{F} - (R + T^2 + 2)\mathcal{F} - C. \tag{3.4.18}$$

Now we decrease further δ (if necessary) so that $16(R+T^2+3)C_1^2\delta^2 < 1$. Hence, we can choose $N > 1$ satisfying

$$\frac{N}{8(LC_1 + NC_1^2\delta^2)} \geq R + T^2 + 3.$$

On the interval $t \in [0, L + NC_1\delta^2]$, we have $|\psi'| \geq 1/4(L + NC_1\delta^2)$. Hence,

$$\frac{N|\psi'|}{2C_1} \geq R + T^2 + 3. \tag{3.4.19}$$

It follows from (3.4.18) and (3.4.19) that

$$F^{pp}|u_p + v_p|^2 + \mathcal{F} \leq C, \tag{3.4.20}$$

where $C = C(A, C_1, L)$. We shall use (3.4.20) to prove that

$$F^{ii} = \frac{S_m^{-1+1/m}(\lambda)}{m}S_{m-1;i}(\lambda) \geq c > 0$$

for some uniform c and for every $1 \le i \le n$. Indeed, since

$$\mathcal{F} = \frac{S_m^{-1+1/m}(\lambda)}{m} \sum_{i=1}^{n} S_{m-1;i}(\lambda) \le C,$$

we have $S_{m-1;i}(\lambda) \le C$ for every $i = 1, ..., n$. By the inequality [85, Proposition 2.1 (4)]

$$\prod_{i=1}^{n} S_{m-1;i}(\lambda) \ge C_{n,m}[S_m(\lambda)]^{n(m-1)/m},$$

where $C_{n,m} > 0$ depends only on n, m. Thus, the desired lower bound for each $S_{m-1;i}(\lambda)$ follows from the equation $(S_m(\lambda))^{\frac{1}{m}} = h > 0$ and the upper bound for $S_{m-1;i}(\lambda)$. We also get the lower bound for each F^{ii}. Finally, from

$$F^{pp}|u_p + v_p|^2 \le C$$

we easily get the a priori gradient bound, $|\nabla u| \le C$.

3.4.3 C^2-estimate

In this section we prove the following estimate

$$\sup_{\bar{\Omega}} |\sqrt{-1}\partial\bar{\partial}u| \le C, \tag{3.4.21}$$

where C depends on $\|u\|_{L^\infty(\bar{\Omega})}$, $\|\nabla u\|_{L^\infty(\bar{\Omega})}$ and the given data.

If $\sup_{\bar{\Omega}} |\partial\bar{\partial}u|$ is attained at an interior point of Ω, then by a result of Székelyhidi [80] (see also Zhang [88]) we have for some $C > 0$, which depends on $\|u\|_\infty$ and the given data,

$$|\sqrt{-1}\partial\bar{\partial}u| \le C(1 + \sup_{\bar{\Omega}} |\nabla u|^2).$$

Therefore, we only need to consider the case when the maximum point P is on the boundary. At this point, following Boucksom [14], we choose a local half-ball coordinate U such that $z(P) = 0$ and r is the defining function for $U \cap \partial\Omega$. Then, $U \cap \Omega = \{r < 0\} \cap U$. We choose the coordinates $z = (z_1, ..., z_n)$, centred at 0, such that the positive x_n axis is the interior normal direction, and near 0 the graph $U \cap \partial\Omega$ is written as

$$r = -x_n + \sum_{j,k=1}^{n} a_{jk} z_j \bar{z}_k + O(|z|^3) = 0. \tag{3.4.22}$$

We refer the reader to the expository paper of Boucksom [14] for more details on this coordinate.

Recall that λ_i's are eigenvalue functions of matrix A, i.e.

$$\lambda(A) = (\lambda_1, ..., \lambda_n).$$

We often represent quantities in the orthonormal coordinates $(w^1, .., w^n)$ in which $\alpha_{i\bar{j}}$ is the identity and A^i_j is diagonal. The following equations will help us in computing quantities in the orthonormal coordinates once we know theirs forms in the fixed coordinates $(z^1, ..., z^n)$.

Suppose at a given point we change the coordinates, $w = Xz$, i.e.

$$w^i = x_{ik} z^k, \quad x_{ik} \in \mathbb{C},$$

and we obtain at that point

$$\begin{aligned} \alpha_{i\bar{j}} \sqrt{-1} dz^i \wedge d\bar{z}^j &= \sum_{a=1}^{n} \sqrt{-1} dw^a \wedge d\bar{w}^a; \\ g_{i\bar{j}} \sqrt{-1} dz^i \wedge d\bar{z}^j &= \sum_{a=1}^{n} \lambda_a \sqrt{-1} dw^a \wedge d\bar{w}^a. \end{aligned}$$

It follows that

$$\alpha_{i\bar{j}} = x_{ai} \overline{x_{aj}}, \quad g_{i\bar{j}} = x_{ai} \lambda_a \overline{x_{aj}}.$$

It is clear that for every $1 \leq i \leq n$,

$$\sum_{a=1}^{n} |x_{ai}|^2 = \alpha_{i\bar{i}} < C.$$

Moreover, the inverse of matrix $\alpha_{i\bar{j}}$ is given by the formula

$$\alpha^{\bar{j}i} = \overline{x^{ja}} x^{ia},$$

where x^{ia} is the inverse of X. Hence,

$$A^i_j = \alpha^{\bar{p}i} g_{j\bar{p}} = x^{ia} \lambda_a x_{aj}.$$

In $\mathbb{C}^{n \times n}$ if we change coordinates $B = XAX^{-1} = (b_{kl})$, then at the considered point B is a diagonal matrix $(\lambda_1, ..., \lambda_n)$. Therefore, λ_a is smooth at the diagonal matrix B (see e.g. [79]) and

$$\frac{\partial F}{\partial b_{kl}} = \frac{\partial f}{\partial \lambda_a} \cdot \frac{\partial \lambda_a}{\partial b_{kl}} = f_a \delta_{ak} \delta_{al};$$

$$\frac{\partial F}{\partial a_{ij}} = \frac{\partial F}{\partial b_{kl}} \frac{\partial b_{kl}}{\partial a_{ij}} = \sum_{k,l} \sum_{a=1}^{n} f_a \delta_{ak} \delta_{al} x_{ki} x^{jl} = x^{ja} f_a x_{ai}.$$

An easy consequence from the above formula is that

$$L^{\bar{p}j} := F^{ij} \alpha^{\bar{p}i} = \overline{x^{pa}} f_a x^{ja},$$

where $F^{ij} = \partial F/\partial a_{ij}$ at A^i_j, is a positive definite Hermitian matrix.

To derive the desired *a priori* estimate we will use the linearised elliptic operator, for a smooth function w,

$$Lw := L^{\bar{p}j} \partial_j \partial_{\bar{p}} w = F^{ij} \alpha^{\bar{p}i} \partial_j \partial_{\bar{p}} w,$$

It is worth to recall that

$$\mathcal{F} := \sum_{1 \le i \le n} f_i$$

where $f_i = \partial f/\partial \lambda_i$ are eigenvalues of F^{ij} with respect to metric α.

Following Guan [39] (c.f. Boucksom [14]) we construct the important barrier function.

Lemma 3.4.1. *Set $b = u - r - \mu r^2$. Then, there exist constants $\mu > 0$ and $\tau > 0$ such that*

$$Lb \le -\frac{1}{2}\mathcal{F}$$

and $b \ge 0$ on the half-ball coordinate U of radius $|r| < \tau$.

Proof. By shrinking the radius of the half coordinate ball U, we have r is plurisubharmonic in U. Then

$$0 \le Lr = L^{\bar{p}j} r_{j\bar{p}} \le C\mathcal{F}. \tag{3.4.23}$$

As $b_{j\bar{p}} := \partial_j \partial_{\bar{p}} b$ is a Hermitian matrix and $\alpha_{i\bar{j}} > 0$, we can represent

$$b_{j\bar{p}} = x_{aj} \gamma_a \overline{x_{ap}},$$

where $\gamma_a \in \mathbb{R}$ are eigenvalues of $b_{i\bar{j}}$ with respect to the matrix $\alpha_{i\bar{j}}$. Hence,

$$Lb = \sum_{a=1}^{n} f_a \gamma_a$$

which does not depend on the choice of coordinates of α. Thus, to verify the desired inequality at a given point, we compute, at this point, in orthonormal coordinates of α and $A^i_j = \alpha^{\bar{p}i}(\chi_{j\bar{p}} + u_{j\bar{p}}) = (\lambda_1, ..., \lambda_n)$ diagonal. So is $L^{\bar{p}i} = (f_1, ..., f_n)$.

