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A gradient estimate in the Calabi–Yau theorem

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Abstract We prove a C^1-estimate for the complex Monge–Ampère equation on a compact Kähler manifold directly from the C^0-estimate, without using a C^2-estimate. This was earlier done only under additional assumption of non-negative bisectional curvature.

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0 Introduction

Let M be a compact Kähler manifold of complex dimension $n \geq 2$ with Kähler form ω. We consider the complex Monge–Ampère equation on M

$$(\omega + dd^c\varphi)^n = f\omega^n, \tag{0.1}$$

we look for solutions satisfying

$$\omega + dd^c\varphi > 0. \tag{0.2}$$

Our main result is the following a priori gradient estimate:

Theorem 1 *Let $\varphi \in C^3(M)$ be a solution of* (0.1) *satisfying* (0.2). *Then*

$$|\nabla\varphi| \leq C_1,$$

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where C_1 is a positive constant depending only on upper bounds for $\operatorname{osc}\varphi$, f, $|\nabla(f^{1/n})|$*, on a lower (negative) bound for the bisectional curvature of M, and on n.*

Here $\operatorname{osc}\varphi = \sup_M \varphi - \inf_M \varphi$ and by a lower bound for the bisectional curvature we mean a constant $-B$, where $B \geq 0$ is such that

$$\sigma(X, Y) \geq -B|X|^2|Y|^2, \quad X, Y \in TM, \tag{0.3}$$

($\sigma(X, Y) = R(X, Y, Y, X) + R(X, JY, JY, X)$ is the bisectional curvature).

Theorem 1 was proved in [2] under the additional assumption $B = 0$. It is new even in the non-degenerate case (when one assumes in addition control of f from below by a positive constant).

One of the main steps in the proof of the Calabi–Yau theorem is a C^2-estimate proved independently by Aubin [1] and Yau [17], which does not make use of any C^1-estimate (and therefore a C^1-estimate is not needed when solving this Monge–Ampère equation). This is a very rare situation in the theory of fully nonlinear elliptic equations of second order. In fact, the main motivation behind Theorem 1 is an upcoming work on complex Hessian equations on compact Kähler manifolds where this Aubin–Yau phenomenon, at least apparently, does not appear, and where the method we use here to prove Theorem 1 will be applied to actually solve an equation.

As observed by Duong Phong, our gradient estimate may be also used to provide an alternative proof of the C^1-estimates obtained by blow-up arguments in Chen's proof of the existence of $C^{1,1}$ geodesic segments in the space of Kähler metrics [7].

By the way, we also revisit the aforementioned C^2-estimate and its generalization to the degenerate case from [2] (which was influenced by P. Guan's work [8] on the degenerate real Monge–Ampère equation, see also [4,9]). We will prove the following, slightly more precise result:

Theorem 2 *For $\varphi \in C^4(M)$ satisfying (0.1) and (0.2) one has*

$$\Delta\varphi \leq C_2,$$

where C_2 is a constant depending only on upper bounds for $\operatorname{osc}\varphi$, f *and the scalar curvature of M, on lower bounds for $f^{1/(n-1)}\Delta(\log f)$ and the bisectional curvature of M, and on n.*

Surprisingly, the proof of Theorem 1, although the method and main ideas are similar, is more complicated than that of Theorem 2. Similarly as in [2], and unlike [17], we do not use covariant derivatives in our proofs. The key however is a local holomorphic change of variables giving (1.2)–(1.4) below.

One can check that the assumptions on f in Theorem 2 are satisfied for example when $||f^{1/(n-1)}||_{C^2(M)}$ is under control.

Dependence on M in Theorems 1 and 2 is quite explicit in terms of its geometry. One of the main ingredients of the fundamental work of Yau [17] was an a priori estimate for $\operatorname{osc}\varphi$. Together with subsequent simplifications due to Kazdan, Aubin and Bourguignon, one can show that the following result holds (see e.g. [15] or [6]): if φ satisfies (0.1) and (0.2) then

$$\operatorname{osc}\varphi \le C_0,$$

where $C_0 > 0$ depends on M and $\sup_M f$. However, the dependence on M here is not so geometric as in Theorems 1 and 2.

