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# On the definition of the Monge-Ampère operator in $\mathbb{C}^2$

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**Abstract.** We show that if $u$ is a plurisubharmonic function defined on an open subset $\Omega$ of $\mathbb{C}^2$ then the Monge-Ampère measure $(dd^c u)^2$ can be well defined if and only if $u$ belongs to the Sobolev space $W^{1,2}_{loc}(\Omega)$.

## 1. Introduction

The complex Monge-Ampére operator for a smooth plurisubharmonic (shortly psh) function $u$ defined on an open subset of $\mathbb{C}^n$ is given by

$$(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u = 4^n n! \det(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}) d\lambda,$$

where $d = \partial + \bar{\partial}$, $d^c := i(\bar{\partial} - \partial)$ and $d\lambda$ is the volume form. By an example due to Shiffman and Taylor (see Siu [10]) the Monge-Ampère operator cannot be well defined as a nonnegative Radon measure for an arbitrary psh function. A simpler example was given by Kiselman [8]: for $z$ near the origin in $\mathbb{C}^n$ he defined

$$u(z) := (-\log |z_1|)^{1/n}(|z_2|^2 + \cdots + |z_n|^2 - 1).$$

Then $u$ is psh near the origin, smooth if $z_1 \neq 0$ but $(dd^c u)^n$ is not integrable near $\{z_1 = 0\}$.

On the other hand, as shown by Bedford and Taylor [3], $(dd^c u)^n$ can be well defined if $u$ is psh and locally bounded. By Demailly [7] it is enough to assume that the set where $u$ is not locally bounded is relatively compact in the domain of definition. In both cases the operator is continuous under decreasing sequences (with weak* topology of Radon measures). It is therefore natural to define the class $\mathcal{D}(\Omega)$ of psh functions in an open $\Omega \subset \mathbb{C}^n$, for which the complex Monge-Ampère operator can be well defined, as follows: a psh function $u$ belongs to $\mathcal{D}(\Omega)$ if there exists a nonnegative Radon measure $\mu$ on $\Omega$ such that if $\Omega' \subset \Omega$

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is open and a sequence $u_j \in PSH \cap C^\infty(\Omega')$ decreases to $u$ in $\Omega'$ then $(dd^c u_j)^n$ tends weakly to $\mu$ in $\Omega'$. The Monge-Ampère measure $\mu$ we then denote by $M(u)$.

It is clear that the definition is purely local and that if $\Omega' \subset \Omega \subset \mathbb{C}^n$ are open, $u \in \mathcal{D}(\Omega)$, then also $u|_{\Omega'} \in \mathcal{D}(\Omega')$ and $M(u|_{\Omega'}) = M(u)|_{\Omega'}$. It follows in particular that the functions from the Shiffman-Taylor and Kiselman examples do not belong to $\mathcal{D}$.

Of course, if $n = 1$ then all subharmonic functions belong to $\mathcal{D}$. The goal of this paper is to give a complete description of the class $\mathcal{D}$ if $n = 2$. We namely show the following result.

**Theorem 1.1.** *If $\Omega$ is an open subset of $\mathbb{C}^2$ then $\mathcal{D}(\Omega) = PSH \cap W^{1,2}_{loc}(\Omega)$.*

Note that in the Kiselman example one has $u \in W^{1,2}_{loc}$ if and only if $n \geq 3$, so the above characterization does not hold in this case.

The fact that the operator $(dd^c)^2$ can be well defined for $u \in PSH \cap W^{1,2}_{loc}(\Omega)$, $\Omega$ open in $\mathbb{C}^2$, is in fact very simple: note that integration by parts gives

$$\int_\Omega \varphi (dd^c u)^2 = -\int_\Omega du \wedge d^c u \wedge dd^c \varphi, \quad \varphi \in C_0^\infty(\Omega), \tag{1.1}$$

if $u$ is smooth and the right hand-side makes sense if $u \in PSH \cap W^{1,2}_{loc}(\Omega)$. This was done already by Bedford and Taylor [2] who in particular solved a Dirichlet problem related to this class. One of the main difficulties for us was to show the continuity of $(dd^c)^2$ in this class for decreasing sequences.

One of the consequences of Theorem 1.1 and Theorem 3.3 below is the following property.