We now compute, as $r \leq 0$,

$$
\begin{aligned}
Lb &= L^{\bar{i}i}u_{i\bar{i}} - Lr - 2\mu r Lr - 2\mu L^{\bar{i}i}|r_i|^2 \\
&= L^{\bar{i}i}g_{i\bar{i}} - L^{\bar{i}i}\chi_{i\bar{i}} - Lr + 2\mu|r|Lr - 2\mu L^{\bar{i}i}r_i^2 \\
&= \sum_{i=1}^{n} f_i\lambda_i + (2\mu|r| - 1)Lr - L^{\bar{i}i}(\chi_{i\bar{i}} + 2\mu r_i^2). \qquad (3.4.24)
\end{aligned}
$$

We have $\sum_{i=1}^n f_i\lambda_i = h$ and

$$(2\mu|r| - 1)Lr \leq 2C\mu|r|\mathcal{F}. \qquad (3.4.25)$$

Notice that $\chi_{i\bar{i}} \geq \alpha_{i\bar{i}} = 1$. The last negative term (3.4.24) will be divided into three parts. First

$$-L^{\bar{i}i}\chi_{i\bar{i}}/2 \leq -\mathcal{F}/2.$$

Next, we use $-L^{\bar{i}i}\chi_{i\bar{i}}/4$ to absorb the right hand side of (3.4.25) (i.e. the second term in (3.4.24)), provided that

$$C\mu|r| \leq 1/8.$$

We will use the part $-L^{\bar{i}i}(\frac{\chi_{i\bar{i}}}{4} + 2\mu r_i^2)$ for μ large to absorb the first term in (3.4.24). We claim that

$$L^{\bar{i}i}(\frac{\chi_{i\bar{i}}}{4} + 2\mu r_i^2) \geq c_0\mu^{\frac{1}{m}} \qquad (3.4.26)$$

for some uniform $c_0 > 0$. In fact, we observe that $|\nabla r| > 0$ at 0. Decreasing τ if necessary,

$$|\nabla r|^2 = \sum_{i=1}^{n} r_i^2 > c_1 \qquad (3.4.27)$$

for a uniform $c_1 > 0$ on U. By Gårding inequality with $\lambda' = (\chi_{1\bar{1}}/4 + 2\mu r_1^2, ..., \chi_{n\bar{n}}/4 + 2\mu r_n^2)$ and

$S_{m-1;i}(\lambda)$, we have

$$\begin{aligned}
\sum_{i=1}^{n}(\frac{\chi_{i\bar{i}}}{4}+2\mu|r_i|^2)S_{m-1;i}(\lambda) &\geq m[S_m(\chi_{i\bar{i}}/4+2\mu|r_i|^2)]^{\frac{1}{m}}[S_m(\lambda)]^{\frac{m-1}{m}} \\
&\geq \frac{m\mu^{\frac{1}{m}}h^{m-1}}{4^{\frac{m-1}{m}}}\left(\sum_{i=1}^{n}2|r_i|^2\prod_{k\neq i}\chi_{k\bar{k}}\right)^{\frac{1}{m}} \\
&\geq \frac{m\mu^{\frac{1}{m}}h^{m-1}}{4^{\frac{m-1}{m}}}\left(\sum_{i=1}^{n}2|r_i|^2\right)^{\frac{1}{m}} \\
&\geq \frac{2^{\frac{1}{m}}m\mu^{\frac{1}{m}}h^{m-1}}{4^{\frac{m-1}{m}}}c_1^{\frac{1}{m}},
\end{aligned}$$

where we used $\chi_{k\bar{k}} \geq 1$ for the third inequality and used (3.4.27) for the last inequality.

To obtain the claim (3.4.26), we only need to notice that

$$L^{\bar{i}i} = f_i = \frac{[S_m(\lambda)]^{(1-m)/m}S_{m-1;i}(\lambda)}{m}.$$

Therefore, the uniform constant we get is $c_0 = C(c_1, h, m) > 0$. So we can choose $\mu > 0$ large enough we get our goal. Therefore we get the required inequality for Lb.

It remains to check that $b \geq 0$. Since $u \geq 0$ it is enough to have that

$$-r-\mu r^2 = |r|(1-\mu|r|) \geq 0.$$

This easily follows by further decreasing (if necessary) the radius τ of the half-ball coordinate. □

We are ready to prove the second order estimates for u at the boundary point $0 \in \partial\Omega$. Following Caffarelli, Nirenberg, Kohn, Spruck [17] (c.f [14]) we set

$$t_1 = x_1, t_2 = y_1, ..., t_{2n-2} = y_{n-1}, t_{2n-1} = y_n, t_{2n} = x_n.$$

Let $D_1, ..., D_{2n}$ be the dual basis of $dt_1, ..., dt_{2n-1}, -dr$, then

$$D_j = \frac{\partial}{\partial t_j} - \frac{r_{t_j}}{r_{x_n}}\frac{\partial}{\partial x_n} \quad \text{for} \quad 1 \leq j < 2n,$$

and

$$D_{2n} = -\frac{1}{r_{x_n}}\frac{\partial}{\partial x_n}.$$

Because $u = 0$ on $\partial\Omega$, we can write, for some positive function σ,

$$u = \sigma r.$$

Then,

$$\partial u / \partial x_n(0) = -\sigma(0). \tag{3.4.28}$$

So, $|\sigma(0)| < C$. Moreover, for $1 \leq j \leq 2n-1$,

$$\frac{\partial^2 u}{\partial t_i \partial t_j}(0) = \sigma(0) \frac{\partial^2 r}{\partial t_i \partial t_j}(0)$$

and hence tangential-tangential derivatives $|\partial_{t_i} \partial_{t_j} u|$ are under control.

Next, we bound normal-tangential derivatives:

Theorem 3.4.1. *We have*

$$\left| \frac{\partial^2 u}{\partial t_j \partial x_n}(0) \right| \leq C \quad \textit{for} \quad j \leq 2n-1,$$

where C depends on $u, |\nabla u|$ and the given datum.

Proof. Without loss of generality we fix $j = 1$ and we shall show that

$$|D_{2n} D_1 u(0)| \leq C.$$

The derivative D_1, acting on functions, is equal to

$$\partial_1 + \partial_{\bar{1}} + \tilde{r}(\partial_n + \partial_{\bar{n}}),$$

where ∂ denotes the usual partial derivatives and $\tilde{r} := -\frac{r_{x_1}}{r_{x_n}}$ is a smooth real-valued function near 0. Recall that we use the subindex to denote usual derivatives in direction $\partial/\partial z_1, ..., \partial/\partial z_n$ and their conjugates if there is no other indication. This gives

$$D_1 u = u_1 + u_{\bar{1}} + \tilde{r}(u_n + u_{\bar{n}}).$$

Following Caffarelli, Nirenberg, Spruck [18] and Guan [39, 40], our goal is to construct a function of form

$$w = D_1 u - \sum_{k<n} |u_k|^2 - |u_n - u_{\bar{n}}|^2 + \mu_1 b + \mu_2 |z|^2,$$

satisfying the following:

(i) $w(0) = 0$;

(ii) $w \geq 0$ on ∂U;

(iii) $Lw = L^{\bar{p}j}\partial_j\partial_{\bar{p}}w \leq 0$ in the interior of U,

where b is the barrier function constructed in Lemma 3.4.1, constants $\mu_1, \mu_2 > 0$ are to be determined later.

To see the first property (i) we note that, for $i < n$,

$$2u_i(0) = \frac{\partial u}{\partial x_i}(0) - \sqrt{-1}\frac{\partial u}{\partial y_i}(0) = 0,$$

and

$$u_n(0) - u_{\bar{n}}(0) = -\sqrt{-1}\frac{\partial u}{\partial y_n}(0) = 0.$$

Moreover, $D_1 u(0) = b(0) = \tilde{r}(0) = 0$. Therefore, the first property follows.

Next, we verify the second property (ii). We claim that there exists a constant $\mu_2 > 0$ such that

$$w \geq 0 \quad \text{on} \quad \partial U.$$

To see this consider two parts $\partial\Omega \cap U$ and $\partial U \setminus (\partial\Omega \cap U)$ of the boundary ∂U separately.

Part 1: On $\partial\Omega \cap U$. We know that $D_1 u = b = 0$, and near 0

$$x_n = \sum_{j,k=1}^{n} a_{jk} z_j \bar{z}_k + O(|z|^3).$$

By writing $x_n = \rho(t_1, ..., t_{2n-1}) = \rho(t)$ we deduce that

$$\rho(t) = \sum_{i,j<2n} k_{ij} t_i t_j + O(|t|^3),$$

where $(k_{ij}) = \left[\frac{\partial^2 x_n}{\partial t_i \partial t_j}(0)\right]$ is uniformly bounded. Since $u(t, \rho(t)) = 0$,

$$\partial u/\partial t_i + \partial u/\partial x_n \cdot \partial\rho/\partial t_i = 0$$

for $i < 2n$. Applying for $y_n = t_{2n-1}$ gives

$$|\partial u/\partial y_n|^2 \leq C|t|^2 \leq C|z|^2.$$

Similarly, for $i < n$,

$$|u_i|^2 \leq C|z|^2.$$

Therefore, $w \geq 0$ on $\partial\Omega \cap U$ for $\mu_2 > 0$ large enough.

Part 2: On $\partial U \setminus (\partial\Omega \cap U)$. On this piece $|z|^2 = \tau^2$ with τ being the radius of U. Since $b \geq 0$ on U, we have $w \geq 0$ as soon as

$$\mu_2\tau^2 \geq |D_1 u| + \sum_{i=1}^{n} |u_i|^2.$$

This is done by choosing $\mu_2 > 0$ large as the right hand side is under control by the C^1-estimate. Thus, the second property is satisfied.

To verify the third property (iii), $L^{\bar{p}j} w_{j\bar{p}} \leq 0$ in the interior of U, we fix an interior point $z_0 \in U$. Below we compute at this fixed point. The estimation will be split into several steps.