The technique used by Yau to estimate $\operatorname{osc}\varphi$ was Moser's iteration. Kołodziej [10] gave another proof using pluripotential theory. In fact, he improved this estimate as follows: for any $p > 1$ the constant C_0 depends on M, $||f||_{L^p(M)}$ and p (see also [5]). He also showed that for any $p > 1$, $f \in L^p(M)$, $f \ge 0$, satisfying the necessary condition

$$\int_M f\omega^n = \int_M \omega^n,$$

there exists $\varphi \in C(M)$ such that (in the weak sense)

$$\omega + dd^c\varphi \ge 0, \quad (\omega + dd^c\varphi)^n = f\omega^n, \quad \sup_M \varphi = 0. \tag{0.4}$$

In [11] it was shown that the solution of (0.4) is unique (see also [3] for a more general uniqueness result with much simpler proof).

The Calabi–Yau theorem may be now viewed as the following regularity of (0.4):

$$f \in C^\infty, \quad f > 0 \;\Rightarrow\; \varphi \in C^\infty.$$

Theorem 2 easily gives (see [2])

$$f^{1/(n-1)} \in C^{1,1} \;\Rightarrow\; \Delta\varphi \in L^\infty \;\Rightarrow\; \varphi \in C^{1,\alpha}, \quad \alpha < 1.$$

Pliś [13], modifying an example of Wang [16], showed that $1/(n-1)$ here, and thus also in Theorem 2, is an optimal exponent. From Theorem 1 we deduce in turn

$$f^{1/n} \in C^{0,1} \;\Rightarrow\; \varphi \in C^{0,1}.$$

Pliś [14] noticed that from [13] it follows that the optimal exponent here is not bigger than $1/(n-2)$. Recently Kołodziej [12] showed the following Hölder regularity:

$$p > 1, \; f \in L^p \;\Rightarrow\; \varphi \in C^{0,q} \text{ for some } q > 0.$$

Here q depends on M, p, and $||f||_{L^p(M)}$.

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1 Proof of Theorem 1

Without loss of generality we may assume that $\inf_M \varphi = 0$, then $\sup_M \varphi = \operatorname{osc} \varphi =: C_0$. Set

$$\beta := |\nabla \varphi|^2$$

and

$$\alpha := \log \beta - \gamma \circ \varphi,$$

where (smooth) $\gamma : [0, C_0] \to \mathbb{R}$ with $\gamma' \geq 0$ will be determined later. Since M is compact, α attains maximum at some $O \in M$. Then

$$\log \beta \leq \log \beta(O) + \gamma \circ \varphi - \gamma(\varphi(O)), \tag{1.1}$$

so it is enough to estimate $\beta(O)$ (provided that γ is under control).

Near O we have $\omega = dd^c g$ for some smooth strongly plurisubharmonic g. By (0.2) the function $u := \varphi + g$ is strongly plurisubharmonic. We claim that there exists a holomorphic chart near O where

$$g_{j\bar{k}}(O) = \delta_{jk}, \tag{1.2}$$

$$g_{j\bar{k}l}(O) = 0, \tag{1.3}$$

$$(u_{p\bar{q}}(O)) \text{ is diagonal}, \tag{1.4}$$

where lower indices denote partial differentiation: $v_j = \partial v/\partial z^j$, $v_{\bar{j}} = \partial v/\partial \bar{z}^j \ldots$ After a linear change of variables we can get (1.2) and (1.4). We then apply another change of variables of the form $F = (F^1, \ldots, F^n)$, where

$$F^l(z) = z^l + \frac{1}{2} a^l_{jk} z^j z^k$$

and $a^l_{jk} = a^l_{kj}$, which changes neither (1.2) nor (1.4). For $\widetilde{g} := g \circ F$ we have

$$\widetilde{g}_p = g_l \circ F\, F^l_p, \quad \widetilde{g}_{p\bar{q}} = g_{l\bar{k}} \circ F\, F^l_p\, \overline{F^k_q},$$

and

$$\widetilde{g}_{p\bar{q}m} = g_{l\bar{k}} \circ F\, F^l_{pm}\, \overline{F^k_q} + g_{l\bar{k}s} \circ F\, F^l_p\, \overline{F^k_q}\, F^s_m.$$

Since $F^l_p(O) = \delta_{lp}$, $F^l_{pm}(O) = a^l_{pm}$, we have

$$\widetilde{g}_{p\bar{q}m}(O) = a^q_{pm} + g_{p\bar{q}m}(O).$$

Therefore, choosing $a^q_{pm} := -g_{p\bar{q}m}(O)$, we will also get (1.3).