**Theorem 1.2.** *If $\Omega$ is open in $\mathbb{C}^2$, $u \in \mathcal{D}(\Omega)$ and $v \in PSH(\Omega)$ is such that $u \leq v$ then $v \in \mathcal{D}(\Omega)$.*

We conjecture that this property holds also in $\mathbb{C}^n$ for $n \geq 3$.

In the author's paper [4] it was conjectured that all locally maximal psh functions are maximal. (Recall that $u \in PSH(\Omega)$, $\Omega$ open in $\mathbb{C}^n$, is called maximal in $\Omega$ if $v \in PSH(\Omega)$, $\{v > u\} \Subset \Omega$ implies that $\{v > u\} = \emptyset$.) Combining Theorem 1.1 with Proposition 2.2 below gives a partial answer to this problem.

**Theorem 1.3.** *If $u$ is a $W^{1,2}_{loc}$ psh function defined on an open subset of $\mathbb{C}^2$ then it is maximal if and only if it is locally maximal.*

This paper was in part motivated by Cegrell's recent paper [6]. In Section 4 we present consequences of our results for the class $\mathcal{E}$ defined by Cegrell [6].

## 2. The class $\mathcal{D}$ in $\mathbb{C}^n$

**Proposition 2.1.** *Let $\Omega$ be an open subset of $\mathbb{C}^n$. If $u_j \in \mathcal{D}(\Omega)$ is a sequence decreasing to $u \in \mathcal{D}(\Omega)$ then $M(u_j)$ tends weakly to $M(u)$.*

*Proof.* Choose a test function $\varphi \in C_0^\infty(\Omega)$ with support contained in an open $\Omega' \Subset \Omega$. We may assume that the regularizations $u_j^k := u_j * \rho_{1/k}$ are defined in $\Omega'$. Since for every fixed $j$ the sequence $u_j^k$ is decreasing to $u_j$ and $M(u_j^k)$ tends weakly to $M(u_j)$, we can find an increasing sequence $k(j)$ such that

$$\left|\int_\Omega \varphi\, M(u_j^{k(j)}) - \int_\Omega \varphi\, M(u_j)\right| \le \tfrac{1}{j}, \tag{2.1}$$

$$||u_j^{k(j)} - u_j||_{L^1(\Omega')} \le \tfrac{1}{j}, \tag{2.2}$$

$$u_j^{k(j)} \le u_{j-1}^{k(j-1)} + \tfrac{1}{j^2}, \quad \text{in } \Omega'. \tag{2.3}$$

Set

$$v_j := u_j^{k(j)} + \sum_{l=j+1}^{\infty} \frac{1}{l^2}.$$

By (2.3) $v_j$ is decreasing. From (2.2) we get

$$||v_j - u_j||_{L^1(\Omega')} \le \frac{1}{j} + \lambda(\Omega') \sum_{l=j+1}^{\infty} \frac{1}{l^2} \to 0$$

and thus $v_j$ converges to $u$ in $\Omega'$. From the definition of $\mathcal{D}(\Omega)$ it now follows that

$$\int_\Omega \varphi\, M(v_j) = \int_\Omega \varphi\, M(u_j^{k(j)}) \to \int_\Omega \varphi\, M(u)$$

and from (2.1) we obtain

$$\int_\Omega \varphi\, M(u_j) \to \int_\Omega \varphi\, M(u). \qquad \square$$

The next result characterizes maximal functions belonging to $\mathcal{D}$. For locally bounded $u$ it follows from the results due to Bedford and Taylor [1], [3].

**Proposition 2.2.** *Let $u \in \mathcal{D}(\Omega)$, $\Omega$ open in $\mathbb{C}^n$. Then $u$ is maximal in $\Omega$ if and only if $M(u) = 0$.*

*Proof.* If $M(u) = 0$ and $u_j := u * \rho_{1/j}$ then $M(u_j)$ tends weakly to 0, by the definition of $\mathcal{D}$. By [4, Theorem 4.4] (see also Sadullaev [9] for the case when $\Omega$ is pseudoconvex) it follows that $u$ is maximal. On the other hand, assume that $u$ is maximal and that $B \Subset B' \Subset \Omega$ are open balls. It follows for example from [4, Proposition 4.1] that the Perron-Bremermann envelopes

$$v_j := \sup\{v \in PSH(B') : v \le u_j \text{ in } B' \setminus B\}$$

satisfy $v_j \in PSH \cap C(B')$ (by Walsh [11]), $v_j = u_j$ in $B' \setminus B$, $v_j$ is maximal in $B$ and $v_j$ is decreasing to $u$ in $B'$. By Bedford-Taylor's solution of the Dirichlet problem [1] we have $M(v_j) = 0$ in $B$ and by Proposition 2.1 we conclude that $M(u) = 0$ in $B$. $\square$

