THE DOMAIN OF DEFINITION OF THE COMPLEX MONGE-AMPÈRE OPERATOR

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Abstract. We give a precise characterization of those plurisubharmonic functions for which one can well define the Monge-Ampère operator as a regular Borel measure.

1. Introduction. For a smooth plurisubharmonic (shortly psh) function u the complex Monge-Ampère operator is given by

$$(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u = 4^n n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) d\lambda, \tag{1.1}$$

where $d = \partial + \overline{\partial}$, $d^c = i(\overline{\partial} - \partial)$ (so that $dd^c = 2i\partial\overline{\partial}$) and $d\lambda$ denotes the volume form. It was first shown by Shiffman and Taylor (see [16]) that one cannot well define $(dd^c u)^n$ as a regular Borel measure for arbitrary psh u if $n \geq 2$. This example was simplified by Kiselman [15]: the function

$$u(z) = (-\log|z_1|)^{1/n}(|z_2|^2 + \ldots + |z_n|^2 - 1)$$

is psh near the origin, smooth away from the hyperplane $z_1 = 0$ but the mass of $(dd^c u)^n$ is unbounded near $z_1 = 0$.

On the other hand, as shown by Bedford and Taylor [3] (see also [10] and [1]) one can well define $(dd^c u)^n$ if u is psh and locally bounded. Moreover, this definition is continuous under decreasing sequences in $PSH \cap L^\infty_{loc}$ (with weak* topology of measures). Demailly [11] (see also [12-14]) extended this to psh functions locally bounded away from a compact set. One thing which distinguishes the unbounded case from the bounded one is non-uniqueness of the Dirichlet problem (see [1, p. 16]).

The choice of monotone sequences for considering continuity of the complex Monge-Ampère operator is also motivated by the following fact: it follows from an example due to Cegrell [6] that one can find a sequence $u_j \in PSH \cap C^\infty$

Manuscript received June 7, 2004.
Research supported in part by KBN Grant #2 P03A 03726.
American Journal of Mathematics 128 (2006), 519–530.

converging to $u \in PSH \cap C^\infty$ in L^p_{loc} for every $p < \infty$ and such that the Monge-Ampère measures of u_j are weakly divergent. Cegrell [7] considered also the following example

$$\begin{aligned} u(z) &= 2\log|z_1 \dots z_n|, \\ u_j(z) &= \log(|z_1 \dots z_n|^2 + 1/j), \\ v_j(z) &= \log(|z_1|^2 + 1/j) + \dots + \log(|z_n|^2 + 1/j). \end{aligned}$$

One can show that on one hand $(dd^c u_j)^n$ tends weakly to 0, whereas $(dd^c v_j)^n$ converges to $n!4^n\delta_0$, where δ_0 is the point mass at the origin. We thus have two decreasing sequences of smooth psh functions converging to u whose Monge-Ampère measures converge to a different limit.

The main goal of this paper is to prove the following result.

THEOREM 1.1. *For a negative $u \in PSH(\Omega)$, where $\Omega \subset \mathbb{C}^n$ is open, the following are equivalent*

(i) *There exists a measure μ in Ω such that if $U \subset \Omega$ is open and a sequence $u_j \in PSH \cap C^\infty(U)$ is decreasing to u in U then $(dd^c u_j)^n$ tends weakly to μ in U;*

(ii) *For every open $U \subset \Omega$ and any sequence $u_j \in PSH \cap C^\infty(U)$ decreasing to u in U the sequence $(dd^c u_j)^n$ is locally weakly bounded in U;*

(iii) *For every open $U \subset \Omega$ and any sequence $u_j \in PSH \cap C^\infty(U)$ decreasing to u in U the sequences*

$$|u_j|^{n-p-2} du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge \omega^{n-p-1}, \quad p = 0, 1, \dots, n-2, \tag{1.2}$$

($\omega := dd^c|z|^2$ is the Kähler form in $\mathbb{C}^n$) are locally weakly bounded in U;

(iv) *For every $z \in \Omega$ there exists an open neighborhood U of z in Ω and a sequence $u_j \in PSH \cap C^\infty(U)$ decreasing to u in U such that the sequences (1.2) are locally weakly bounded in U.*

The equivalence of (iii) and (iv) means that one has to check the local weak boundedness of sequences (1.2) for arbitrary sequence of smooth psh functions decreasing to u, for example the standard regularizations of u. The following example shows that the condition on local weak boundedness of sequences (1.2) cannot be improved.