We have $\beta = g^{j\bar{k}}\varphi_j\varphi_{\bar{k}}$, where $(g^{j\bar{k}})$ is the inverse transposed of $(g_{j\bar{k}})$, that is

$$g_{j\bar{k}}g^{j\bar{l}} = \delta_{kl}. \tag{1.5}$$

We get

$$\beta_p = (g^{j\bar{k}})_p\varphi_j\varphi_{\bar{k}} + g^{j\bar{k}}\varphi_{jp}\varphi_{\bar{k}} + g^{j\bar{k}}\varphi_j\varphi_{p\bar{k}}.$$

Since

$$(g^{j\bar{k}})_p = -g^{j\bar{t}}g^{s\bar{k}}g_{s\bar{t}p} \tag{1.6}$$

(which we obtain differentiating (1.5)), from (1.2) to (1.4) at O we infer

$$\beta = \sum_j |\varphi_j|^2,$$

$$\beta_p = \sum_j \varphi_{jp}\varphi_{\bar{j}} + \varphi_p(u_{p\bar{p}} - 1), \tag{1.7}$$

$$\beta_{p\bar{p}} = (g^{j\bar{k}})_{p\bar{p}}\varphi_j\varphi_{\bar{k}} + 2\mathrm{Re}\sum_j u_{p\bar{p}j}\varphi_{\bar{j}} + \sum_j |\varphi_{jp}|^2 + \varphi_{p\bar{p}}^2. \tag{1.8}$$

At O we have

$$(g^{j\bar{k}})_{p\bar{p}} = -g_{j\bar{k}p\bar{p}} = R_{j\bar{k}p\bar{p}}, \tag{1.9}$$

provided that we normalize the definition of d^c to

$$d^c := \frac{1}{4}i(\overline{\partial} - \partial), \tag{1.10}$$

so that $dd^c = i\partial\overline{\partial}/2$ and

$$\omega = \frac{i}{2}g_{j\bar{k}}dz^j \wedge d\overline{z^k}.$$

Note that (0.3) is pointwise equivalent to

$$R_{j\bar{k}p\bar{q}}a^j\overline{a^k}b^p\overline{b^q} \geq -B|a|^2|b|^2, \quad a, b \in \mathbb{C}^n. \tag{1.11}$$

On the other hand

$$\alpha_p = \frac{\beta_p}{\beta} - \gamma' \circ \varphi\,\varphi_p.$$

Since $\alpha_p(O) = 0$, at O we have

$$\alpha_{p\bar{p}} = \frac{\beta_{p\bar{p}}}{\beta} - \frac{|\beta_p|^2}{\beta^2} - \gamma''|\varphi_p|^2 - \gamma'\varphi_{p\bar{p}}$$
$$= \frac{\beta_{p\bar{p}}}{\beta} - \left[(\gamma')^2 + \gamma''\right]|\varphi_p|^2 - \gamma'\varphi_{p\bar{p}}, \tag{1.12}$$

where with some abuse of notation we denote $\gamma' = \gamma'(\varphi(O))$, $\gamma'' = \gamma''(\varphi(O))$.

Near O we have

$$\det(u_{p\bar{q}}) = f\ \det(g_{p\bar{q}}) =: \widetilde{f}.$$

Differentiating both sides and dividing by $\widetilde{f}$ we get

$$u^{p\bar{q}}u_{p\bar{q}j} = (\log \widetilde{f})_j. \tag{1.13}$$

By (1.2)–(1.4) this means at O

$$\sum_p \frac{u_{p\bar{p}j}}{u_{p\bar{p}}} = (\log f)_j.$$

Combining this with (1.8), (1.9), (1.11), (1.12), and the fact that α has maximum at O, we obtain

$$0 \geq \sum_p \frac{\alpha_{p\bar{p}}}{u_{p\bar{p}}} \geq (\gamma' - B)\sum_p \frac{1}{u_{p\bar{p}}} + \frac{2}{\beta}\mathrm{Re}\sum_j (\log f)_j\varphi_{\bar{j}} + \frac{1}{\beta}\sum_{j,p}\frac{|\varphi_{jp}|^2}{u_{p\bar{p}}}$$
$$-[(\gamma')^2 + \gamma'']\sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - n\gamma'. \tag{1.14}$$

The proof so far was essentially the same as in [2] (except for the way we changed variables). If $B = 0$ then in order to get rid of $u_{p\bar{p}}$'s in (1.14) one could take $\gamma(t) = 2^{-1}\log(2t+1)$ (then $\gamma' \geq (2C_0+1)^{-1}$, $-[\gamma''+(\gamma')^2] \geq (2C_0+1)^{-2}$), and we could easily estimate β. However, one cannot construct γ with $\gamma' \geq B$ and $-[\gamma''+(\gamma')^2] \geq 0$ if $B > 1/C_0$.