(1) Estimate for $D_1 u$. We start by computing

$$\begin{aligned} L^{\bar{p}j}(D_1 u)_{j\bar{p}} &= L^{\bar{p}j}\left[u_1 + u_{\bar{1}} + \tilde{r}(u_n + u_{\bar{n}})\right]_{j\bar{p}} \\ &= L^{\bar{p}j}[u_{1j\bar{p}} + u_{\bar{1}j\bar{p}} + \tilde{r}(u_{ni\bar{p}} + u_{\bar{n}j\bar{p}})] \\ &\quad + L^{\bar{p}j}[\tilde{r}_j(u_n + u_{\bar{n}})_{\bar{p}} + \tilde{r}_{\bar{p}}(u_n + u_{\bar{n}})_j] \\ &\quad + L^{\bar{p}j}\tilde{r}_{j\bar{p}}(u_n + u_{\bar{n}}) \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.4.29}$$

Let us denote $K := \sup_\Omega |\nabla u|^2$, which is bounded by the C^1-estimate.

Lemma 3.4.2. *There exists a constant C depending only on α such that for any fixed j, q,*

$$|L^{\bar{p}j} g_{q\bar{p}}| \leq C \sum_{i=1}^{n} f_i |\lambda_i|.$$

Similarly,

$$|L^{\bar{j}p} g_{p\bar{q}}| \leq C \sum_{i=1}^{n} f_i |\lambda_i|.$$

Proof. Recall that we have $\alpha_{i\bar{j}} = x_{ai}\overline{x_{aj}}$, $g_{i\bar{j}} = x_{ai}\lambda_a\overline{x_{aj}}$, and $L^{\bar{p}j} = \overline{x^{pa}}f_a x^{ja}$. Therefore,

$$L^{\bar{p}j}g_{q\bar{p}} = \overline{x^{pa}}f_a x^{ja}x_{bq}\lambda_b\overline{x_{bp}} = x^{ja}f_a\lambda_a x_{aq}.$$

Thus, the conclusion follows. The second inequality is proved in the same way. □

(1a) Estimate I_2 and I_3. We first easily have

$$\begin{aligned} |I_3| = |L^{\bar{p}j}\tilde{r}_{j\bar{p}}(u_n + u_{\bar{n}})| &\le CK^{\frac{1}{2}}\mathcal{F} \\ &\le C\mathcal{F}. \end{aligned} \tag{3.4.30}$$

Since two terms in I_2 are conjugate, so we will estimate one of them. We proceed as follows:

$$\begin{aligned} \tilde{r}_j(u_n + u_{\bar{n}})_{\bar{p}} &= \tilde{r}_j[2u_n - (u_n - u_{\bar{n}})]_{\bar{p}} \\ &= 2\tilde{r}_j u_{n\bar{p}} - \tilde{r}_j(u_n - u_{\bar{n}})_{\bar{p}} \\ &= 2\tilde{r}_j g_{n\bar{p}} - 2\tilde{r}_j\chi_{n\bar{p}} - \tilde{r}_j V_{\bar{p}}, \end{aligned}$$

where we wrote $V = u_n - u_{\bar{n}}$.

By Lemma 3.4.2, we have for $\mathcal{F}|\lambda| := \sum_i f_i|\lambda_i|$,

$$|2L^{\bar{p}j}\tilde{r}_j g_{n\bar{p}}| \le C|L^{\bar{p}j}g_{n\bar{p}}| \le C\mathcal{F}|\lambda|.$$

A straightforward estimate gives

$$|2L^{\bar{p}j}\tilde{r}_j\chi_{n\bar{p}}| \le C\mathcal{F}.$$

Cauchy-Schwarz's inequality implies that

$$\begin{aligned} |L^{\bar{p}j}\tilde{r}_j V_{\bar{p}}| &\le \frac{1}{2}L^{\bar{p}j}\tilde{r}_j\tilde{r}_{\bar{p}} + \frac{1}{2}L^{\bar{p}j}(\bar{V})_j V_{\bar{p}} \\ &\le C\mathcal{F} + \frac{1}{2}L^{\bar{p}j}(\bar{V})_j V_{\bar{p}}. \end{aligned}$$

Thus, the above estimates give

$$|I_2| \le C(\mathcal{F} + \mathcal{F}|\lambda|) + L^{\bar{p}j}(\bar{V})_j V_{\bar{p}}. \tag{3.4.31}$$

(1b) Estimate I_1. We have

$$u_{1j\bar{p}} = u_{j\bar{p}1} = g_{j\bar{p}1} - \chi_{j\bar{p}1}.$$

Covariant differentiation in direction $\partial/\partial z_1$ of the equation $F(A) = h$ gives

$$F^{ij}\alpha^{\bar{p}i}\nabla_1 g_{j\bar{p}} = L^{\bar{p}j}\left[g_{j\bar{p}1} - \Gamma^q_{1j} g_{q\bar{p}}\right] = h_1. \tag{3.4.32}$$

It follows that

$$\begin{aligned} |L^{\bar{p}j} u_{1j\bar{p}}| &= |L^{\bar{p}j}(g_{j\bar{p}1} - \chi_{j\bar{p}1})| \\ &= |h_k + L^{\bar{p}j}\Gamma^q_{1j} g_{q\bar{p}} - L^{\bar{p}j}\chi_{j\bar{p}1}| \\ &\le C(1+\mathcal{F}) + |L^{\bar{p}j}\Gamma^q_{1j} g_{q\bar{p}}| \\ &\le C(1+\mathcal{F}+\mathcal{F}|\lambda|), \end{aligned} \tag{3.4.33}$$

where we used Lemma 3.4.2 for the last inequality.

The remaining terms in I_1 are estimated similarly, when the index 1 is replaced by $\bar{1}, \bar{n}$ or n. Therefore,

$$|I_1| \le C(1+\mathcal{F}+\mathcal{F}|\lambda|). \tag{3.4.34}$$

Combining (3.4.30), (3.4.31) and (3.4.34) yields

$$|L^{\bar{p}j}(D_1 u)_{j\bar{p}}| \le C(1+\mathcal{F}+\mathcal{F}|\lambda|) + L^{\bar{p}j}(\bar{V})_j V_{\bar{p}}. \tag{3.4.35}$$

We continue to estimate the other terms in the formula for w.

(2) Estimate for $-\sum_{k<n}|u_k|^2$**.** By computing

$$(u_k u_{\bar{k}})_{j\bar{p}} = u_{kj\bar{p}} u_{\bar{k}} + u_k u_{\bar{k}j\bar{p}} + u_{kj} u_{\bar{k}\bar{p}} + u_{k\bar{p}} u_{\bar{k}j}. \tag{3.4.36}$$

Similarly to the estimation of I_1, we have

$$\begin{aligned} \sum_{k<n} |L^{\bar{p}j}(u_{kj\bar{p}} u_{\bar{k}} + u_k u_{\bar{k}j\bar{p}})| &\le CK^{\frac{1}{2}}(1+\mathcal{F}+\mathcal{F}|\lambda|) \\ &\le C(1+\mathcal{F}+\mathcal{F}|\lambda|). \end{aligned}$$

For the third term, with k fixed, $L^{\bar{p}j} u_{kj} u_{\bar{k}\bar{p}} \ge 0$. The last term in (3.4.36) will give a good positive term. By using Lemma 3.4.2,

$$\begin{aligned} L^{\bar{p}j} u_{k\bar{p}} u_{\bar{k}j} &= L^{\bar{p}j}(g_{k\bar{p}} - \chi_{k\bar{p}})(g_{j\bar{k}} - \chi_{j\bar{k}}) \\ &\ge L^{\bar{p}j} g_{k\bar{p}} g_{j\bar{k}} - C(\mathcal{F}+\mathcal{F}|\lambda|). \end{aligned} \tag{3.4.37}$$

The following result is similar to Guan's [40, Proposition 2.19] in the real case.

Lemma 3.4.3. *There exists an index s such that*

$$\sum_{k<n} L^{\bar{p}j} g_{k\bar{p}} g_{j\bar{k}} \geq \frac{\min_i \tau_i}{2} \sum_{i \neq s} f_i \lambda_i^2,$$

where τ_i's are the eigenvalues of the matrix $\alpha_{i\bar{j}}$.