Example. For a fixed $p_0 = 0, 1, \dots, n-2$ set

$$\begin{aligned} u(z) &:= \log(|z_1|^2 + \dots + |z_{p_0+1}|^2), \\ u_j(z) &:= \log(|z_1|^2 + \dots + |z_{p_0+1}|^2 + 1/j). \end{aligned}$$

Then near the origin the sequence (1.2) vanishes if $p > p_0$, is weakly unbounded for $p = p_0$ and weakly bounded for $p < p_0$.

Condition (i) in Theorem 1.1 in a natural way provides the domain of definition $\mathcal{D}$ of the operator $(dd^c)^n$. One can easily show (see [5, Proposition 2.1]) that $\mathcal{D}$ is the biggest subclass of the class of psh functions where the complex Monge-Ampère operator can be defined as a regular Borel measure in such a way that (1.1) holds for smooth functions and so that the operator is continuous for decreasing sequences in $\mathcal{D}$.

We will also show that the class $\mathcal{D}$ has the following property, thus answering in the affirmative a conjecture from [5] for $n \geq 3$.

THEOREM 1.2. *If Ω is open in $\mathbb{C}^n$, $u \in \mathcal{D}(\Omega)$, $v \in PSH(\Omega)$ and $u \leq v$ outside a compact subset of Ω then $v \in \mathcal{D}(\Omega)$.*

This implies in particular that the Monge-Ampère operator can be well defined for psh functions that are locally bounded outside a compact set (see [11]).

Theorem 1.1 was proved in [5] for $n = 2$, then of course any of the conditions (iii) and (iv) means precisely that $u \in PSH \cap W^{1,2}_{loc}$. The thing which obviously distinguishes this case from the general one is the lack of zero-th and second order terms in (1.2). The fact that $(dd^c u)^2$ can be well defined for $u \in PSH \cap W^{1,2}_{loc}$ is quite simple and was already observed in [1, p.3] (see also [2]). In [5] it was shown that the operator $(dd^c)^2$ is continuous under decreasing sequences on $PSH \cap W^{1,2}_{loc}$. In the proof the potential theory in $\mathbb{R}^4$ was in fact used (see also [9] which was a follow-up to [5]). In this paper the implication (iii)$\Rightarrow$(i) is proved without the use of the real potential theory (see Section 4) and thus we also obtain a different proof of the continuity of $(dd^c)^2$ for decreasing sequences.

For a bounded hyperconvex domain Ω in $\mathbb{C}^n$ (a ball is an example of a hyperconvex domain) Cegrell [8] introduced the following class of psh functions. One says that a negative $u \in PSH(\Omega)$ belongs to $\mathcal{E}(\Omega)$ if for every $z_0 \in \Omega$ one can find an open neighborhood $U \Subset \Omega$ of z_0 and a decreasing sequence $u_j \in PSH \cap L^\infty(\Omega)$ such that u_j converges to u in U, $\lim_{z \to \partial\Omega} u_j(z) = 0$ and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$. It was shown in [8] that $\mathcal{E}(\Omega)$ is the biggest subclass $\mathcal{K}$ of $PSH(\Omega)$ satisfying

$$\mathcal{K} \ni u \leq v \in \mathcal{PSH}(\Omega) \Rightarrow v \in \mathcal{K}, \tag{1.3}$$

where the Monge-Ampère operator can be well defined and is continuous under decreasing sequences.

The strategy of the proof of Theorem 1.1 is the following. We first show that the conditions (iii) and (iv) are equivalent. Moreover, if Ω is hyperconvex then u satisfies (iii) (or (iv)) if and only if $u \in \mathcal{E}(\Omega)$. (It shows by the way that to belong to the Cegrell class $\mathcal{E}$ is a local property—that is, if $\Omega = \bigcup_\iota \Omega_\iota$ then $u \in \mathcal{E}(\Omega)$ if and only if $u|_{\Omega_\iota} \in \mathcal{E}(\Omega_\iota)$ for every ι.) Using the Cegrell result we then get the implication (iii)$\Rightarrow$(i), or in other words, that $\mathcal{E} \subset \mathcal{D}$. To show that we in fact have the equality it remains to prove the implication (ii)$\Rightarrow$(iii). We remark that it would then be very simple if we already knew that $\mathcal{D}$ satisfies (1.3) (or Theorem 1.2), which we do not *a priori* assume.