## 3. The Monge-Ampére operator in $\mathbb{C}^2$

For $u \in PSH \cap W^{1,2}_{loc}$ the measure $(dd^c u)^2$ is well defined by (1.1). This formula also easily gives the following estimate.

**Proposition 3.1.** *Let $u \in PSH \cap W^{1,2}(\Omega)$, $\Omega$ open in $\mathbb{C}^2$. Then*

$$\int_{\Omega'} (dd^c u)^2 \leq C(\Omega', \Omega) ||\nabla u||^2_{L^2(\Omega)}, \quad \Omega' \Subset \Omega. \qquad \square$$

It is also easy to prove that the operator $(dd^c)^2$ is continuous under sequences converging in $W^{1,2}_{loc}$.

**Proposition 3.2.** *If $u_j$ is a sequence of $W^{1,2}$ psh functions defined on an open subset $\Omega$ of $\mathbb{C}^2$ converging to a psh $u$ in the $W^{1,2}(\Omega)$ norm then $(dd^c u_j)^2$ tends weakly to $(dd^c u)^2$.*

*Proof.* For $\varphi \in C_0^\infty(\Omega)$ we have

$$\left| \int_\Omega \varphi (dd^c u_j)^2 - \int_\Omega \varphi (dd^c u)^2 \right| = \left| \int_\Omega \varphi dd^c(u_j - u) \wedge dd^c(u_j + u) \right|$$
$$= \left| \int_\Omega d(u_j - u) \wedge d^c(u_j + u) \wedge dd^c \varphi \right| \leq C \left( \int_\Omega |\nabla(u_j - u)|^2 d\lambda \right)^{1/2},$$

where $C$ is independent of $j$. $\square$

In order to show that $(dd^c)^2$ is continuous also under decreasing sequences we need the following estimate for subharmonic functions.

**Theorem 3.3.** *Let $\Omega$ be an open subset of $\mathbb{R}^m$. Suppose that $u \in W^{1,2}(\Omega)$ is a negative subharmonic function in $\Omega$. Then for every subharmonic function $v$ in $\Omega$ satisfying $u \leq v < 0$ we have $v \in W^{1,2}_{loc}(\Omega)$ and*

$$||\nabla v||_{L^2(\Omega')} \leq C(\Omega', \Omega) \left( ||u||_{L^2(\Omega)} + ||\nabla u||_{L^2(\Omega)} \right), \quad \Omega' \Subset \Omega.$$

*Proof.* After regularizing $u$ and $v$ we may assume that $u$, $v$, $\Omega$, $\Omega'$ are $C^\infty$ smooth, $\Omega$ is bounded and that $u, v$ are defined in a neighborhood of $\overline{\Omega}$. Solving the Dirichlet problem in $\Omega \setminus \overline{\Omega'}$ we obtain subharmonic functions $\widetilde{u}, \widetilde{v}$ in $\Omega$, continuous on $\overline{\Omega}$, harmonic in $\Omega \setminus \overline{\Omega'}$ and such that $\widetilde{u} = \widetilde{v} = 0$ on $\partial\Omega$, $\widetilde{u} = u$, $\widetilde{v} = v$ on $\overline{\Omega'}$. The functions $\widetilde{u}, \widetilde{v}$ are smooth on $\overline{\Omega} \setminus \partial\Omega'$ and they belong to $C^{0,1}(\overline{\Omega})$. Integration by parts gives

$$\int_\Omega |\nabla \widetilde{u}|^2 d\lambda - \int_\Omega |\nabla \widetilde{v}|^2 d\lambda = \int_\Omega \langle \nabla(\widetilde{u} - \widetilde{v}), \nabla(\widetilde{u} + \widetilde{v}) \rangle d\lambda$$
$$= \int_\Omega (\widetilde{v} - \widetilde{u}) \Delta(\widetilde{u} + \widetilde{v}) \geq 0.$$