To improve the method from [2] we will in addition estimate the term

$$\frac{1}{\beta}\sum_{j,p}\frac{|\varphi_{jp}|^2}{u_{p\bar{p}}}$$

from below. Since $\alpha_p(O) = 0$, at O we have $\beta_p = \gamma'\beta\varphi_p$. Using (1.7) this means that (at O)

$$\sum_j \varphi_{jp}\varphi_{\bar{j}} = (\gamma'\beta + 1 - u_{p\bar{p}})\varphi_p.$$

By the Schwarz inequality

$$\left|\sum_j \varphi_{jp}\varphi_{\bar{j}}\right|^2 \le \beta \sum_j |\varphi_{jp}|^2$$

and thus (at O)

$$\begin{aligned}\frac{1}{\beta}\sum_{j,p}\frac{|\varphi_{jp}|^2}{u_{p\bar{p}}} &\ge \frac{1}{\beta^2}\sum_p \frac{(\gamma'\beta + 1 - u_{p\bar{p}})^2|\varphi_p|^2}{u_{p\bar{p}}}\\ &\ge (\gamma')^2\sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - 2\gamma' - \frac{2}{\beta},\end{aligned} \tag{1.15}$$

where we have used the inequality

$$(\gamma'\beta + 1 - u_{p\bar{p}})^2 \ge (\gamma')^2\beta^2 - 2u_{p\bar{p}} - 2\gamma'\beta u_{p\bar{p}}$$

(and the fact that $\gamma' \ge 0$). Moreover,

$$|\nabla(\log f)| = n\frac{|\nabla(f^{1/n})|}{f^{1/n}} \le \frac{F_1}{2f^{1/n}},$$

where

$$F_1 := \frac{2}{n}\sup_M |\nabla(f^{1/n})|.$$

Assuming in addition that $\beta(O) \ge 1$

$$\frac{2}{\beta}\mathrm{Re}\sum_j (\log f)_j\varphi_{\bar{j}} \ge -\frac{2|\nabla(\log f)|}{\sqrt{\beta}} \ge -2|\nabla(\log f)| \ge -\frac{F_1}{f^{1/n}} \ge -F_1\sum_p \frac{1}{u_{p\bar{p}}}$$

by the inequality between arithmetic and geometric means. Combining this with (1.14) and (1.15)

$$0 \ge (\gamma' - B - F_1)\sum_p \frac{1}{u_{p\bar{p}}} - \gamma''\sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - (n+2)\gamma' - 2.$$

This time we can easily find a right γ, for example set

$$\gamma(t) := (B + F_1 + 3)t - \frac{1}{C_0}t^2, \quad 0 \le t \le C_0.$$

Then

$$B + F_1 + 1 \le \gamma' \le B + F_1 + 3, \quad -\gamma'' = \frac{2}{C_0}.$$

Therefore

$$\sum_p \frac{1}{u_{p\bar{p}}} + \frac{2}{C_0} \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} \le D := 2 + (n+2)(B + F_1 + 3). \tag{1.16}$$

Thus

$$\frac{1}{u_{p\bar{p}}} \le D.$$

Combining this with $u_{1\bar{1}} \dots u_{n\bar{n}} \le F_0 := \sup_M f$, we get

$$u_{p\bar{p}} \le F_0 D^{n-1}.$$

Using (1.16) again

$$|\varphi_p|^2 \le \frac{C_0 F_0 D^n}{2}$$

and therefore

$$\beta(O) \le \max\left(1, \frac{n C_0 F_0 D^n}{2}\right).$$

Combining this with (1.1), and since $\operatorname{osc} \gamma = \gamma(C_0) = C_0(B + F_1 + 2)$, we get Theorem 1 with

$$C_1^2 = e^{C_0(B+F_1+2)} \max\left(1, \frac{n C_0 F_0}{2} [2 + (n+2)(B + F_1 + 3)]^n\right)$$

(the constants are of course subjected to the normalization given by (1.10)).

2 Proof of Theorem 2

By (0.2)

$$\Delta\varphi = g^{j\bar{k}} \varphi_{j\bar{k}} > -n$$

(Δ denotes the complex Laplacian which is the double of the real one). We keep the previous notation, except that this time we set

$$\alpha := \log(\Delta\varphi + n) - A\varphi,$$

where a constant $A > 0$ will be determined later. As before, the function $u = g + \varphi$ is defined near O where α attains maximum. We have

$$\alpha_p = \frac{(\Delta u)_p}{\Delta u} - A\varphi_p,$$
$$\alpha_{p\bar p} = \frac{(\Delta u)_{p\bar p}}{\Delta u} - \frac{|(\Delta u)_p|^2}{(\Delta u)^2} + A - Au_{p\bar p}. \tag{2.1}$$