Acknowledgments. The author would like to thank Urban Cegrell for inspiring discussions on the subject.

2. The basic estimates. If $n = 2$ then it is clear that conditions (iii) and (iv) in Theorem 1.1 are equivalent, they mean precisely that u belongs to the Sobolev space $W^{1,2}_{loc}$. The goal of this section is to show that they are also equivalent for $n \geq 3$ and that for functions satisfying them one can well define the operator $(dd^c)^n$.

PROPOSITION 2.1. *Let $\Omega' \Subset \Omega$ be domains in $\mathbb{C}^n$. Assume that $2 \leq m \leq n$ and that either $r \leq 0$ or $r \geq 1$. Then for any $u \in PSH \cap C(\Omega)$, $u < 0$, we have*

$$\int_{\Omega'} |u|^r (dd^c u)^m \wedge \omega^{n-m} \leq C \int_{\Omega} |u|^r du \wedge d^c u \wedge (dd^c u)^{m-2} \wedge \omega^{n-m+1},$$

where C is a positive constant depending only on Ω' and Ω.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ be equal to 1 in a neighborhood of $\overline{\Omega'}$ and $0 \leq \varphi \leq 1$ elsewhere. Set $T := (dd^c u)^{m-2} \wedge \omega^{n-m}$. Integrating by parts we obtain

$$\int_{\Omega'} |u|^r (dd^c u)^2 \wedge T \leq \int_{\Omega} \varphi |u|^r (dd^c u)^2 \wedge T = -\int_{\Omega} du \wedge d^c u \wedge dd^c(\varphi |u|^r) \wedge T.$$

We also have

$$\begin{aligned} -du \wedge d^c u \wedge dd^c(\varphi|u|^r) \wedge T &= -|u|^r du \wedge d^c u \wedge dd^c \varphi \wedge T \\ &\quad - r(r-1)|u|^{r-2} du \wedge d^c u \wedge dd^c u \wedge T \\ &\leq C|u|^r du \wedge d^c u \wedge T \wedge \omega \end{aligned}$$

which completes the proof. □

The crucial step is the following estimate:

THEOREM 2.2. *Let $\Omega' \Subset \Omega$ be domains in $\mathbb{C}^n$. Assume that $2 \leq m \leq n$ and $r \geq 0$. Then for $u, v \in PSH \cap C(\Omega)$ with $u \leq v < 0$ one has*

$$\begin{aligned} &\int_{\Omega'} |v|^r \quad dv \wedge d^c v \wedge (dd^c v)^{m-2} \wedge \omega^{n-m+1} \\ &\qquad \leq C \left(\int_{\Omega} |u|^{m+r} \omega^n + \sum_{p=0}^{m-2} \int_{\Omega} |u|^{m-p+r-2} du \wedge d^c u \wedge (dd^c u)^p \wedge \omega^{n-p-1} \right), \end{aligned}$$

where C is a constant depending only on Ω', Ω and r.

Proof. Let S_1, S_2 be arbitrary currents of the form

$$S_1 = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1} \wedge \omega^{n-m}$$
$$S_2 = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-2} \wedge \omega^{n-m},$$

where $u_1, \ldots, u_{m-1} \in PSH \cap C(\Omega)$. By C we will denote possibly different constants depending only on Ω', Ω and r. The desired estimate can be easily deduced from the following three inequalities

$$\int_{\Omega'} |v|^r dv \wedge d^c v \wedge S_1 \leq C \int_\Omega |u|^r \big(u^2\omega + du \wedge d^c u\big) \wedge S_1, \tag{2.1}$$

$$\int_{\Omega'} |u|^{r+1} dd^c v \wedge S_1 \leq C \int_\Omega |u|^r \big(u^2\omega + du \wedge d^c u\big) \wedge S_1, \tag{2.2}$$

$$\begin{aligned} &\int_{\Omega'} |u|^r du \wedge d^c u \wedge dd^c v \wedge S_2 \\ &\quad \leq C \int_\Omega |u|^r \big(|u|^3\omega^2 + |u|\, du \wedge d^c u \wedge \omega + du \wedge d^c u \wedge dd^c u\big) \wedge S_2. \end{aligned} \tag{2.3}$$