Proof. First at the given point let U be a unitary matrix such that $\alpha = U^t \Lambda \bar{U}$, where $\Lambda = diag(\tau_1, \ldots, \tau_n)$. Without loss of generality, we can assume $X = \Lambda^{\frac{1}{2}} U$, so that $\alpha = X^t \bar{X}$ and $x_{ij} = \tau_i^{\frac{1}{2}} u_{ij}$. Again we have formulas $\alpha_{i\bar{j}} = x_{ai} \overline{x_{aj}}$ and $\alpha^{\bar{i}j} = \overline{x^{ia}} x^{ja}$. Moreover,

$$L^{\bar{p}j} = \overline{x^{pa}} f_a x^{ja}, \quad g_{i\bar{j}} = x_{ib} \lambda_b \overline{x_{bj}}.$$

Thus, for a fixed $k < n$,

$$\begin{aligned} L^{\bar{p}j} g_{k\bar{p}} g_{j\bar{k}} &= \overline{x^{pa}} f_a x^{ja} \; x_{bk} \lambda_b \overline{x_{bp}} \; x_{cj} \lambda_c \overline{x_{ck}} \\ &= \sum_{i=1}^n f_i \lambda_i^2 |x_{ik}|^2. \end{aligned}$$

As

$$\sum_{k<n} |x_{ik}|^2 = \sum_{k=1}^n |x_{ik}|^2 - |x_{in}|^2 = \tau_i (1 - |u_{in}|^2),$$

we have

$$S := \sum_{k<n} L^{\bar{p}j} g_{k\bar{p}} g_{j\bar{k}} = \sum_{i=1}^n f_i \lambda_i^2 \tau_i (1 - |u_{in}|^2).$$

If for every $1 \leq i \leq n$ we have $|u_{in}|^2 \leq \frac{1}{2}$, then

$$S \geq \frac{\min_i \tau_i}{2} \sum_{i=1}^n f_i \lambda_i^2.$$

Otherwise, there exists an index s such that $|u_{sn}|^2 > \frac{1}{2}$. It follows that

$$\sum_{i \neq s} |u_{in}|^2 \leq \frac{1}{2}.$$

Then,

$$S = \sum_{i=1}^n f_i \lambda_i^2 \tau_i (1 - |u_{in}|^2) \geq \sum_{i \neq s} f_i \lambda_i^2 \tau_i (1 - |u_{in}|^2) \geq \frac{\min_i \tau_i}{2} \sum_{i \neq s} f_i \lambda_i^2.$$

Thus, the lemma follows. □

It follows from Lemma 3.4.3 and (3.4.37) that for some index s,

$$\sum_{k<n} L^{\bar{p}j} u_{k\bar{p}} u_{\bar{k}j} \geq \frac{\min_i \tau_i}{2} \sum_{i \neq s} f_i \lambda_i^2 - C(\mathcal{F} + \mathcal{F}|\lambda|).$$

Therefore,

$$L(-\sum_{k<n} |u_k|^2) \leq -\frac{\min_i \tau_i}{2} \sum_{i \neq s} f_i \lambda_i^2 + C(\mathcal{F} + \mathcal{F}|\lambda|). \tag{3.4.38}$$

(3) Estimate for $-|V|^2 = -|u_n - u_{\bar{n}}|^2$. We compute

$$(V\bar{V})_{j\bar{p}} = (u_{nj\bar{p}} - u_{\bar{n}j\bar{p}})\bar{V} + V(u_{\bar{n}j\bar{p}} - u_{nj\bar{p}})$$
$$+ V_j(\bar{V})_{\bar{p}} + (\bar{V})_j V_{\bar{p}}.$$

Since $L^{\bar{p}j} V_j (\bar{V})_{\bar{p}} \geq 0$, we get, similarly to (3.4.33), the following

$$L^{\bar{p}j}(-|V|^2)_{j\bar{p}} \leq -L^{\bar{p}j}(\bar{V})_j V_{\bar{p}} + C(1 + \mathcal{F} + \mathcal{F}|\lambda|). \tag{3.4.39}$$

Combining (3.4.35), (3.4.38) and (3.4.39) gives us

$$Lw \leq -\frac{\min_i \tau_i}{2} \sum_{i \neq r} f_i \lambda_i^2 + C(1 + \mathcal{F} + \mathcal{F}|\lambda|) + \mu_1 Lb + \mu_2 L(|z|^2).$$

By this and Lemma 3.4.1 we get that, for some index s,

$$Lw \leq -\frac{\min_i \tau_i}{2} \sum_{i \neq s} f_i \lambda_i^2 + C\mathcal{F}|\lambda| + (C + \mu_2 - \frac{\mu_1}{2})\mathcal{F}. \tag{3.4.40}$$

Recall that μ_2 was chosen to have the property (ii) and $\mu_1 > 0$ can be chosen freely. To achive the third property of w we need the following

Lemma 3.4.4. *Let $\varepsilon > 0$. There is a constant $C_\varepsilon > 0$ such that for any index s,*

$$\mathcal{F}|\lambda| = \sum_{i=1}^n f_i |\lambda_i| \leq \varepsilon \sum_{i \neq s} f_i \lambda_i^2 + C_\varepsilon \mathcal{F}.$$

Proof. Since $\sum_{i=1}^n f_i \lambda_i = h$, we have

$$\begin{aligned}
\mathcal{F}|\lambda| &\leq 2 \sum_{i \neq s} f_i |\lambda_i| + h \\
&\leq \sum_{i \neq s} f_i (\varepsilon \lambda_i^2 + \frac{1}{\varepsilon}) + h \\
&\leq \varepsilon \sum_{i \neq s} f_i \lambda_i^2 + C_\varepsilon \mathcal{F},
\end{aligned}$$

where we used the fact that $\mathcal{F}$ is uniformly bounded below by a positive constant. ∎

Using Lemma 3.4.4 we get from (3.4.40) that

$$Lw \leq (-\frac{\min_i \tau_i}{2} + C\varepsilon)\sum_{i\neq s} f_i\lambda_i^2 + (\mu_2 + C + C_\varepsilon - \frac{\mu_1}{2})\mathcal{F}.$$

Thus, we choose ε so small that the first term on the right hand side is negative, and then choose μ_1 so large that the second term is also negative. The third property (iii) is proved.

We are ready to conclude the bound for tangential-normal second derivatives. By the maximum principle we have $w \geq 0$ on U. Therefore, as $w(0) = 0$, $D_{2n}w(0) \geq 0$. It follows that

$$D_{2n}D_1u(0) \geq -C.$$

The properties $(i), (ii)$ and (iii) also hold, with the same argument, if we replace w by the function

$$\tilde{w} = -D_1u - \sum_{k<n}|u_k|^2 - |u_n - u_{\bar{n}}|^2 + \mu_1 b + \mu_2|z|^2.$$

Therefore, $D_{2n}D_1u(0) \leq C$. Thus, we get the desired bound for $|D_{2n}D_1u(0)|$. □

The last estimate we need is the normal-normal derivative bound.

Lemma 3.4.5. *We have*

$$\left|\frac{\partial^2 u}{\partial x_n^2}(0)\right| \leq C,$$

where C depends on $h, C_0, C_1,$ and the bounds of tangential-normal derivatives.

Proof. Since $4u_{n\bar{n}} = \partial^2u/\partial x_n^2 + \partial^2u/\partial y_n^2$, the normal-normal estimate is equivalent to

$$|u_{n\bar{n}}(0)| \leq C.$$

Moreover, as $|u_{i\bar{j}}| < C$ with $i + j < 2n$, we get that for $j < n$,

$$|A_j^i| = |\alpha^{\bar{p}i}(\chi_{j\bar{p}} + u_{j\bar{p}})| < C.$$

Hence, it follows from

$$\sum_{i=1}^n A_i^i = \sum_{i=1}^n \lambda_i \geq 0$$

that $A^n_n \geq -C$, so is $g_{n\bar{n}} \geq -C$. It implies that $u_{n\bar{n}} \geq -C$. Therefore, it remains to prove that $u_{n\bar{n}} \leq C$. By $u = \sigma r$, with $\sigma > 0$, we have for $j, k < n$,

$$u_{j\bar{k}}(0) = \sigma(0) r_{j\bar{k}}(0).$$

Let S be a $(n-1) \times (n-1)$ unitary matrix diagonalising $[u_{j\bar{k}}]_{j,k<n}$. It means that for $j, k < n$,

$$u_{j\bar{k}}(0) = \sum_p S^*_{jp} d_p S_{pk}$$

Since r is strictly plurisubharmonic in U, we get that $d_p > 0$, $p = 1, ..., n-1$. By elementary matrix computation we have for $D = (d_1, .., d_{n-1})$ a diagonal matrix and the column vector $V = (u_{1\bar{n}}, ..., u_{(n-1)\bar{n}})^t$,

$$\begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \times [u_{i\bar{j}}]_{i,j\leq n} \times \begin{pmatrix} S^* & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} D & SV \\ V^* S^* & u_{n\bar{n}} \end{pmatrix}.$$

By $|u_{j\bar{n}}|, |u_{n\bar{j}}| < C$ for $j < n$ and $\chi_{i\bar{j}} > 0$, we may assume that $u_{n\bar{n}}$ is so large (otherwise we are done) that $g_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}(0) > 0$, i.e., positive definite. So

$$\lambda_i(A) > 0$$

for every $i = 1, .., n$. Hence,

$$(\det A)^{\frac{1}{n}} \leq C_{m,n} [S_m(\lambda(A))]^{\frac{1}{m}} = C_{m,n} h.$$

By $\det g_{i\bar{j}} = \det \alpha_{i\bar{j}} \cdot \det A^i_j$ we get that $\det g_{i\bar{j}} \leq C$. Since

$$[g_{i\bar{j}}]_{i,j<n} \geq [\chi_{i\bar{j}}]_{i,j<n} > 0$$

and

$$\det g_{i\bar{j}} = g_{n\bar{n}} \det([g_{i\bar{j}}]_{i,j<n}) + O(1),$$

we have $g_{n\bar{n}} \leq C$. Thus, the normal-normal derivative bound at a boundary point is proven. □

Altogether, we have proven the C^2-estimate (3.4.21) and completed the proof of Theorem 1.0.2.