Therefore

$$\int_{\Omega'} |\nabla v|^2 d\lambda \le \int_{\Omega} |\nabla \widetilde{v}|^2 d\lambda \le \int_{\Omega} |\nabla \widetilde{u}|^2 d\lambda$$
$$= \int_{\Omega'} |\nabla u|^2 d\lambda + \int_{\Omega\setminus\Omega'} |\nabla \widetilde{u}|^2 d\lambda. \tag{3.1}$$

Let $\varphi \in C_0^\infty(\Omega)$ be such that $0 \le \varphi \le 1$ and $\varphi = 1$ in a neighborhood of $\overline{\Omega'}$. Then

$$\int_{\Omega\setminus\Omega'} |\nabla \widetilde{u}|^2 d\lambda = \int_{\partial(\Omega\setminus\Omega')} \widetilde{u}\frac{\partial \widetilde{u}}{\partial n} d\sigma = \int_{\partial(\Omega\setminus\Omega')} \varphi u \frac{\partial \widetilde{u}}{\partial n} d\sigma = \int_{\Omega\setminus\Omega'} \langle \nabla(\varphi u), \nabla \widetilde{u}\rangle d\lambda$$
$$\le \left(\int_{\Omega\setminus\Omega'} |\nabla(\varphi u)|^2 d\lambda\right)^{1/2} \left(\int_{\Omega\setminus\Omega'} |\nabla \widetilde{u}|^2 d\lambda\right)^{1/2}.$$

Combining this with (3.1) we easily get the required estimate. □

We do not know if every sequence of subharmonic functions in open $\Omega \subset \mathbb{R}^m$ decreasing to a $W^{1,2}(\Omega)$ subharmonic function converges also in the $W^{1,2}(\Omega)$ norm. This would immediately imply the next result.

**Theorem 3.4.** *Assume that $u_j$ is a decreasing sequence of psh functions defined in an open subset of $\mathbb{C}^2$ decreasing to $u \in PSH \cap W^{1,2}_{loc}$. Then $u_j \in W^{1,2}_{loc}$ and $(dd^c u_j)^2$ tends weakly to $(dd^c u)^2$.*

*Proof.* The first part follows from Theorem 3.3. By Proposition 2.1 we may assume that $u_j$ are continuous and negative. Let $B_1 \Subset B_2$ be concentric open balls in $\mathbb{C}^2$. We may assume that $u_j$ are defined in a neighborhood of $\overline{B}_2$ and negative there.

For a negative $v \in PSH(\Omega)$ we set

$$\widetilde{v} := \sup\{w \in PSH(B_2) : w < 0 \text{ in } B_2,\ w \le v \text{ in } B_1\}.$$

It follows easily that $\widetilde{v} \in PSH(B_2)$, $\widetilde{v} < 0$ in $B_2$, $\widetilde{v} = v$ in $B_1$, $\widetilde{v}$ is maximal in $B_2 \setminus \overline{B}_1$. If $v_j$ is a sequence of negative psh functions in $B_2$ decreasing to $v$ then $\widetilde{v}_j$ is decreasing to $\widetilde{v}$. Moreover, by Walsh [11] if $v$ is continuous on $\overline{B}_1$ then $\widetilde{v}$ is continuous on $\overline{B}_2$ and $\widetilde{v} = 0$ on $\partial B_2$.

By Theorem 3.3 and Proposition 3.1 we have

$$\int_{B_2} (dd^c \widetilde{u}_j)^2 = \int_{\overline{B}_1} (dd^c \widetilde{u}_j)^2 \le C, \tag{3.2}$$

where $C$ is independent of $j$. Theorem 3.4 will follow from Cegrell [6, Theorem 4.2] (because $\widetilde{u} \in \mathcal{F}(B_2)$ and $\widetilde{u}_j = u_j$ in $B_1$) combined with Proposition 3.2 applied to the regularizations of $u$. We will repeat Cegrell's argument for the convenience of the reader. Take $\psi \in PSH(B_2) \cap C(\overline{B}_2)$ with $\psi = 0$ on $\partial B_2$. Integrating by parts we get