Let φ be as in the proof of Proposition 2.1. We first prove (2.2). Integrating by parts we get

$$\int_{\Omega'} |u|^{r+1} dd^c v \wedge S_1 \leq \int_\Omega \varphi |u|^{r+1} dd^c v \wedge S_1 = -\int_\Omega |v|\, dd^c\big(\varphi |u|^{r+1}\big) \wedge S_1. \tag{2.4}$$

For any constant t we have $d(u + t\varphi) \wedge d^c(u + t\varphi) \geq 0$ and therefore

$$\pm u(du \wedge d^c\varphi + d\varphi \wedge d^c u) \leq du \wedge d^c u + u^2 d\varphi \wedge d^c\varphi.$$

Using this we get

$$\begin{aligned} \mp dd^c\varphi |u|^r &= \mp |u|^r dd^c\varphi \pm r\,|u|^{r-1}(du \wedge d^c\varphi + d\varphi \wedge d^c u) \\ &\quad - r(r-1)\varphi\, du \wedge d^c u + r\,\varphi\,|u|\, dd^c u \\ &\leq C|u|^{r-2}\big(u^2\omega + du \wedge d^c u\big) \pm r\,\varphi\,|u|^{r-1} dd^c u. \end{aligned} \tag{2.5}$$

Moreover,

$$\begin{aligned} \int_\Omega \varphi |u|^{r+1} dd^c u \wedge S_1 &= (r+2)^{-1} \int_\Omega \varphi\big(dd^c |u|^{r+2} + (r+1)^{-1} du \wedge d^c u\big) \wedge S_1 \\ &= (r+2)^{-1} \int_\Omega |u|^r \big(u^2 dd^c\varphi + (r+1)^{-1}\varphi\, du \wedge d^c u\big) \wedge S_1. \end{aligned}$$

Combining this with (2.4) and (2.5) we get (2.2).

To show (2.1) we estimate

$$\begin{aligned}\int_{\Omega'} |v|^r dv \wedge d^c v \wedge S_1 &\leq \int_\Omega \varphi |u|^r dv \wedge d^c v \wedge S_1 \\ &= \int_\Omega \varphi |u|^r \left(\frac{1}{2} dd^c v^2 - v dd^c v\right) \wedge S_1 \\ &\leq \frac{1}{2}\int_\Omega v^2 dd^c(\varphi|u|^r) \wedge S_1 + \int_\Omega \varphi |u|^{r+1} dd^c v \wedge S_1\end{aligned}$$

and (2.1) follows from (2.5) and (2.2).

Further

$$\begin{aligned}\int_{\Omega'} |u|^r du \wedge d^c u \wedge dd^c v \wedge S_2 &\leq \int_\Omega \varphi |u|^r du \wedge d^c u \wedge dd^c v \wedge S_2 \\ &= \int_\Omega \varphi |u|^r \left(\frac{1}{2} dd^c u^2 - u dd^c u\right) \wedge dd^c v \wedge S_2 \\ &= \int_\Omega |v|\, \alpha \wedge S_2,\end{aligned}$$

where

$$\begin{aligned}\alpha &= -\frac{1}{2} dd^c(\varphi|u|^r) \wedge dd^c u^2 - dd^c(\varphi|u|^{r+1}) \wedge dd^c u \\ &= -dd^c(\varphi|u|^r) \wedge du \wedge d^c u - dd^c(\varphi|u|^{r+1}) \wedge dd^c u + |u| dd^c(\varphi|u|^r) \wedge dd^c u \\ &\leq C\left[|u|^{r-1} du \wedge d^c u \wedge (|u|\omega + dd^c u) + |u|^{r+1}\omega \wedge dd^c u + |u|^r (dd^c u)^2\right]\end{aligned}$$

on the support of φ, by (2.5). Now (2.3) can be deduced from (2.2) applied for $v = u$. □

Theorem 2.2 implies in particular that conditions (iii) and (iv) in Theorem 1.1 are equivalent. More generally we have the following local result (we consider the germs of functions).

