EXTREMAL FUNCTIONS AND EQUILIBRIUM MEASURES FOR BOREL SETS

by

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Abstract. We extend some results on extremal functions and equilibrium measures from compact to Borel subsets of $\mathbb{C}^N$.

0. Introduction.

Let E be a bounded Borel set with closure $\overline{E}$ contained in a bounded, hyperconvex domain Ω in $\mathbb{C}^N$ (i.e., there exists a continuous, negative plurisubharmonic (psh) exhaustion function for Ω). We let

$$u_E(z) := \sup\{u(z) : u \text{ psh in } \Omega,\ u \leq 0,\ u \leq -1 \text{ on } E\}$$

and call $u_E^*(z) := \limsup_{\zeta\to z} u_E(\zeta)$ the *relative extremal function of* E (relative to Ω). Similarly, letting

$$V_E(z) := \sup\{u(z) : u \in \mathrm{L},\ u \leq 0 \text{ on } E\}$$

where

$$\mathrm{L} := \{u \text{ psh in } \mathbb{C}^N :\ u(z) - \log|z| = 0(1),\ |z| \to \infty\},$$

we call $V_E^*(z) := \limsup_{\zeta\to z} V_E(\zeta)$ the *global extremal function of* E. It is well-known that $u_E^* \equiv 0 \iff V_E^* \equiv +\infty \iff E$ is *pluripolar*; i.e., there exists u psh in $\mathbb{C}^N$ with $E \subset \{z \in \mathbb{C}^N : u(z) = -\infty\}$. If E is not pluripolar, then, using the complex Monge-Ampere operator $(dd^c(\cdot))^N$ for locally bounded psh functions, we can define the relative and global *equilibrium measures* $(dd^c u_E^*)^N$ and $(dd^c V_E^*)^N$ for E. It is known (cf., [BT1] or [K]) that these measures are supported in $\overline{E}$ and, in the case where E is compact and the polynomially convex hull $\widehat{E}$ of E is contained in Ω, $(dd^c u_E^*)^N$ and $(dd^c V_E^*)^N$ are mutually absolutely continuous [L]. Moreover, one can define a nonnegative function $C(E)$ on the Borel subsets E of Ω via

$$C(E) := \sup\{\int_E (dd^c u)^N : u \text{ psh on } \Omega,\ 0 \leq u \leq 1\}.$$

For Borel sets we have (Proposition 4.7.2 [K])

$$C(E) = \int_\Omega (dd^c u_E^*)^N.$$

In fact, from Proposition 10.1 [BT1] it follows that

$$C(E) = \int_\Omega -u_E^*(dd^c u_E^*)^N. \tag{0.1}$$

The purpose of this note is to give more precise information on the behavior of the extremal functions and extremal measures for Borel sets. First we prove the following equivalences.

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Theorem 1. *Let E and F be Borel sets with $\overline{E} \subset \Omega$ and $F \subset E$. The following statements are equivalent.*
1. $C(E) = C(F)$
2. $u_E^* = u_F^*$
3. $V_E^* = V_F^*$.

Next we generalize the mutual absolute continuity of the relative and global equilibrium measures.

Theorem 2. *Let E be a nonpluripolar Borel set with $\widehat{\overline{E}} \subset \Omega$ where Ω is hyperconvex. Then*

$$(\sup_{\partial\Omega} V_E)^{-N}(dd^c V_E^*)^N \leq (dd^c u_E^*)^N \leq (\inf_{\partial\Omega} V_E)^{-N}(dd^c V_E^*)^N. \tag{0.2}$$

Note that we have $0 < \inf_{\partial\Omega} V_E \leq \sup_{\partial\Omega} V_E < \infty$ (cf., Proposition 5.3.3 [K]).

Theorem 1 was stated in [BT2] for E compact and also proved in this case by [Z]. The question of the validity of the theorem for Borel sets E was raised by Tom Bloom and is used in [B]. An alternate proof for the planar case ($N = 1$) can be found in [ST], pp. 226-227. The mutual absolute continuity of $(dd^c u_E^*)^N$ and $(dd^c V_E^*)^N$ for E compact was proved in [L].

1. Proof of Theorem 1.

Note that Theorem 1 is trivial if E is pluripolar; thus we assume for the remainder of the discussion that E is nonpluripolar.