3.5 Appendix

The results in this section are classical. It is a natural generalisation of properties of subharmonic functions (see e.g [52]). However, we could not find the the precise forms that we need in the literature. Some of them have been pointed out recently by Harvey-Lawson [47]. Our setup here is simpler than the one in [47], therefore we have several finer properties. We emphasize here the use of a theorem of Littman [64]. For the readers' convenience we give results with proofs here.

3.5.1 Littman's theorem

We briefly recall a simpler version of a result of Littman [63, 64]. Roughly speaking it allows to approximate a generalised subharmonic function (with respect to a uniformly elliptic operator L) in a constructive way.

Let D be a smoothly bounded domain in $\mathbb{R}^n$, $n \geq 3$. Consider the partial differential operator L defined by

$$Lu = \left(b^{ij}u\right)_{x_i x_j} - \left(b^i(x)u\right)_{x_i}$$

and assumed to be uniformly elliptic there. Its formal adjoint L^* is given by

$$L^*v = b^{ij}(x)v_{x_i x_j} + b^i(x)v_{x_i},$$

where coefficients $b^{ij}(x)$, $b^i(x)$ are smooth function on D.

We say that $u \in L^1_{loc}(D)$ satisfies $Lu \geq 0$ weakly if

$$\int u(x)L^*v(x) \geq 0 \tag{3.5.1}$$

for any non-negative function v in $C^2(D)$ with compact support in D. The natural question is to find a sequence of smooth functions u_j such that $Lu_j \geq 0$ and u_j decreases to u. The usual convolution with a smooth kernel will not give us the desired sequence.

Before stating Littman's theorem let us introduce some notations. We denote by $g(x, y)$ the Green function of the operator L_x with respect to domain D and with singularity at $y \in D$; as constructed for example in [71]. The subindex x means that L acts on functions of x. The basic properties of g are:

$$L_x^* g(x,y) = 0 \quad \text{on } D \setminus \{y\},$$

and

$$g(x,y) = O\big(|x-y|^{2-n}\big) \text{ as } x \to y.$$

In particular, $g(x,y) \to \infty$ as $x \to y$. Furthermore, let us denote $\Delta = \{(x,x) : x \in \bar{D}\}$, then

$$g \in C^0(\bar{D} \times \bar{D} \setminus \Delta) \cap C^2(D \times D \setminus \Delta);$$

also $g(x,y) = 0$ for $x \in \partial D$ and a fixed $y \in D$. If we denote $r = |x-y|$. Then

$$g(x,y) = O(r^{2-n}), \quad g_{x_i} = O(r^{1-n}), \quad g_{x_i x_j} = O(r^{-n}).$$

Fix a function $p(t) = 1 - t^2$ for $t \in \mathbb{R}$. So $p(0) = 1$ and $p(t) \geq p_0 > 0$ for $|t| < \delta_0$ small enough. It is easy to see that

$$L_x^* p(|x-y|) < 0 \quad \text{for } |x-y| < 2\delta_0 \text{ and } x, y \in D.$$

Let $\Phi(t) \geq 0$ be a smooth function vanishing outside $(0,1)$ and positive in the interior, such that

$$\Phi(t) \to 0 \quad \text{exponetially, as } t \to 0, 1;$$
$$\int_{-\infty}^{+\infty} \Phi(t)dt = 1.$$

For $\delta > 0$ we write $D_\delta = \{z \in D : dist(z, \partial D) > \delta\}$. For $h \geq 0$, $x \in D$, $y \in D_{2\delta}$ we define a function $G_h(x,y)$ on $D \times D_{2\delta}$ by letting

$$G_h(x,y) := 0 \quad \text{for} \quad |x-y| \geq 2\delta,$$

and for $|x-y| < 2\delta$,

$$G_h(x,y) := \int_{-\infty}^{+\infty} \Phi(s-h) \max\{g(x,y) - sp(|x-y|), 0\} ds.$$

Notice that $G_h(x,y) = 0$ for $|x-y| \geq \delta$, if we take $h \geq h_\delta$, where

$$h_\delta := \frac{1}{p_0} \max\big\{g(x,y) : \delta \leq |x-y| \leq 2\delta, (x,y) \in D \times D_{2\delta}\big\}. \tag{3.5.2}$$

Another remark is that

$$G_h(x,y) - g(x,y) = -g(x,y)\int_{g/p}^{+\infty}\Phi(s-h)ds - p\int_{-\infty}^{g/p}\Phi(s-h)sds \tag{3.5.3}$$

is continuous for $x \in \bar{D}$ and $y \in D_{2\delta}$ and it belongs to $C^2(D \times D_{2\delta})$ as the rate of $g(x,y)$ growing to $+\infty$ is polynomially while $\Phi(t) \to 0$ exponentially. In particular, $G_h(x,y) \to +\infty$ as $x \to y$ with the same order of growth as $g(x,y)$.

By a direct computation we get that

$$\frac{\partial G_h}{\partial h} = -p\int_{-\infty}^{g/p}\Phi(s-h)ds \leq 0. \tag{3.5.4}$$

The formula also shows that $\frac{\partial G_h}{\partial h} \in C^2(D)$ and compactly supported as a function of x. Hence,

$$J_h := \int L_x^* G_h dx = 1$$

for every $h \geq h_\delta$. Indeed, by the property [64, 4.f] we have $\lim_{h\to\infty} J_h = 1$, and for any constant c we have $Lc = 0$. Therefore,

$$\partial J_h/\partial h = \int L_x^*\big(\partial G_h/\partial h\big)dx = \int \partial G_h/\partial h \; L_x 1 = 0.$$

Since coefficients $b^{ij}(x), b^i(x)$ are smooth, we have

$$G_h(x,y) - g(x,y) \in C^2(D_{2\delta})$$

as a function of y uniformly with respect to x (c.f [64, 4.e]). Hence, G_h is a *Levi function* satisfying

$$L_x^* G_h(x,y) = O(|x-y|^{\lambda-n})$$

for any $0 < \lambda \leq 1$ (c.f [71, (8.5) p. 18]). Therefore, for $u \in L^1_{loc}(D)$ and $h \geq h_\delta$,

$$u_h(y) = \int u(x)L_x^*G_h(x,y)dx = \int_{|x-y|\leq\delta} u(x)L_x^*G_h(x,y)dx \tag{3.5.5}$$

is well defined. Notice that the support of $G_h(x,y)$, as a function in x, shrinks to y as $h \to +\infty$.

We are ready to state a theorem of Littman [64].

Theorem 3.5.1 (Littman). *Let $u \in L^1_{loc}(D)$ be such that $Lu \geq 0$ weakly in D in the sense of* (3.5.1). *Then, $\{u_h(x)\}_{h \geq h_\delta}$, defined by* (3.5.5), *are smooth functions satisfying:*

- *$Lu_h \geq 0$;*
- *u_h is a nonincreasing sequence as $h \to +\infty$, u_h converges to u in $L^1(D_{2\delta})$;*
- *$U(x) := \lim_{h\to\infty} u_h(x)$ is upper semicontinuous, and $U(x) = u(x)$ almost everywhere.*

3.5.2 Properties of $\omega-$subharmonic functions

Let ω be a Hermitian metric on a bounded open set $\Omega \subset \mathbb{C}^n$. Let us denote

$$\Delta_\omega := \omega^{\bar{j}i}(z)\frac{\partial^2}{\partial z^i \partial \bar{z}^j}. \tag{3.5.6}$$

We first recall

Definition 3.5.2. A function $u : \Omega \to [-\infty, +\infty[$ is called $\omega-$subharmonic if

(a) u is upper semicontinuous and $u \in L^1_{loc}(\Omega)$.

(b) for every relatively compact open set $D \subset\subset \Omega$ and every $h \in C^0(\bar{D})$ satisfying $\Delta_\omega h = 0$ in D, if $h \geq u$ on ∂D, then $h \geq u$ on $\bar{D}$.

As in the case of subharmonic functions we have the following properties.

Proposition 3.5.1. *Let Ω be a bounded open set in $\mathbb{C}^n$.*

(a) If $u_1 \geq u_2 \geq \cdots$ is a decreasing sequence of $\omega-$subharmonic functions, then $u := \lim_{j\to+\infty} u_j$ is either $\omega-$subharmonic or $\equiv -\infty$.

(b) If u, v belong to $SH(\omega)$, then so does $\max\{u, v\}$.

Proof. (a) is obvious. We shall prove (b). It is rather standard (see [45]), but probably it is not so well known. We include the proof for the sake of completeness. Observe that

$$\max\{u, v\} = \lim_{j\to+\infty} \frac{\log(e^{ju} + e^{jv})}{j}$$

By a simple computation we get that

$$dd^c \log(e^u + e^v) = \frac{e^u dd^c u + e^v dd^c v}{(e^u + e^v)} + \frac{e^{u+v} d(u-v) \wedge d^c(u-v)}{(e^u + e^v)^2}.$$

It follows that $\frac{1}{j}\log(e^{ju} + e^{jv})$ is ω-subharmonic. So is $\max\{u, v\}$. $\square$

The subharmonicity is a local notion meaning that a function is subharmonic in a open set if and only if at every point there exists a neighbourhood such that the function is subharmonic in that neighbourhood. The precise statement is

Proposition 3.5.2. *The following statements are equivalent for an upper semicontinuous and locally integrable function u in Ω.*

(1) u is an $\omega-$subharmonic function in Ω.