COROLLARY 2.3. *Assume that $2 \leq m \leq n$ and that either $r = 0$ or $r \geq 1$. Let u be a negative psh function such that there exists a sequence $\widetilde{u}_j \in PSH \cap C$ decreasing to u such that the sequences of measures*

$$|\widetilde{u}_j|^{r+m-p-2} d\widetilde{u}_j \wedge d^c \widetilde{u}_j \wedge (dd^c \widetilde{u}_j)^p \wedge \omega^{n-p-1}, \quad p = 0, 1, \ldots, m-2,$$

are locally weakly bounded. Then for every sequence $u_j \in PSH \cap C$ decreasing to u the sequences of measures

$$|u_j|^a du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge \omega^{n-p-1}, \quad p = 0, 1, \ldots, m-2, \quad 0 \leq a \leq r+m-p-2,$$
$$|u_j|^b (dd^c u_j)^q \wedge \omega^{n-q}, \quad q = 0, 1, \ldots, m, \quad 0 \leq b \leq r+m-q,$$

are locally weakly bounded.

Proof. Since the problem is purely local and on every compact set the function u is bounded above by a negative constant, we may assume that v_j, u_j are bounded above by -1 for every j. It is thus enough to consider the cases $a = r+m-p-2$ and $b = r+m-q$. Since without loss of generality we may add to u_j a sequence of constants decreasing to 0, and also choose a subsequence of $\widetilde{u}_j$ if necessary, we may assume that $\widetilde{u}_j \le u_j$. It is now enough to use Theorem 2.2 to get the first assertion and Proposition 2.1 to deduce the second one. □

We are now able to characterize the Cegrell class $\mathcal{E}$.

THEOREM 2.4. *If Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ and u is a negative psh function in Ω then $u \in \mathcal{E}(\Omega)$ if and only if u satisfies condition (iii) in Theorem 1.1.*

Proof. If $u \in PSH(\Omega)$ is negative then by [8] there exists a sequence $u_j \in PSH(\Omega) \cap C(\overline{\Omega})$ decreasing to u in Ω and vanishing on $\partial\Omega$. For a ball $B \Subset \Omega$ we set

$$\widetilde{u}_j := \sup\{v \in PSH(\Omega) : v < 0 \text{ in } \Omega, v \le u_j \text{ in } B\}.$$

Then $\widetilde{u}_j \in PSH(\Omega) \cap C(\overline{\Omega})$ (by [17]), $\widetilde{u}_j = 0$ on $\partial\Omega$, $\widetilde{u}_j = u_j$ in B and $(dd^c\widetilde{u}_j)^n = 0$ in $\Omega \setminus \overline{B}$. First assume that u satisfies iii) in Theorem 1.1. Let $\varphi \in C_0^\infty(\Omega)$ be equal to 1 on $\overline{B}$. Then

$$\int_\Omega (dd^c\widetilde{u}_j)^n = \int_{\overline{B}} \varphi (dd^c\widetilde{u}_j)^n = -\int_\Omega d\widetilde{u}_j \wedge d^c\widetilde{u}_j \wedge (dd^c\widetilde{u}_j)^{n-2} \wedge dd^c\varphi$$

and it follows from Theorem 2.2 (applied in Ω' such that $B \Subset \Omega' \Subset \Omega$) that this sequence is bounded.

On the other hand, if $u \in \mathcal{E}(\Omega)$ and $\psi \in PSH(\Omega) \cap C(\overline{\Omega})$ is such that $\psi = 0$ on $\partial\Omega$ and $dd^c\psi \ge \omega$ in B then for $p = 0, 1, \dots, n-2$ we get