Lemma 1.1. *Let E be a nonpluripolar Borel set with $\overline{E} \subset \Omega$. Define*

$$E' := \{z \in \Omega : u_E^*(z) = -1\}.$$

Then
(1) $E' = \{z \in \Omega : V_E^*(z) = 0\}$*;*
(2) $u_E^* = u_{E'}^*$ *and* $V_E^* = V_{E'}^*$*.*

Proof. (1) follows from Proposition 5.3.3 [K]. For (2), we prove $u_E^* = u_{E'}^*$; the proof for $V_E^* = V_{E'}^*$ is similar. First of all, from the definition of E', we have $u_E^* \leq u_{E'} \leq u_{E'}^*$. Since $u_E^* = -1$ on E except perhaps a pluripolar set (cf., Theorem 4.7.6 [K]), we also have that $E \subset E' \cup A$ where A is pluripolar. By Proposition 5.2.5 [K], $u_{E'}^* = u_{E' \cup A}^* \leq u_E^*$ and equality holds. ♣

Proof of Theorem 1. *1. implies 2.*: This argument was shown to us by Urban Cegrell. Suppose $C(E) = C(F)$. Since $F \subset E$, $u_E^* \leq u_F^*$. Using this inequality, (0.1), and Stokes' theorem (recall that $\overline{E} \subset \Omega$ so that $u_E^*, u_F^* = 0$ on $\partial\Omega$ (cf. [K], Proposition 4.5.2)), we obtain

$$\begin{aligned} C(E) = \int_\Omega -u_E^*(dd^c u_E^*)^N &\geq \int_\Omega -u_F^*(dd^c u_E^*)^N \\ &= \int_\Omega -u_E^* dd^c u_F^* \wedge (dd^c u_E^*)^{N-1} \\ &\geq \int_\Omega -u_F^* dd^c u_F^* \wedge (dd^c u_E^*)^{N-1} \end{aligned}$$

$$\cdots \geq \int_\Omega -u_F^*(dd^c u_F^*)^N = C(F).$$

Thus equality holds throughout; in particular, from the first line we have

$$\int_\Omega (u_E^* - u_F^*)(dd^c u_E^*)^N = 0; \text{ i.e.,}$$

$$\int_{\{u_E^* < u_F^*\}} (dd^c u_E^*)^N = 0.$$

By the comparison principle (Corollary 3.7.5 [K]), $u_E^* \geq u_F^*$ and equality holds.

2. implies 1.: This follows from (0.1).

2. $\iff$ 3.: Let $E' := \{z \in \Omega : u_E^*(z) = -1\}$ and $F' := \{z \in \Omega : u_F^*(z) = -1\}$. From (1) and (2) of Lemma 1.1 it suffices to show that

$$u_E^* = u_F^* \iff E' = F'$$

(the proof that $V_E^* = V_F^* \iff E' = F'$ is similar). The implication $u_E^* = u_F^*$ implies $E' = F'$ is obvious. For the reverse implication, if $E' = F'$, then $u_F^* = -1$ on $F' = E'$ so that $u_F^* \leq u_{E'} \leq u_{E'}^* = u_E^*$ (the last equality is from (2) of Lemma 1.1). The reverse inequality follows since $F \subset E$. ♣

2. Proof of Theorem 2.

In this section, we prove Theorem 2, the mutual absolute continuity of the equilibrium measures $(dd^c u_E^*)^N$ and $(dd^c V_E^*)^N$ when E is a nonpluripolar Borel set. The main tool will be the following result.

Lemma 2.1. *Let E be a compact subset of a bounded domain Ω in $\mathbb{C}^N$. Let u_1, u_2 be nonnegative continuous functions on $\overline{\Omega}$ which are psh on Ω. If*
(1) $u_1 = u_2 = 0$ *on* E;
(2) $u_1 \geq u_2$ *on* Ω;
(3) $(dd^c u_1)^N = (dd^c u_2)^N = 0$ *on* $\Omega \setminus E$;
(4) $u_2 > 0$ *on* $\partial\Omega$,
then $(dd^c u_1)^N \geq (dd^c u_2)^N$; *i.e., for all* $\phi \in C_0^\infty(\Omega)$ *with* $\phi \geq 0$,

$$\int_\Omega \phi (dd^c u_1)^N \geq \int_\Omega \phi (dd^c u_2)^N.$$

Proof. This lemma follows easily from Theorem 5.6.5 [K] (see also [L]). For let ω be a domain containing E such that $\overline{\omega} \subset \Omega$ and $u_2 > 0$ on $\partial\omega$. Take any positive t with $t < 1$. Then we have $u_1 \geq t u_2 + \eta$ on $\partial\omega$ for some $\eta > 0$. By Theorem 5.6.5 [K], $(dd^c u_1)^N \geq (dd^c(t u_2))^N$ and the lemma follows. ♣

We shall also need two simple lemmas.

