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EQULIBRIUM MEASURE OF A PRODUCT SUBSET OF $\mathbb{C}^n$

ZBIGNIEW BŁOCKI

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ABSTRACT. In this note we show that an equilibrium measure of a product of two subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, is a product of their equilibrium measures. We also obtain a formula for $(dd^c \max\{u,v\})^p$, where u,v are locally bounded plurisubharmonic functions and $2 \leq p \leq n$.

INTRODUCTION

Let E be a bounded subset of $\mathbb{C}^n$. The function

$$V_E := \sup\{u \in PSH(\mathbb{C}^n) : u|_E \leq 0, \sup_{z\in\mathbb{C}^n}(u(z) - \log^+|z|) < \infty\}$$

is called a *global extremal function* (or the *Siciak extremal function*) of E. It is known that V_E^*, the upper regularization of V_E, is plurisubharmonic in $\mathbb{C}^n$ if and only if E is not pluripolar. In such a case, by [BT1], $(dd^c V_E^*)^n$ is a well defined nonnegative Borel measure and it is called an *equilibrium measure* of E. We refer to [Kl] for a detailed exposition of this topic.

In this note we shall show

Theorem 1. *Let E and F be nonpluripolar bounded subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then*

$$V_{E\times F}^* = \max\{V_E^*, V_F^*\} \tag{1}$$

and

$$(dd^c V_{E\times F}^*)^{n+m} = (dd^c V_E^*)^n \wedge (dd^c V_F^*)^m. \tag{2}$$

Note that here we treat V_E^* (resp. V_F^*) as a function of $\mathbb{C}^{n+m}$ independent of the last m (respectively first n) variables.

The formula (1) was proved by Siciak (see [Si]) for E, F compact (see also [Ze] for a proof using the theory of the complex Monge-Ampère operator). For $n = m = 1$ the proof of (2) can be found in [BT2].

If $E \subset D$, where D is a bounded domain in $\mathbb{C}^n$, then the function

$$u_{E,D} := \sup\{v \in PSH(D) : v \leq 0, v|_E \leq -1\}$$

is called a *relative extremal function* of E. Combining our methods of the proof of Theorem 1 with a result from [EP] we can also obtain

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Theorem 2. *Let D be a bounded domain in $\mathbb{C}^n$ and G a bounded domain in $\mathbb{C}^m$. Then for arbitrary subsets $E \subset D$, $F \subset G$ we have*

$$u^*_{E\times F, D\times G} = \max\{u^*_{E,D}, u^*_{F,G}\} \tag{3}$$

and

$$(dd^c u^*_{E\times F, D\times G})^{n+m} = (dd^c u^*_{E,D})^n \wedge (dd^c u^*_{F,G})^m.$$

The *relative Monge-Ampère capacity* of $E \subset D$ is defined by

$$c(E,D) := \sup\{\int_E (dd^c u)^n : u \in PSH(D), -1 \le u \le 0\},$$

provided that E is Borel. If $E \subset D$ is arbitrary, then, as usual, we can define

$$c^*(E,D) := \inf_{E\subset U,\ U \text{ open}} c(U,D),$$
$$c_*(E,D) := \sup_{K\subset E,\ K \text{ compact}} c(K,D).$$

By [BT1], if $E \Subset D$ and D is hyperconvex (that is $(u_{E,D})_* = 0$ on ∂D), then

$$c^*(E,D) = \int_D (dd^c u^*_{E,D})^n.$$

Moreover, $c^*(E,D) = c(E,D) = c_*(E,D)$ if E is Borel. Theorem 2 thus gives

Theorem 3. *Assume that D and G are bounded hyperconvex domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then for $E \Subset D$, $F \Subset G$ we have*

$$c^*(E\times F, D\times G) = c^*(E,D)c^*(F,G). \quad \square$$

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Proofs

If Ω is an open subset of $\mathbb{C}^n$ and $1 \le p \le n$, then by [BT1] the mapping

$$(u_1, \dots, u_p) \longmapsto dd^c u_1 \wedge \dots \wedge dd^c u_p \tag{4}$$

is well defined on the set $(PSH \cap L^\infty_{loc}(\Omega))^p$ and its values are nonnegative currents of bidegree (p,p). Moreover, (4) is symmetric and continuous with respect to decreasing sequences. First, we shall prove

Theorem 4. *Let u, v be locally bounded plurisuharmonic functions. Then, if $2 \le p \le n$, we have*

$$(dd^c \max\{u,v\})^p$$
$$= dd^c \max\{u,v\} \wedge \sum_{k=0}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-1-k} - \sum_{k=1}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-k}.$$

Proof. We leave it as an exercise to the reader to show that a simple inductive argument reduces the proof to the case $p = 2$. By the continuity of (4) under decreasing sequences we may also assume that u, v are smooth.