$$\begin{aligned}
&\int_B |\widetilde{u}_j|^{n-p-2} d\widetilde{u}_j \wedge d^c\widetilde{u}_j \wedge (dd^c\widetilde{u}_j)^p \wedge \omega^{n-p-1} \\
&\qquad \le \int_\Omega |\widetilde{u}_j|^{n-p-2} d\widetilde{u}_j \wedge d^c\widetilde{u}_j \wedge (dd^c\widetilde{u}_j)^p \wedge (dd^c\psi)^{n-p-1} \\
&\qquad = (n-p-1)^{-1} \int_\Omega |\widetilde{u}_j|^{n-p-1} (dd^c\widetilde{u}_j)^{p+1} \wedge (dd^c\psi)^{n-p-1} \\
&\qquad \le (n-p-2)!\ ||\psi||_{L^\infty(\Omega)}^{n-p-1} \int_\Omega (dd^c\widetilde{u}_j)^n,
\end{aligned}$$

where the last inequality follows by successive integration by parts (as in [4], see also the proof of Proposition 3.1 below). □

3. Proofs of the main results. In this section we will complete the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We have already proven that (iii)⇔(iv) and it is clear that (i)⇒(ii). By Theorem 2.4 and Cegrell's theorem [8, Theorem 4.2] it follows that (iii)⇒(i). For the convenience of the reader we will now provide a direct proof of this implication not using the class $\mathcal{E}$. The argument will be similar to those of Cegrell [8]. Let $B_1 \Subset B_2 \Subset B_3$ be open balls in U. For $v \in PSH \cap C(B_3)$, $v < 0$, we set

$$\widetilde{v} := \sup\{w \in PSH(B_2) : w < 0 \text{ in } B_2,\ w < v \text{ in } B_1\}.$$

By well-known facts from pluripotential theory (see [3], [17]) we then have $\widetilde{v} \in PSH(B_2) \cap C(\overline{B}_2)$, $\widetilde{v} = 0$ on ∂B_2, $\widetilde{v} = v$ on $\overline{B}_1$ and $(dd^c\widetilde{v})^n = 0$ in $B_2 \setminus \overline{B}_1$. If u satisfies iii), from Theorem 2.2 and Proposition 2.1 it follows that for any sequence $u_j \in PSH \cap C^\infty(B_3)$ decreasing to u in B_3 one has

$$\sup_j \int_{B_2} (dd^c\widetilde{u}_j)^n < \infty. \tag{3.1}$$

To prove that (i) holds it is enough to show that if $\varphi \in C_0^\infty(B_1)$ then the sequence

$$\int_{B_1} \varphi(dd^c u_j)^n = \int_{B_1} \varphi(dd^c\widetilde{u}_j)^n \tag{3.2}$$

is convergent and its limit is independent of the choice of u_j. Suppose this is not the case. Since we can write $\varphi = \psi_1 - \psi_2$, where $\psi_1, \psi_2 \in PSH(B_2) \cap C^\infty(\overline{B}_2)$ are such that $\psi_1 = \psi_2 = 0$ on ∂B_2, from (3.1) it follows that we may replace φ by ψ_1 in (3.2). Passing to subsequences and subtracting small constants if necessary, we can therefore find appropriate sequences u_j and v_j such that $u_j \le v_j$ and

$$\lim_{j\to\infty} \int_{B_2} \psi_1(dd^c\widetilde{u}_j)^n > \lim_{j\to\infty} \int_{B_2} \psi_1(dd^c\widetilde{v}_j)^n.$$

However, integration by part easily leads to contradiction:

$$\begin{aligned} \int_{B_2} \psi_1(dd^c\widetilde{u}_j)^n &= \textstyle\int_{B_2} \widetilde{u}_j dd^c\psi_1 \wedge (dd^c\widetilde{u}_j)^{n-1} \\ &\le \textstyle\int_{B_2} \widetilde{v}_j dd^c\psi_1 \wedge (dd^c\widetilde{u}_j)^{n-1} \\ &\le \cdots \le \textstyle\int_{B_2} \psi_1(dd^c\widetilde{v}_j)^n. \end{aligned}$$

This proves the implication (iii)⇒(i)