(2) In a neighbourhood U of a given point a, if $q \in C^2(U)$ such that $q - u \geq 0$ and $q(a) = u(a)$, then

$$\Delta_\omega q(a) \geq 0.$$

Proof. (1) $\Rightarrow$ (2). We argue by contradiction. Suppose that there exist a neighbourhood U of a point a and $q \in C^2(U)$ satisfying $q \geq u$ and $q(a) = u(a)$, but

$$\Delta_\omega q(a) < 0.$$

By Taylor's formula we may assume that q is quadratic and there exists $\varepsilon > 0$ such that $\Delta_\omega q < -\varepsilon$ on a small ball $B(a, r)$. Solve

$$\Delta_\omega v(z) = -\Delta_\omega q, \quad v = 0 \quad \text{on } \partial B(a, r).$$

Notice that by maximum principle we get that $v(a) < 0$. Let $h = v + q$. Then, $\Delta_\omega h = 0$, and $h \geq u$ on $\bar{B}(a, r)$. However, $h(a) = u(a) + v(a) < u(a)$, which is impossible. The first direction follows.

(2) $\Rightarrow$ (1). We also argue by contradiction. Suppose that there exist an open set $D \subset\subset \Omega$ and a function $h \in C^0(\bar{D})$ and $\Delta_\omega h = 0$ in D, which satisfies $u \leq h$ on ∂D, such that $\{u > h\}$ is

non-empty. Without loss of generality we may assume that D is a small ball B and h is continuous on $\bar{B}$. Set for $\varepsilon > 0$

$$v_\varepsilon(z) = h(z) - \varepsilon |z|^2.$$

Then, the upper semicontinuous function $(u - v_\varepsilon)$ takes its maximum at a point $a \in B$, so

$$u(z) \leq v_\varepsilon(z) + u(a) - v_\varepsilon(a) \quad \text{for } z \in B.$$

By Taylor's formula

$$\begin{aligned} h(z) &= h(a) + \Re(P(z)) + \frac{1}{2}\frac{\partial^2 h}{\partial z_i \partial \bar{z}_j}(a)(z_i - a_i)\overline{(z_j - a_j)} + O(|z-a|^3) \\ &=: H(z) + O(|z-a|^3), \end{aligned}$$

where $P(z)$ is a holomorphic polynomial. Therefore, $\Delta_\omega H(a) = 0$. Consider the function

$$q(z) = u(a) - v_\varepsilon(a) + H(z) - \varepsilon|z|^2 + \frac{\varepsilon}{2}|z-a|^2.$$

Then, it is easy to check that $\Delta_\omega q(a) < 0$, $q(a) = u(a)$ and $q(z) \geq u(z)$ in a neighbourhood of a. This is impossible and the proof is completed. $\square$

Since $\omega-$subharmonicity is a local property we easily get the gluing lemma.

Lemma 3.5.1. *Let $U \subset V$ be two open sets. Let $u \in SH(\omega, U)$ and $v \in SH(\omega, V)$. Assume that*

$$\limsup_{z \to \zeta} u(z) \leq v(\zeta) \quad \forall \zeta \in \partial U \cap V. \tag{3.5.7}$$

Then, $\tilde{u} \in SH(\omega, V)$, where

$$\tilde{u} = \begin{cases} \max\{u, v\} & \text{on } U, \\ v & \text{on } V \setminus U. \end{cases}$$

Proof. Consider

$$u_\varepsilon = \begin{cases} \max\{u, v + \varepsilon\} & \text{on } U, \\ v + \varepsilon & \text{on } V \setminus U. \end{cases}$$

If $x \in U$, then there is a small ball $B(x,r) \subset U$. Hence,

$$u_\varepsilon = \max\{u, v+\varepsilon\} \tag{3.5.8}$$

is ω-subharmonic in $B(x,r)$. Similarly, for $x \in V \setminus U$ by the assumption on $\partial U \cap V$, there is $B(x,r) \subset V$ such that $u_\varepsilon = v + \varepsilon$ on $B(x,r)$. Thus, $u_\varepsilon \in SH(\omega, V)$. Since $u_\varepsilon \searrow u$ we can apply Proposition 3.5.1 getting the lemma. □

The proposition above shows that we only need to check the $\omega-$subharmonicity of a function on a small ball, but it is not clear whether sum of two subarmonic functions is again subharmonic. We shall need another criterion.

By linear PDEs potential theory, e.g. see [71], for any ball $B(a,r)$, there exists a Poisson kernel $P_{a,r}$ for the operator Δ_ω. Namely, for every continuous function φ on $\partial B(a,r)$, the function

$$h(z) = \int_{\partial B(a,r)} \varphi(w) P_{a,r}(z,w) d\sigma_r(w),$$

is the unique continuous solution to the Dirichlet problem

$$\Delta_\omega h(z) = 0 \quad \text{in } B(a,r), \quad h = \varphi \text{ on } \partial B(a,r),$$

where $d\sigma_r(z)$ is the standard surface measure on $\partial B(a,r)$.

Lemma 3.5.2. *Let $u : \Omega \to [-\infty, +\infty[$ be a locally integrable upper semicontinuous function. For $\Omega_\delta = \{z \in \Omega : dist(z, \partial\Omega) > \delta\}$, $\delta > 0$, consider the function*

$$M(u,a,r) = \int_{\partial B(a,r)} u(z) P_{a,r}(a,z) d\sigma_r(z), \quad a \in \Omega_\delta,$$

where $r \in [0, \delta]$. Then, u is an $\omega-$subharmonic function if and only if

$$u(a) \le M(u,a,r)$$

for $a \in \Omega_\delta$, $r \in [0,\delta]$. Furthermore, $M(u,a,r)$ decreases to $u(a)$ as r goes to 0.

Proof. We first prove that it is a necessary condition. Take $\phi \geq u$ to be a continuous function on $\partial B(a, r)$. Then,

$$h(z) = \int_{\partial B(a,r)} \phi(w) P_{a,r}(z, w) d\sigma_r(w)$$

satisfies $\Delta_\omega h = 0$ and $h = \phi \geq u$ on $\partial B(a, r)$. It follows from definition that $h \geq u$ on $B(a, r)$. In particular,

$$u(a) \leq \int_{\partial B(a,r)} \phi(w) P_{a,r}(a, w) d\sigma_r(w).$$

As u is upper semicontinuous, we can let $\phi \searrow u$. By monotone convergence theorem we get the desired inequality.

Now we prove the other direction by contradiction. Assume that there exist a relatively compact open set $D \subset \Omega$, $h \in C^0(\bar{D})$ with $\Delta_\omega h = 0$ and $h \geq u$ on ∂D, but

$$c := \sup_{\bar{D}} (u - h) > 0.$$

As $v = u - h$ is upper semicontinuous, c is finite and $F := \{v = c\}$ is a compact set in D. We choose $a \in F$ such that it is the closest point to the boundary ∂D. Assume that $dist(a, \partial D) = 2\delta > 0$. Since there exists $x \in B(a, \delta)$ such that $v(x) < c$, so there is $B(x, \epsilon') \subset \{v < c - \epsilon\} \cap B(a, \delta)$ for some $\epsilon, \epsilon' > 0$. It follows from $\Delta_\omega h = 0$ on D that

$$v(a) \leq M(v, a, r) \quad \forall z \in B(a, r), \forall r \leq \delta.$$

Notice that in our case $\Delta_\omega 1 = 0$ and

$$\int_{\partial B(a,r)} P_{a,r}(z, w) d\sigma_r(w) = 1.$$

Integrating from 0 to δ we get that

$$\delta v(a) \leq \int_{[0,\delta]} \int_{\partial B(a,r)} v(z) P_r(a, z) d\sigma_r(z) dr < \delta c.$$

This is impossible. Thus, the sufficient condition is proved.

For the last assertion, let $0 \leq r < \delta$. Fix a continuous function $\phi \geq u$ on $\partial B(0, \delta)$. As $\Delta_\omega h = 0$ in $B(a, \delta)$ for

$$h(z) = \int_{\partial B(a,\delta)} \phi(w) P_\delta(z, w) d\sigma_\delta(w),$$

we get that $u(z) \leq h(z)$ on $B(a, \delta)$. Therefore,

$$M(u, a, r) \leq \int_{\partial B(a,r)} h(w) P_r(a, w) d\sigma_r(w) = h(a). \tag{3.5.9}$$

Moreover,

$$h(a) = \int_{\partial B(a,\delta)} \phi(w) P_\delta(a, w) d\sigma_\delta(w).$$

Letting $\phi \searrow u$, we get the monotocity of $M(u, a, r)$ in $r \in [0, \delta]$. Moreover, as u is upper semicontinuous,

$$\lim_{r \to 0} M(u, a, r) \leq u^*(a) = u(a),$$

where we used the fact above that $\int_{\partial B(a,r)} P_r d\sigma_r = 1$. □

An immediate consequence of the last assertion in the above lemma is

Corollary 3.5.1. *If two $\omega-$subharmonic functions are equal almost everywhere, then they are equal everywhere.*

We are ready to state a consequence of Littman's theorem, which says that we can always find a smooth approximation for $\omega-$subharmonic functions.