By (1.2)–(1.4), (1.6), (1.9) at O we will get

$$(\Delta u)_p = \sum_j u_{j\bar j p},$$
$$(\Delta u)_{p\bar p} = \sum_j R_{j\bar j p\bar p} u_{j\bar j} + \sum_j u_{j\bar j p\bar p}. \tag{2.2}$$

Differentiating (1.13) and using (1.6) for u we obtain

$$u^{p\bar q} u_{p\bar q j\bar j} = (\log \widetilde{f})_{j\bar j} + u^{p\bar t} u^{s\bar q} u_{p\bar q j} u_{s\bar t \bar j}.$$

Similarly

$$(\log \widetilde{f})_{j\bar j} = (\log f)_{j\bar j} + g^{p\bar q} g_{p\bar q j\bar j} - g^{p\bar t} g^{s\bar q} g_{p\bar q j} g_{s\bar t\bar j}.$$

At O by (1.2)–(1.4) we thus get

$$\sum_{j,p} \frac{u_{j\bar j p\bar p}}{u_{p\bar p}} = \Delta(\log f) - \sum_{j,p} R_{j\bar j p\bar p} + \sum_{j,p,q} \frac{|u_{p\bar q j}|^2}{u_{p\bar p} u_{q\bar q}}$$
$$\geq -\frac{F_2}{f^{1/(n-1)}} - S + \sum_{j,p} \frac{|u_{j\bar j p}|^2}{u_{p\bar p} u_{j\bar j}}, \tag{2.3}$$

where

$$F_2 := \max\left(0, -\inf_M \left[f^{1/(n-1)} \Delta(\log f)\right]\right)$$

and S denotes an upper bound for the scalar curvature.

By the Schwarz inequality

$$|(\Delta u)_p|^2 = \left|\sum_j u_{j\bar j p}\right|^2 \leq \Delta u \sum_j \frac{|u_{j\bar j p}|^2}{u_{j\bar j}}. \tag{2.4}$$

Combining (2.1)–(2.4) and using the fact that α has maximum at O, we get there

$$0 \geq \sum_p \frac{\alpha_{p\bar{p}}}{u_{p\bar{p}}} \geq \frac{1}{\Delta u}\left[\sum_{j,p} R_{j\bar{j}p\bar{p}} \frac{u_{j\bar{j}}}{u_{p\bar{p}}} - \frac{F_2}{f^{1/(n-1)}} - S\right] + A\sum_p \frac{1}{u_{p\bar{p}}} - nA. \quad (2.5)$$

We also have

$$\sum_{j,p} R_{j\bar{j}p\bar{p}} \frac{u_{j\bar{j}}}{u_{p\bar{p}}} \geq -B\sum_{j,p} \frac{u_{j\bar{j}}}{u_{p\bar{p}}} = -B\Delta u \sum_p \frac{1}{u_{p\bar{p}}}, \quad (2.6)$$

$$\sum_p \frac{1}{u_{p\bar{p}}} \geq (n-1)^{1/(n-1)} (\Delta u/f)^{1/(n-1)} \geq (\Delta u/f)^{1/(n-1)}. \quad (2.7)$$

From (2.5) to (2.7) with $A := B + 1$ we get

$$\begin{aligned} 0 &\geq (\Delta u)^{n/(n-1)} - n(B+1) f^{1/(n-1)} \Delta u - F_2 - S f^{1/(n-1)} \\ &\geq (\Delta u)^{n/(n-1)} - n(B+1) F_0^{1/(n-1)} \Delta u - F_2 - S F_0^{1/(n-1)}. \end{aligned}$$

Since for $x, a, b, \varepsilon > 0$

$$x^{1+\varepsilon} - ax - b \leq 0 \quad \Rightarrow \quad x \leq \max\left((2a)^{1/\varepsilon}, (2b)^{1/(1+\varepsilon)}\right),$$

we get

$$\Delta u(O) \leq \max\left([2n(B+1)]^{n-1} F_0, [2(F_2 + S F_0^{1/(n-1)})]^{(n-1)/n}\right).$$

But

$$\Delta\varphi + n \leq e^{\alpha(O) + A\varphi} \leq e^{AC_0} \Delta u(O)$$

and we obtain Theorem 2 with

$$C_2 = e^{C_0(B+1)} \max\left([2n(B+1)]^{n-1} F_0, [2(F_2 + S F_0^{1/(n-1)})]^{(n-1)/n}\right) - n.$$

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