$$\begin{aligned}\int_{B_2} \psi (dd^c\widetilde{u}_j)^2 &= \int_{B_2} \widetilde{u}_j dd^c\psi \wedge dd^c\widetilde{u}_j \geq \int_{B_2} \widetilde{u}_{j+1} dd^c\psi \wedge dd^c\widetilde{u}_j \\ &= \int_{B_2} \widetilde{u}_j dd^c\psi \wedge dd^c\widetilde{u}_{j+1} \geq \int_{B_2} \widetilde{u}_{j+1} dd^c\psi \wedge dd^c\widetilde{u}_{j+1} \\ &= \int_{B_2} \psi (dd^c\widetilde{u}_{j+1})^2.\end{aligned}$$

It follows that the sequence $\int_{B_2} \psi (dd^c\widetilde{u}_j)^2$ is decreasing and by (3.2) it converges to some $a \in \mathbb{R}$. If $v_j \in PSH \cap C(B_2)$ is another sequence decreasing to $u$ in $B_2$ then, by the same argument, $\int_{B_2} \psi (dd^c\widetilde{v}_j)^2$ converges to some $b \in \mathbb{R}$. Fix $\varepsilon > 0$. For every $j$ we can find $k(j)$ such that for every $k \geq k(j)$ one has $u_k \leq v_j + \varepsilon$ in $B_1$. Then

$$\widetilde{v}_j \geq \widetilde{(u_k - \varepsilon)} \geq \widetilde{u}_k + \widetilde{(-\varepsilon)} = \widetilde{u}_k + \varepsilon\widetilde{(-1)} \quad \text{in } B_3.$$

Integrating by parts as before and using the superadditivity of the operator $(dd^c)^2$ we get

$$\begin{aligned}\int_{B_2} \psi (dd^c\widetilde{v}_j)^2 &\geq \int_{B_2} \psi (dd^c(\widetilde{u}_k + \varepsilon\widetilde{(-1)}))^2 \\ &\geq \int_{B_2} \psi (dd^c\widetilde{u}_k)^2 - \varepsilon^2 C(B_1, B_2, \psi).\end{aligned}$$

From this it easily follows that $b \geq a$ and in the same way we get the reverse inequality. Choose $v_j := u * \rho_{1/j}$, the classical regularizations of $u$. For $\varphi \in C_0^\infty(B_1)$ we can easily find $\psi_1, \psi_2 \in PSH(B_2) \cap C^\infty(\overline{B}_2)$ such that $\varphi = \psi_1 - \psi_2$ in $B_2$ and $\psi_1 = \psi_2 = 0$ on $\partial B_2$. Therefore by Proposition 3.1

$$\begin{aligned}\int_{B_1} \varphi (dd^c v)^2 &= \lim_{j\to\infty} \int_{B_1} \varphi (dd^c\widetilde{v}_j)^2 \\ &= \lim_{j\to\infty} \int_{B_2} \psi_1 (dd^c\widetilde{v}_j)^2 - \lim_{j\to\infty} \int_{B_2} \psi_2 (dd^c\widetilde{v}_j)^2 \\ &= \lim_{j\to\infty} \int_{B_2} \psi_1 (dd^c\widetilde{u}_j)^2 - \lim_{j\to\infty} \int_{B_2} \psi_2 (dd^c\widetilde{u}_j)^2 \\ &= \lim_{j\to\infty} \int_{B_1} \varphi (dd^c u_j)^2.\end{aligned}$$
□

Theorem 3.4 (applied for smooth $u_j$) implies that $PSH \cap W^{1,2}_{loc} \subset \mathcal{D}$ in Theorem 1.1. The reverse inclusion follows from the next result.

**Theorem 3.5.** *Assume that* $\Omega$ *is an open subset of* $\mathbb{C}^2$ *and* $u \in PSH(\Omega)\setminus W^{1,2}_{loc}(\Omega)$. *Then one can find open balls* $B_1 \Subset B_2 \Subset \Omega$ *and a sequence* $u_j \in PSH(B_2) \cap C(\overline{B}_2)$ *decreasing to* $u$ *on* $\overline{B}_2$ *such that*