Corollary 3.5.2. *Let $u \in SH(\omega, \Omega)$ and $\Omega' \subset\subset \Omega$. There exists a sequence of smooth $\omega-$subharmonic functions $[u]_\varepsilon$ decreasing to u as $\varepsilon \to 0$ on Ω'.*

Proof. We simply choose a smooth domain $D \subset \Omega$ and $\delta > 0$ small such that $\Omega' \subset D_{2\delta}$ and let

$$[u]_\varepsilon(z) := u_h(z) \tag{3.5.10}$$

where $u_h(z)$, $h = 1/\varepsilon > h_\delta$, is defined in Theorem 3.5.1. As $U(z) := \lim_\varepsilon [u]_\varepsilon$ is equal to $u(z)$ almost everywhere and u is also $\omega-$subharmonic, it follows from Corollary 3.5.1 that $U = u$ everywhere. □

Corollary 3.5.3. *Let $\{u_\alpha\}_{\alpha \in I} \subset SH(\omega)$ be a family that is locally bounded from above. Let $u(z) := \sup_\alpha u_\alpha(z)$. Then, the upper semicontinuous regularisation u^* is $\omega-$subharmonic.*

Proof. By Choquet's lemma one can choose an increasing sequence $u_j \in SH(\omega)$ such that $u = \lim_j u_j$. Then, by Littman's theorem and the notation in Corollary 3.5.2, $\lim_\varepsilon [u]_\varepsilon = U \in SH(\omega)$ and $u = U$ almost everywhere. As $u_j \in SH(\omega)$ we have

$$u_j \leq [u_j]_\varepsilon \to [u]_\varepsilon \quad \text{as } j \to +\infty$$

uniformly on compact subsets of Ω. It follows that $u \leq U$. By upper semicontinuous of U we have $u^* \leq U$. By the formula (3.5.5) and $J_h = 1$, $\lim_\varepsilon [u]_\varepsilon \leq u^*$. Thus, $u^* = U$. □

Lemma 3.5.3. *Let u be an $\omega-$subharmonic function in Ω. Then,*

$$\Delta_\omega u \geq 0$$

in the distributional sense. Conversely, if $v \in L^1_{loc}(\Omega)$ and $\Delta_\omega v \geq 0$ (as a distribution), then there exists a unique function $V \in SH(\omega)$ such that $V = v$ in $L^1_{loc}(\Omega)$.

Proof. Let $[u]_\varepsilon$, $\varepsilon > 0$, be the smooth decreasing approximation of u. As $\Delta_\omega [u]_\varepsilon \geq 0$ and the family weakly converges to $\Delta_\omega u$, we get the first statement. Conversely, by Littman's theorem we know that $V(z) = \lim_{\varepsilon \to 0} [v]_\varepsilon(z) \in SH(\omega)$ and $V(z) = v(z)$ almost everywhere. Therefore, we get the existence. The uniqueness follows from the fact that two $\omega-$subharmonic functions are equal almost everywhere. □

The following result is rather simple but it is important.

Lemma 3.5.4. *Let $u \in SH(\omega)$. Let $K \subset\subset D \subset\subset \Omega$ be a compact set and an open set. Then,*

$$\int_K dd^c u \wedge \omega^{n-1} \leq C(D, \Omega) \|u\|_{L^1(D)}.$$

Proof. Let ϕ be a cut-off function of K and $supp\phi \subset D$. Then,

$$\begin{aligned}\int_K dd^c u \wedge \omega^{n-1} &\leq \int \phi dd^c u \wedge \omega^{n-1} \\ &= \int u dd^c(\phi \omega^{n-1}) \\ &\leq C(D, \Omega) \|u\|_{L^1(D)},\end{aligned}$$

where we used that ϕ is smooth and has compact support in Ω. □

Lemma 3.5.5. *The convex cone $SH(\omega)$ is closed in $L^1_{loc}(\Omega)$, and it has a property that every bounded subset is relatively compact.*

Proof. Let u_j be a sequence in $SH(\omega)$. If $u_j \to u$ in $L^1_{loc}(\Omega)$, then $\Delta_\omega u_j \to \Delta_\omega u$ in weak topology of distributions, hence $\Delta_\omega u \geq 0$, and u can be represented by an $\omega-$subharmonic function thanks to Lemma 3.5.3.

Now suppose that $\|u_j\|_{L^1(K)}$ is uniformly bounded for every compact subset K of Ω. Let $\mu_j = \Delta_\omega u_j \geq 0$. Let ψ be a test function such that $0 \leq \psi \leq 1$ and $\psi = 1$ on K. Then, by Lemma 3.5.4

$$\mu_j(K) \leq \int_\Omega \psi \Delta_\omega u_j \leq C\|u_j\|_{L^1(K')},$$

where $K' = \text{Supp}\ \psi$. By weak compactness μ_j weakly converges to a positive measure μ. Let $G(x, y)$ be the Green kernel for the smooth domain D, where $K' \subset D \subset \Omega$. Consider

$$h_j := u_j(z) - \int G(z, w)\psi\mu_j(w).$$

Notice that since $G(x, y) \in L^1(d\lambda(z))$ and ψ has compact support in D,

$$\int G(z, w)\psi\mu_j(w) \to \int G(z, w)\psi\mu(w)$$

in L^1 as j goes to $+\infty$. Therefore, $\Delta_\omega h_j = 0$ in K and $\|h_j\|_{L^1} \leq C$. Since

$$h_j(z) = \int_{\partial D} h_j(w)P(z, w)d\sigma(w),$$

it follows that $\|h_j\|_{C^1} \leq C$. Then, there exists a subsequene h_j converging to h uniformly. Therefore,

$$h_j + \int G(z, w)\psi\mu_j(w) \to u = h + \int G(z, w)\psi\mu(w)$$

in $L^1(K)$ as j goes to ∞. $\square$

Lemma 3.5.6. *Let u_j be a sequence of $\omega-$subharmonic functions which are uniformly bounded above. If u is an $\omega-$subharmonic function and $u_j \to u$ in $\mathcal{D}'(\Omega)$, then $u_j \to u$ in $L^1_{loc}(\Omega)$, and*

$$\overline{\lim}_{j\to\infty} u_j(z) \leq u(z), \quad z \in \Omega,$$

(where two sides are equal and finite almost everywhere).

Proof. By Corollary 3.5.2 for $\varepsilon > 0$ small enough,

$$u_j \leq [u_j]_\varepsilon \to [u]_\varepsilon \tag{3.5.11}$$

uniformly on compact sets in Ω as $j \to \infty$. If $0 \leq \phi \in C_c^\infty$ then

$$\int ([u]_\varepsilon + \delta - u_j)\phi d\lambda(z) \to \int ([u]_\varepsilon + \delta - u)\phi d\lambda(z)$$

as $j \to \infty$ and if $\delta > 0$ the integrand is positive for j large. Hence,

$$\overline{\lim}_{j\to\infty} \int |u - u_j|\phi d\lambda(z) \leq 2 \int |[u]_\varepsilon + \delta - u|\phi d\lambda(z).$$

Since ε, δ are arbitrary it follows that $u_j \to u$ in L^1_{loc}.

By (3.5.11) it is easy to see that $\overline{\lim}_{j\to\infty} u_j \leq u$ in Ω. Furthermore, Fatou's lemma gives

$$\int \overline{\lim}\, u_j \phi d\lambda \geq \overline{\lim} \int u_j \phi d\lambda = \int u \phi d\lambda,$$

so we conclude that $\overline{\lim}_j\, u_j = u$ almost everywhere. □

Lemma 3.5.7 (Hartogs). *Let f be a continuous function on Ω and $K \subset\subset \Omega$ be a compact set. Suppose that $\{v_j\}_{j\geq 1} \subset SH(\omega)$ decrease point-wise to $v \in SH(\omega)$. Then, for any $\delta > 0$, there exists j_δ such that*

$$\sup_K (v_j - f) \leq \sup_K (v - f) + \delta$$

for $j \geq j_\delta$.