Let $\chi : \mathbb{R} \to [0, +\infty)$ be smooth and such that $\chi(x) = 0$ if $x \le -1$, $\chi(x) = x$ if $x \ge 1$ and $0 \le \chi' \le 1$, $\chi'' \ge 0$ everywhere. Define

$$\psi_j := v + \frac{1}{j}\chi(j(u-v)).$$

Denote for simplicity $w = \max\{u, v\}$ and $\alpha = u - v$. We can easily check that $\psi_j \downarrow w$ as $j \uparrow \infty$. An easy computation gives

$$dd^c(\chi(j\alpha)/j) = \chi'(j\alpha)dd^c\alpha + j\chi''(j\alpha)d\alpha \wedge d^c\alpha. \tag{5}$$

Therefore

$$dd^c\psi_j = \chi'(j\alpha)dd^c u + (1 - \chi'(j\alpha))dd^c v + j\chi''(j\alpha)d\alpha \wedge d^c\alpha$$

and, in particular, ψ_j is plurisubharmonic.

From the definition of ψ_j we obtain

$$(dd^c\psi_j)^2 = (dd^c v)^2 + 2dd^c(\chi(j\alpha)/j) \wedge dd^c v + (dd^c(\chi(j\alpha)/j))^2. \tag{6}$$

We have weak convergences

$$\begin{aligned} (dd^c\psi_j)^2 &\longrightarrow (dd^c w)^2, \\ dd^c(\chi(j\alpha)/j) \wedge dd^c v &\longrightarrow dd^c(w - v) \wedge dd^c v, \end{aligned} \tag{7}$$

so it remains to analyze the third term of the right-hand side of (6). Using (5) and the fact that $(d\alpha \wedge d^c\alpha)^2 = 0$, we compute

$$\begin{aligned} (dd^c(\chi(j\alpha)/j))^2 &= (\chi'(j\alpha))^2(dd^c\alpha)^2 + 2j\chi'(j\alpha)\chi''(j\alpha)d\alpha \wedge d^c\alpha \wedge dd^c\alpha \\ &= d\left[(\chi'(j\alpha))^2 d^c\alpha \wedge dd^c\alpha\right] \\ &= dd^c(\gamma(j\alpha)/j) \wedge dd^c\alpha, \end{aligned}$$

where $\gamma : \mathbb{R} \to \mathbb{R}$ is such that $\gamma' = (\chi')^2$. In fact, if γ is chosen so that $\gamma(-1) = 0$, then $\gamma(jx)/j \downarrow \max\{0, x\}$ as $j \uparrow \infty$ and

$$(dd^c(\chi(j\alpha)/j))^2 \longrightarrow dd^c(w - v) \wedge dd^c\alpha$$

weakly. Combining this with (6) and (7) we conclude

$$\begin{aligned} (dd^c w)^2 &= (dd^c v)^2 + 2dd^c(w - v) \wedge dd^c v + dd^c(w - v) \wedge dd^c(u - v) \\ &= dd^c w \wedge (dd^c u + dd^c v) - dd^c u \wedge dd^c v \end{aligned}$$

which completes the proof of Theorem 4. □

From Theorem 4 we can immediately get the following two consequences:

Corollary 5. *If u is locally bounded, plurisubharmonic and h is pluriharmonic, then*

$$(dd^c \max\{u, h\})^p = dd^c \max\{u, h\} \wedge (dd^c u)^{p-1}.$$ □

Corollary 6. *Suppose u, v are locally bounded plurisubharmonic functions with $(dd^c u)^p = 0$ and $(dd^c v)^q = 0$, where $1 \leq p, q \leq n$ and $p + q \leq n$. Then $(dd^c \max\{u, v\})^{p+q} = 0$.* □

The main part of the proof of (2) will be contained in

Theorem 7. *Let D be open in $\mathbb{C}^n$ and G open in $\mathbb{C}^m$. Assume that u, v are nonnegative plurisubharmonic functions in D and G, respectively, such that*

$$\int_{\{u>0\}} (dd^c u)^n = 0 \quad \textit{and} \quad \int_{\{v>0\}} (dd^c v)^m = 0.$$

Then, treating u, v as functions on $D \times G$, we have

$$(dd^c \max\{u, v\})^{n+m} = (dd^c u)^n \wedge (dd^c v)^m.$$

Proof. Let w, χ and ψ_j be defined in the same way as in the proof of Theorem 4. By Theorem 4 and since $(dd^cu)^{n+1} = 0$, $(dd^cv)^{m+1} = 0$, we have

$$(dd^cw)^{n+m} = dd^cw \wedge \left[(dd^cu)^{n-1} \wedge (dd^cv)^m + (dd^cu)^n \wedge (dd^cv)^{m-1}\right] - (dd^cu)^n \wedge (dd^cv)^m. \tag{8}$$