$$\lim_{j\to\infty}\int_{\overline{B}_1}(dd^c u_j)^2 = \infty.$$

*Proof.* Let $B_1 \Subset B_2 \Subset B_3 \Subset \Omega$ be open balls such that $u \notin W^{1,2}(B_1)$. Let $v_j \in PSH(B_3) \cap C(\overline{B}_3)$ be a sequence decreasing to $u$ on $\overline{B}_3$. We may assume that

$$\sup_j \int_{\overline{B}_2}(dd^c v_j)^2 < \infty, \tag{3.3}$$

since otherwise we are done. For every $j$ we can find increasing $k = k(j) \geq j$ such that

$$\int_{B_1} |\nabla(v_j - v_k)|^2 d\lambda \geq j. \tag{3.4}$$

We then set

$$u_j := \sup\{v \in PSH(B_2) : v \leq v_j \text{ in } B_2, v \leq v_k \text{ in } B_1\}.$$

By $h_j$ denote the harmonic function in $B_2 \setminus \overline{B}_1$, continuous on $\overline{B}_2 \setminus B_1$ with $h_j = v_k$ on $\partial B_1$ and $h_j = v_j$ on $\partial B_2$. Then, if

$$\widetilde{h}_j := \begin{cases} v_k & \text{on } \overline{B}_1 \\ h_j & \text{on } \overline{B}_2 \setminus \overline{B}_1 \end{cases} \in C(\overline{B}_2),$$

we have

$$u_j = \sup\{v \in PSH(B_2) : v \leq \widetilde{h}_j\}.$$

By Walsh [11] the function $u_j$ is continuous on $B_2$ and we easily show that it is continuously extendable to $\overline{B}_2$. We thus have $u_j \in PSH(B_2) \cap C(\overline{B}_2)$, $v_k \leq u_j \leq v_j$ on $\overline{B}_2$, $u_j = v_k$ on $\overline{B}_1$ and $u_j = v_j$ on $\partial B_2$.

We claim that we also have

$$(dd^c u_j)^2 \leq (dd^c v_j)^2 \quad \text{in } B_2 \setminus \overline{B}_1. \tag{3.5}$$

Indeed, on the set $\{u_j < v_j\} \cap (B_2 \setminus \overline{B}_1)$ the function $u_j$ is maximal and thus (3.5) holds there. Take a compact $K \subset \{u_j = v_j\} \cap (B_2 \setminus \overline{B}_1)$. Then for $\varepsilon > 0$

$$\int_K (dd^c u_j)^2 = \int_K (dd^c \max\{u_j + \varepsilon, v_j\})^2$$

and (3.5) follows from the weak convergence $(dd^c \max\{u_j + \varepsilon, v_j\})^2 \to (dd^c v_j)^2$ as $\varepsilon \to 0$.

Set $\psi(z) := |z - z_0|^2 - R^2$, where $z_0$ is the center of $B_2$ and $R$ is its radius. By (3.4) and (3.5) we then have

$$\begin{aligned}4j &\le 4\int_{B_2} |\nabla(v_j - v_k)|^2 d\lambda = \int_{B_2} d(v_j - u_j) \wedge d^c(v_j - u_j) \wedge dd^c\psi \\ &= \int_{B_2} (v_j - u_j) dd^c(u_j - v_j) \wedge dd^c\psi \le \int_{B_2} (v_j - u_j) dd^c u_j \wedge dd^c\psi \\ &= \int_{B_2} \psi dd^c u_j \wedge dd^c(v_j - u_j) \le \int_{B_2} |\psi| (dd^c u_j)^2 \\ &\le R^2 \left( \int_{\overline{B}_1} (dd^c u_j)^2 + \int_{B_2 \setminus \overline{B}_1} (dd^c v_j)^2 \right).\end{aligned}$$

It is now sufficient to use (3.3). □

## 4. The class $\mathcal{E}$ in $\mathbb{C}^2$

If $\Omega$ is a bounded hyperconvex domain in $\mathbb{C}^n$ (this means that there exists a negative psh $u$ in $\Omega$ such that $\lim_{z\to\partial\Omega} u(z) = 0$) then following Cegrell [6] by $\mathcal{E}(\Omega)$ we denote the class of plurisubharmonic functions in $\Omega$ such that for every $z_0 \in \Omega$ there exists a neighborhood $U \Subset \Omega$ of $z_0$ and a decreasing sequence $u_j$ of negative locally bounded psh functions in $\Omega$ such that $u_j$ converges to $u$ in $U$, $\lim_{z\to\partial\Omega} u_j(z) = 0$ and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$. The following result shows in particular that in $\mathbb{C}^2$ this definition is local. One can also apply it to the examples given by Cegrell in [5].