It now remains to show that (ii)⇒(iv). We will generalize a construction used in the proof of [5, Theorem 3.5]. Suppose that u does not satisfy (iv) and let $p_0 = 0, 1, \dots, n-2$ be such that the sequences of measures (1.2) are locally weakly bounded in Ω for $p < p_0$ and that for a ball $B \Subset \Omega$ we have

$$\lim_{k\to\infty} \int_B |v_k|^{n-p_0-2} dv_k \wedge d^c v_k \wedge (dd^c v_k)^{p_0} \wedge \omega^{n-p_0-1} = \infty, \tag{3.3}$$

where $v_k = \lambda_k u * \rho_{1/k}$ and $u * \rho_{1/k}$ are the standard regularizations of u whereas λ_k is a sequence of positive numbers (strictly) increasing to 1. We claim that there exists an increasing sequence $k = k(j) \geq j+1$ such that for every j

$$\int_B |v_j - v_k|^{n-p_0-2} d(v_j - v_k) \wedge d^c(v_j - v_k) \wedge (dd^c v_k)^{p_0} \wedge \omega^{n-p_0-1} \geq j. \tag{3.4}$$

For $k \geq j+1$ we have $|v_j - v_k| \geq (1 - \lambda_j/\lambda_{j+1})|v_k|$ and to show (3.4) it is enough to prove that for every fixed j one has

$$\lim_{k\to\infty} \int_B |v_k|^{n-p_0-2} d(v_j - v_k) \wedge d^c(v_j - v_k) \wedge (dd^c v_k)^{p_0} \wedge \omega^{n-p_0-1} = \infty. \tag{3.5}$$

We have

$$\begin{aligned} &\left(\int_B |v_k|^{n-p_0-2} d(v_j - v_k) \wedge d^c(v_j - v_k) \wedge (dd^c v_k)^{p_0} \wedge \omega^{n-p_0-1}\right)^{1/2} \\ &\quad \geq \left(\int_B |v_k|^{n-p_0-2} dv_k \wedge d^c v_k \wedge (dd^c v_k)^{p_0} \wedge \omega^{n-p_0-1}\right)^{1/2} \\ &\quad\quad - \left(\int_B |v_k|^{n-p_0-2} dv_j \wedge d^c v_j \wedge (dd^c v_k)^{p_0} \wedge \omega^{n-p_0-1}\right)^{1/2}. \end{aligned} \tag{3.6}$$

Since the sequences (1.2) are bounded for $p < p_0$, we may use Corollary 2.3 with $m = p_0 + 1$ and $r = n - m$. It follows in particular that for $b \leq n - p_0$ the sequence of measures $|v_k|^b (dd^c v_k)^{p_0} \wedge \omega^{n-p_0}$ is locally weakly bounded in Ω. Therefore, the last term in (3.6) is bounded in k and by (3.3) we obtain (3.5). Hence (3.4) holds.

Let B' be a ball satisfying $B \Subset B' \Subset \Omega$. We set

$$\begin{aligned} u_j &:= \sup\{w \in PSH(B') : w \leq v_j \text{ in } B',\ w \leq v_k \text{ in } B\} \\ &= \sup\{w \in PSH(B') : w \leq h_j\}, \end{aligned}$$

where $h_j \in C(\overline{B'})$ is defined by $h_j = v_k$ in $\overline{B}$, $h_j = v_j$ on $\partial B'$ and h_j is harmonic in $B' \setminus \overline{B}$. By [17] $u_j \in PSH(B') \cap C(\overline{B'})$. It is clear that u_j is decreasing to u in B' and therefore by (ii) (by approximation it follows that we can use this condition also for sequences of continuous functions) we have

$$\sup_j \int_{\overline{B}} (dd^c u_j)^n < \infty.$$

We also have $(dd^c u_j)^n = 0$ in $\{u_j < v_j\}$, and, since $u_j \leq v_j$, it follows that $(dd^c u_j)^n \leq (dd^c v_j)^n$ on $\{u_j = v_j\}$. (It is a general fact that if u, v are psh and continuous then $(dd^c \max\{u, v\})^n \geq (dd^c v)^n$ on the set $\{u \leq v\}$ - see e.g. [1].) By another application of (ii), this time to the sequence v_j, we obtain therefore

$$\sup_j \int_{B'} (dd^c u_j)^n < \infty.$$

However, combining (3.4) with the following estimate we will arrive at contradiction, which will finish the proof of Theorem 1.1.