Using the hypothesis on u, v we may compute

$$\begin{aligned} dd^c\psi_j \wedge (dd^cu)^{n-1} \wedge (dd^cv)^m & \\ &= \left[\chi'(0)(dd^cu)^n + j\chi''(ju)du \wedge d^cu \wedge (dd^cu)^{n-1}\right] \wedge (dd^cv)^m \\ &= dd^c(\chi(ju)/j) \wedge (dd^cu)^{n-1} \wedge (dd^cv)^m. \end{aligned}$$

Since $\chi(ju)/j \downarrow u$ as $j \uparrow \infty$, it follows that

$$dd^cw \wedge (dd^cu)^{n-1} \wedge (dd^cv)^m = (dd^cu)^n \wedge (dd^cv)^m$$

and, similarly,

$$dd^cw \wedge (dd^cu)^n \wedge (dd^cv)^{m-1} = (dd^cu)^n \wedge (dd^cv)^m.$$

This, together with (8), finishes the proof. □

For the proof of Theorem 1 we need a lemma which is an extension of a result from [Sa].

Lemma 8. *Let E, F, D, G be as in Theorem 2. For $\varepsilon > 0$ set*

$$E_\varepsilon := \{V_E^* < \varepsilon\},\ F_\varepsilon := \{V_F^* < \varepsilon\},$$

$$\widetilde{E}_\varepsilon := \{u_{E,D}^* < -1 + \varepsilon\},\ \widetilde{F}_\varepsilon := \{u_{F,G}^* < -1 + \varepsilon\}.$$

Then

$$V_{E_\varepsilon}^* \uparrow V_E^*,\ V_{F_\varepsilon}^* \uparrow V_F^*,\ V_{E_\varepsilon \times F_\varepsilon}^* \uparrow V_{E\times F}^*, \tag{9}$$

$$u_{\widetilde{E}_\varepsilon,D}^* \uparrow u_{E,D}^*,\ u_{\widetilde{F}_\varepsilon,G}^* \uparrow u_{F,G}^*,\ u_{\widetilde{E}_\varepsilon \times \widetilde{F}_\varepsilon, D\times G}^* \uparrow u_{E\times F, D\times G}^*, \tag{10}$$

as $\varepsilon \downarrow 0$, and every convergence is uniform.

Proof. The set $E \setminus E_\varepsilon = E \cap \{V_E^* \geq \varepsilon\}$ is pluripolar by Bedford-Taylor's theorem on negligible sets (see [BT1]). It follows that

$$V_E^* - \varepsilon \leq V_{E_\varepsilon} = V_{E_\varepsilon}^* \leq V_E^*$$

which gives the first two convergences of (9). In order to show the third one, observe that

$$\max\{V_E, V_F\} \leq V_{E\times F} \leq V_E + V_F. \tag{11}$$

Indeed, the first inequality in (11) follows easily from the definition of extremal function. Fixing one of the variables $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$, we see that the second inequality in (11) is satisfied, first on the cross $(E \times \mathbb{C}^m) \cup (\mathbb{C}^n \times F)$, and then everywhere.

By (11) $V_{E\times F} \leq 2\varepsilon$ on $E_\varepsilon \times F_\varepsilon$. On the other hand, by (11) the set $(E \times F) \setminus (E_\varepsilon \times F_\varepsilon)$ is contained in $(E \times F) \cap \{V_{E\times F}^* \geq \varepsilon\}$ and is thus pluripolar. Therefore

$$V_{E\times F}^* - 2\varepsilon \leq V_{E_\varepsilon \times F_\varepsilon} = V_{E_\varepsilon \times F_\varepsilon}^* \leq V_{E\times F}^*$$

and this gives (9).

Similarly as in (11) we can show

$$\max\{u_{E,D}, u_{F,G}\} \le u_{E\times F, D\times G} \le -u_{E,D}u_{F,G}.$$

Now the proof of (10) is parallel to that of (9). □

Proof of Theorem 1. If E, F are compact and *L-regular* (that is, V_E and V_F are continuous), then (1) was shown in [Si] and (2) follows immediately from Theorem 7. For E, F open we can find sequences of compact, L-regular sets with $E_j \uparrow E$ and $F_j \uparrow F$. Then $V_{E_j} \downarrow V_E, V_{F_j} \downarrow V_F$ and $V_{E_j \times F_j} \downarrow V_{E\times F}$ as $j \uparrow \infty$. This gives (1) and (2) for open sets. The general case can now be deduced from Lemma 8. □

Proof of Theorem 2. The proof of (3) for open subsets can be found in [EP]. Now the proof is the same as the proof of Theorem 1. □

Remark. Although (3) is stated in [EP] for arbitrary subsets E, F, the way from open subsets to the general case is not so straightforward as the authors claim—one needs Lemma 8.

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Institute of Mathematics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

E-mail address: `blocki@im.uj.edu.pl`