Lemma 2.2. *Let E and Ω be as in Theorem 2. Then*

$$\sup_{\partial\Omega} V_E = \sup_{\overline{\Omega}} V_E^*.$$

Proof. The inequality $\sup_{\partial\Omega} V_E \leq \sup_{\overline{\Omega}} V_E^*$ is obvious. To show the reverse inequality take any $u \in \mathrm{L}$ with $u \leq 0$ on E. Then $u \leq \sup_{\partial\Omega} V_E$ on $\overline{\Omega}$; hence $V_E \leq \sup_{\partial\Omega} V_E$ on $\overline{\Omega}$ so that $\sup_{\overline{\Omega}} V_E^* \leq \sup_{\partial\Omega} V_E$. ♣

Lemma 2.3. *Let $\{f_j\}$ be a sequence of lower (resp. upper) semicontinuous functions defined on a compact set K which increase (resp. decrease) to a bounded function f. Then*

$$\lim_{j\to\infty}(\inf_K f_j) = \inf_K f \qquad (\textit{resp.}\ \ \lim_{j\to\infty}(\sup_K f_j) = \sup_K f).$$

Proof. We have $\inf_K f_j \uparrow a \leq \inf_K f$. To prove the reverse inequality, assume that $a < b < \inf_K f$ for some b. From the lower semicontinuity of the $\{f_j\}$ it follows that the nonempty sets $\{f_j \leq b\}$ are compact. However, these sets decrease to the empty set since $\inf_K f > b$; this is a contradiction. The corresponding statement for a decreasing sequence of upper semicontinuous functions $\{f_j\}$ follows from the previous argument applied to the functions $\{-f_j\}$. ♣

Proof of Theorem 2. First assume that E is compact and L-regular; i.e., $V_E = V_E^*$. Then V_E and u_E are continuous in $\overline{\Omega}$ and (0.2) follows from Lemma 2.1, since

$$V_E / \sup_{\partial\Omega} V_E \leq u_E + 1 \leq V_E / \inf_{\partial\Omega} V_E$$

(cf., Proposition 5.3.3 [K]).

Now suppose that E is compact but not necessarily L-regular. For $j = 1, 2, \ldots$ define $E_j := \{z \in E : \operatorname{dist}(z, E) \leq 1/j\}$. Then for j sufficiently large $\widehat{E}_j \subset \Omega$ and E_j is L-regular (Corollary 5.1.5 [K]). Furthermore, $E_j \downarrow E$, $u_{E_j} \uparrow u_E$, and $V_{E_j} \uparrow V_E$ as $j \uparrow \infty$. Moreover, $\sup_{\partial\Omega} V_{E_j} \leq \sup_{\partial\Omega} V_E$ and, by Lemma 2.3, $\lim_{j\to\infty}(\inf_{\partial\Omega} V_{E_j}) = \inf_{\partial\Omega} V_E$. From the previous case and the continuity of the Monge-Ampère operator under monotone increasing limits (cf., Theorem 3.6.1 [K] or Proposition 5.2 [BT1]), we get (0.2) for general nonpluripolar compact sets.

Finally, let E be an arbitrary nonpluripolar Borel set. Then from Corollary 8.5 [BT1] it follows that there exist compact sets E_j, $j = 1, 2, \ldots$ and an F_σ set F such that $E_j \uparrow F \subset E$, $u_{E_j}^* \downarrow u_F^* = u_E^*$, and $V_{E_j}^* \downarrow V_F^* = V_E^*$. Then $\inf_{\partial\Omega} V_{E_j} \geq \inf_{\partial\Omega} V_E$ and by Lemmas 2.2 and 2.3,

$$\lim_{j\to\infty}(\sup_{\partial\Omega} V_{E_j}) = \lim_{j\to\infty}(\sup_{\overline{\Omega}} V_{E_j}^*) = \sup_{\overline{\Omega}} V_E^* = \sup_{\partial\Omega} V_E.$$

Using the continuity of the Monge-Ampère operator under monotone decreasing limits (cf., Theorem 3.4.3 [K] or Theorem 2.1 [BT1]), we conclude the proof of the theorem. ♣

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