Proof. Let $[v_j]_\varepsilon$ and $[v]_\varepsilon$ be decreasing approximations defined in Corollary 3.5.2 for v_j and v, respectively. As v_j converges to v in $L^1_{loc}(\Omega)$, for any fixed $\varepsilon > 0$,

$$[v_j]_\varepsilon \to [v]_\varepsilon \tag{3.5.12}$$

uniformly on compact sets of Ω as j goes to $+\infty$. Since $v_j \leq [v_j]_\varepsilon$, we have

$$\sup_K (v_j - f) \leq \sup_K ([v_j]_\varepsilon - f)$$

Let $M := \sup_K (v - f)$. By Dini's theorem $\max\{M, [v]_\varepsilon(z) - f(z)\}$ decreases uniformly to M on Ω as ε goes to 0. Hence, for $\varepsilon > 0$ small enough,

$$\sup_K([v]_\varepsilon - f) \leq M + \delta/2.$$

Let us fix such a small ε. By uniform convergence (3.5.12), for $j \geq j_1$

$$\sup_K([v_j]_\varepsilon - f) \leq \sup_K([v]_\varepsilon - f) + \delta/2.$$

Thus, altogether we get the desired inequality. □

A direct consequence of this lemma is

Corollary 3.5.4. *Let γ be a real $(1,1)-$form in Ω. Let $v \in SH_{\gamma,1}(\omega) \cap L^\infty(\Omega)$. Let $\{v_j\}_{j\geq 1} \subset SH_{\gamma,1}(\omega) \cap L^\infty(\Omega)$ be such that*

$$\lim_{j\to+\infty} v_j(z) = v(z) \quad \forall z \in \Omega.$$

Let $K \subset \Omega$ be a compact set and $\delta > 0$. Then, there exists j_δ such that for $j \geq j_\delta$,

$$v_j(z) \leq \sup_K v + \delta.$$

Proof. We can find a smooth function w in Ω such that

$$dd^c w \wedge \omega^{n-1} = \gamma \wedge \omega^{n-1}.$$

As $u_j = v_j + w$ and $u = v + w$ satisfy assumptions of Lemma 3.5.7, we can apply it for $f = w$ to get the statement of the corollary. □

Corollary 3.5.5. *Let $\{u_j\}_{j\geq 1} \subset SH(\omega)$ be a sequence that is locally uniformly bounded above. Define $u(z) = \limsup_{j\to+\infty} u_j(z)$. Then, the upper semicontinuous regularisation u^* is either $\omega-$subharmonic or $\equiv -\infty$.*

Proof. Let $v_k = \sup_{j\geq k} u_j$. Thanks to Corollary 3.5.3, $v_k^* \in SH(\omega)$ and v_k^* decreases to $v \in SH(\omega)$ or $\equiv -\infty$. Clearly, $v \geq u$, and thus $v \geq u^* \geq u$. Since $v_k = v_k^*$ almost everywhere, so $v = u$ almost everywhere. Furthermore, it is easy to see that $\Delta_\omega u \geq 0$. By Lemma 3.5.6

$$v = \lim_{\varepsilon} [v]_\varepsilon \leq \limsup_{\varepsilon} [u]_\varepsilon \leq u^*. \tag{3.5.13}$$

Therefore, $v = u^*$ everywhere. □

We now prove that our definition is indeed equivalent to the definition given by Lu-Nguyen [68, Definition 2.3], (see also Dinew-Lu [31]).

Lemma 3.5.8. *A function $u : \Omega \to [-\infty, +\infty[$ is $\omega-$subharmonic if and only if it satisfies the following two conditions:*

(i) upper semicontinuous, locally integrable and $\Delta_\omega u \geq 0$ in Ω.

(ii) if v satisfies the condition (i) and $v \geq u$ almost everywhere, then $v \geq u$ everywhere.

Proof. We first show that it is a necessary conditions. The only thing that remains to be checked is the condition (ii). Pick v satisfying (i) and $v \geq u$ almost everywehre, we wish to show that $v \geq u$ everywhere. As $J_h = 1$, it follows from the formulas (3.5.5), (3.5.10), and the upper semicontinuity of v that

$$\lim_{\varepsilon \to 0} [v]_\varepsilon(z) \leq v(z).$$

Since $[v]_\varepsilon \geq [u]_\varepsilon$ for $\varepsilon > 0$, letting $\varepsilon \to 0$, we get that $v \geq u$ everywhere.

Suppose that u satisfies (i) and (ii) above. By Littman's theorem $U(z) = \lim_\varepsilon [u]_\varepsilon = u(z)$ almost everywhere, where $U(z)$ is an $\omega-$subharmonic function, which also satisfies (i). Hence, $u(z) \leq U(z)$ everywhere in Ω. Moreover, using the upper semicontinuity of u as above, we have $u(z) \geq U(z)$ in Ω. □

We define the capacity for Borel sets $E \subset \Omega$,

$$\mathbf{c}_1(E) = \sup \left\{ \int_E dd^c v \wedge \omega^{n-1} : 0 \leq v \leq 1, v \in SH(\omega) \right\}.$$

According to Lemma 3.5.4 $\mathbf{c}_1(E)$ is finite as long as E is relatively compact in Ω.

The quasi-continuity of $\omega-$subharmonic functions was used in [60]. We give here the details of the proof. First, the decreasing convergence implies the convergence in capacity.

Lemma 3.5.9. *Suppose that $u_j \in SH(\omega) \cap L^\infty(\Omega)$ and $u_j \searrow u \in SH(\omega) \cap L^\infty(\Omega)$. Then, for any compact $K \subset \Omega$ and $\delta > 0$,*

$$\lim_{j \to +\infty} \mathbf{c}_1(\{u_j > u + \delta\} \cap K) = 0.$$

Proof. Applying the localisation principle [57, p. 7], we assume that Ω is a ball and $u_j = u = h$ outside a neighbourhood of K. Let $0 \leq v \leq 1$ be $\omega-$ subharmonic in Ω. We have

$$\int_{\{u+\delta<u_j\}\cap K} dd^c v \wedge \omega^{n-1} \leq \frac{1}{\delta} \int (u_j - u) dd^c v \wedge \omega^{n-1}.$$

By Stokes's theorem,

$$\begin{aligned} \int (u_j - u) dd^c v \wedge \omega^{n-1} &= - \int d(u_j - u) \wedge d^c v \wedge \omega^{n-1} \\ &\quad + \int (u_j - u) d^c v \wedge d\omega^{n-1}. \end{aligned} \tag{3.5.14}$$

We shall show that both integrals on the right hand side tend to 0 as j goes to $+\infty$. Hence, we get the lemma. The second one is easier. Indeed, by Schwarz's inequality [73],

$$\begin{aligned} \left| \int (u_j - u) d^c v \wedge d\omega^{n-1} \right| &\leq C \left(\int (u_j - u) dv \wedge d^c v \wedge \omega^{n-1} \right)^{\frac{1}{2}} \times \\ &\quad \times \left(\int (u_j - u) \omega^n \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore the second integral of the right hand side in (3.5.14) goes to 0 as $j \to +\infty$.

Similarly, we use the Schwarz inequality for the first integral in (3.5.14). Let $K \subset D \subset\subset \Omega$ such that $u_j = u$ on $\Omega \setminus D$.

$$\begin{aligned} \left| \int d(u_j - u) \wedge d^c v \wedge \omega^{n-1} \right| &\leq C \left(\int d(u_j - u) \wedge d^c (u_j - u) \wedge \omega^{n-1} \right)^{\frac{1}{2}} \times \\ &\quad \times \left(\int_D dv \wedge d^c v \wedge \omega^{n-1} \right)^{\frac{1}{2}}. \end{aligned}$$

Again by Stokes's theorem

$$\begin{aligned}
&\int d(u_j-u)\wedge d^c(u_j-u)\wedge\omega^{n-1}\\
&=-\int(u_j-u)dd^c(u_j-u)\wedge\omega^{n-1}+\int(u_j-u)d^c(u_j-u)\wedge d\omega^{n-1}\\
&=\int(u_j-u)dd^cu\wedge\omega^{n-1}-\int(u_j-u)dd^cu_j\wedge\omega^{n-1}\\
&\quad+\frac{1}{2}\int d^c(u_j-u)^2\wedge d\omega^{n-1}\\
&\leq\int(u_j-u)dd^cu\wedge\omega^{n-1}+\frac{1}{2}\int d^c(u_j-u)^2\wedge d\omega^{n-1}.
\end{aligned}$$

Thus, the fist integral goes to 0 as $j\to+\infty$ by the Lebesgue dominated convergence theorem. For the second integral we use Stokes' theorem once more

$$\begin{aligned}
\int d^c(u_j-u)^2\wedge d\omega^{n-1}&=-\int(u_j-u)^2dd^c\omega^{n-1}\\
&\leq C\int(u_j-u)^2\omega^n.
\end{aligned}$$

The right hand side also goes to 0 as $j\to\infty$. Thus, we get the lemma. □

Lemma 3.5.10. *Let $u\in SH(\omega)\cap L^\infty(\Omega)$. Then for each $\varepsilon>0$, there is an open subset $\mathcal{O}$ of Ω such that $\mathbf{c}_1(\mathcal{O},\Omega)<\varepsilon$ and u is continuous on $\Omega\setminus\mathcal{O}$.*

Proof. We may assume that Ω is a small ball because of the properties of capacity:

- if $E\subset\Omega_1\subset\Omega_2$, then $\mathbf{c}_1(E,\Omega_2)\leq\mathbf{c}_1(E,\Omega_1)$.
- $\mathbf{c}_1(\bigcup_j E_j)\leq\sum_j\mathbf{c}_1(E_j)$.

Let $SH(\omega)\cap C^\infty(\Omega)\ni u_j\searrow u$ and fix a compact set $K\subset\Omega$. By Lemma 3.5.9 there exists an integer j_k and an open set

$$\mathcal{O}_l=\{u_{j_l}>u+\frac{1}{l}\}\subset\Omega,\tag{3.5.15}$$

such that $\mathbf{c}_1(\mathcal{O}_k\cap K,\Omega)<2^{-k}$. If $G_k=\cup_{l>k}\mathcal{O}_l$. Then, u_{j_k} decreases uniformly to u on $K\setminus G_k$. Hence, u is continuous on $K\setminus G_k$.

Applying the argument above for a sequence of compact sets K_j increasing to Ω we get open sets G_j that $\mathbf{c}_1(G_j,\Omega)<\varepsilon 2^{-j}$. Let $\mathcal{O}=\cup_j G_j$, the lemma follows. □

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