**Theorem 4.1.** *If $\Omega$ is a bounded hyperconvex domain in $\mathbb{C}^2$ then*

$$\mathcal{E}(\Omega) = \{u \in PSH \cap W^{1,2}_{loc}(\Omega) : u < 0\}.$$

*Proof.* First take $u \in \mathcal{E}(\Omega)$. By Cegrell [6, Theorem 2.1] the functions $u_j$ in the definition of $\mathcal{E}(\Omega)$ can be chosen to be continuous on $\overline{\Omega}$. Moreover, we may assume that $(dd^c u_j)^2 = 0$ in $\Omega \setminus \overline{U}$. Let $\psi \in PSH(\Omega) \cap C(\overline{\Omega})$ be such that $\psi = 0$ on $\partial\Omega$ and $dd^c\psi \ge dd^c|z|^2$ in $U$. Then

$$4\int_B |\nabla u_j|^2 d\lambda \le \int_\Omega du_j \wedge d^c u_j \wedge dd^c\psi = \int_\Omega |\psi| (dd^c u_j)^2 \le C$$

and thus $u \in W^{1,2}_{loc}(\Omega)$.

On the other hand, let $u \in PSH \cap W^{1,2}_{loc}(\Omega)$ be negative. Again by Cegrell [6, Theorem 2.1] we can find a sequence $v_j \in PSH(\Omega) \cap C(\overline{\Omega})$ decreasing to $u$ in $\Omega$, vanishing on $\partial\Omega$. For a fixed open ball $B \Subset \Omega$ set

$$\widetilde{v}_j := \{v \in PSH(\Omega) : v < 0 \text{ in } \Omega,\ v \le v_j \text{ in } B\}.$$

Then $\widetilde{v}_j$ decreases to a psh $\widetilde{u} \geq u$ in $\Omega$. By the results of Section 3 the sequence of measures $(dd^c\widetilde{v}_j)^2$ is weakly convergent in $\Omega$ and since they are supported on $\overline{B}$ we get $\sup_j \int_\Omega (dd^c\widetilde{v}_j)^2 < \infty$. But $\widetilde{v}_j = u_j$ decreases to $u$ in $B$, thus $u \in \mathcal{E}(\Omega)$. □

## References


1. Bedford, E., Taylor, B.A.: The Dirichlet problem for a complex Monge-Ampère equation. Invent. Math. **37**, 1–44 (1976)
2. Bedford, E., Taylor, B.A.: Variational properties of the complex Monge-Ampère equation I. Dirichlet principle. Duke. Math. J. **45**, 375–403 (1978)
3. Bedford, E., Taylor, B.A.: A new capacity for plurisubharmonic functions. Acta Math. **149**, 1–41 (1982)
4. Błocki, Z.: Estimates for the complex Monge-Ampère operator. Bull. Pol. Acad. Sci. **41**, 151–157 (1993)
5. Cegrell, U.: Explicit calculation of a Monge-Ampère measure. In: Raby G, Symesak F. (eds.) Actes des recontres d'analyse complexe, Atlantique, Universite de Poitiers, 2001
6. Cegrell, U.: The general definition of the complex Monge-Ampère operator. To appear in Ann. Inst. Fourier
7. Demailly, J.-P.: Monge-Ampère operators, Lelong numbers and intersection theory. Complex analysis and geometry, Univ. Ser. Math., Plenum, New York, 1993, pp. 115–193
8. Kiselman, C.O.: Sur la définition de l'opérateur de Monge-Ampère complexe. Proc. Analyse Complexe, Toulouse 1983, Lect. Notes in Math. **1094**, pp. 139–150
9. Sadullaev, A.: Plurisubharmonic measures and capacities on complex manifolds. Russian Math. Surv. **36**, 61–119 (1981)
10. Siu, Y.-T.: Extension of meromorphic maps into Kähler manifolds. Ann. Math. **102**, 421–462 (1975)
11. Walsh, J.B.: Continuity of envelopes of plurisubharmonic functions. J. Math. Mech. **18**, 143–148 (1968)