PROPOSITION 3.1. *Assume that* $0 \leq p \leq n-2$ *and that* Ω *is a bounded domain in* $\mathbb{C}^n$. *Let* $u, v \in PSH(\Omega) \cap C(\overline{\Omega})$ *be such that* $u \leq v$ *in* Ω *and* $u = v$ *on* $\partial\Omega$. *Then*

$$\int_\Omega (v-u)^{n-p-2} d(v-u) \wedge d^c(v-u) \wedge (dd^c u)^p \wedge \omega^{n-p-1} \leq C \int_\Omega (dd^c u)^n,$$

where C is a constant depending only on n and on an upper bound for the diameter of Ω.

Proof. It will be similar to that of [4, Theorem 2.1]. For $\varepsilon > 0$ set $v_\varepsilon := \max\{u, v-\varepsilon\}$. Then by the weak convergence we have

$$\begin{aligned}\int_\Omega & (v-u)^{m-p-2} d(v-u) \wedge d^c(v-u) \wedge (dd^c u)^p \wedge \omega^{n-p-1} \\ &\leq \liminf_{\varepsilon\to 0} \int_\Omega (v_\varepsilon - u)^{m-p-2} d(v_\varepsilon - u) \wedge d^c(v_\varepsilon - u) \wedge (dd^c u)^p \wedge \omega^{n-p-1}.\end{aligned}$$

We may thus assume that $u = v$ near $\partial\Omega$.

Set $\psi(z) := |z - z_0|^2 - M$, where $z_0 \in \Omega$ and M is so big that $\psi \leq 0$ in Ω. We then have

$$\begin{aligned}(n-p-1) & \int_\Omega (v-u)^{n-p-2} d(v-u) \wedge d^c(v-u) \wedge (dd^c u)^p \wedge \omega^{n-p-1} \\ &= \int_\Omega d(v-u)^{n-p-1} \wedge d^c(v-u) \wedge (dd^c u)^p \wedge \omega^{n-p-1} \\ &\leq \int_\Omega (v-u)^{n-p-1} (dd^c u)^{p+1} \wedge \omega^{n-p-1} \\ &= \int_\Omega \psi\, dd^c(v-u)^{n-p-1} \wedge (dd^c u)^{p+1} \wedge \omega^{n-p-2},\end{aligned}$$

and

$$-dd^c(v-u)^{n-p-1} \le (n-p-1)(v-u)^{n-p-2}dd^c u.$$

Therefore

$$\int_\Omega (v-u)^{n-p-1}(dd^c u)^{p+1} \wedge \omega^{n-p-1} \le M(n-p-1)\int_\Omega (v-u)^{n-p-2}(dd^c u)^{p+2} \wedge \omega^{n-p-2}.$$

Successive application of this inequality for $p = 0, 1, \dots, n-2$ gives the desired estimate. □

Proof of Theorem 1.2. Let K be a compact subset of Ω such that $u \le v$ in $\Omega \setminus K$. By Theorem 2.2 (and Theorem 1.1) we have $v \in \mathcal{D}(\Omega \setminus \mathcal{K})$. Let Ω', Ω'' be smooth domains satisfying $K \subset \Omega' \Subset \Omega'' \Subset \Omega$. We need to show that $v \in \mathcal{D}(\Omega')$. Let $v_j \in PSH \cap C^\infty(\Omega'')$ be a sequence decreasing to v in Ω''. We can find a sequence $u_j \in PSH \cap C^\infty(\Omega'')$ decreasing to u in Ω'' and such that $u_j \le v_j + 1/j$ near $\partial\Omega'$. Set $\widehat{u}_j := \max\{u_j, v_j + 2/j\}$. Then $\widehat{u}_j$ decreases to $\max\{u, v\} \in \mathcal{D}(\Omega'')$ and $\widehat{u}_j = v_j + 2/j$ near $\partial\Omega'$. We thus have

$$\int_{\Omega'} (dd^c v_j)^n = \int_{\Omega'} (dd^c \widehat{u}_j)^n.$$

By Theorem 1.1 we obtain that for every such a sequence v_j

$$\sup_j \int_{\Omega'} (dd^c v_j)^n < \infty.$$

This means that the function v satisfies a slightly weaker condition than (ii) in Theorem 1.1: only for sequences defined on a neighborhood of K. However, the proof of implication (ii)⇒(iii) can be repeated in this case with only one modification: B' has to be chosen as Ω''. □

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