On Nazarov's Complex Analytic Approach to the Mahler Conjecture and the Bourgain-Milman Inequality

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Abstract We survey the several complex variables approach to the Mahler conjecture from convex analysis due to Nazarov. We also show, although only numerically, that his proof of the Bourgain-Milman inequality using estimates for the Bergman kernel for tube domains cannot be improved to obtain the Mahler conjecture which would be the optimal version of this inequality.

Keywords Mahler conjecture · Bergman kernel · Pluricomplex Green function

1 Introduction

Let K be a convex symmetric body in $\mathbb{R}^n$. This means that $K = -K$, K is convex, bounded, closed and has non-empty interior. The dual (or polar) body of K is given by

$$K' = \{y \in \mathbb{R}^n : x \cdot y \le 1 \text{ for all } x \in K\},$$

where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$. The Mahler volume of K is defined by

$$M(K) = \lambda_n(K)\lambda_n(K'),$$

where λ_n denotes the Lebesgue measure in $\mathbb{R}^n$. It is easy to see that it is independent of linear transformations and thus also on the inner product in $\mathbb{R}^n$. The Mahler volume is therefore an invariant of the Banach space $(\mathbb{R}^n, q_K)$, where q_K is the Minkowski functional of K:

$$q_K(x) = \inf\{t > 0 : t^{-1}x \in K\} = \sup\{x \cdot y : y \in K'\}.$$

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The Blaschke-Santaló inequality says that the Mahler volume is maximal for balls:

$$\lambda_n(K)\lambda_n(K') \leq (\lambda_n(\mathbb{B}_n^2))^2,$$

where for $p \geq 1$ we denote

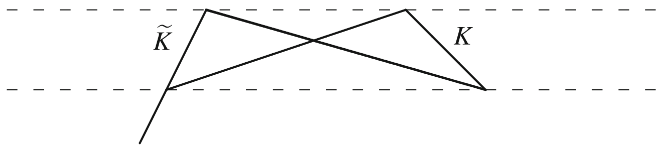
$$\mathbb{B}_n^p = \{x \in \mathbb{R}^n : |x_1|^p + \cdots + |x_n|^p \leq 1\}.$$

In fact, it holds without the assumption of symmetry but one has to assume that the interior of K contains the origin. Moreover, one has equality if and only if K is an ellipsoid, that is a linear image of $\mathbb{B}_n^2$. It was proved by Blaschke [B1, B2] for $n = 2$, $n = 3$, and by Santaló [S1] for arbitrary n (see also [SR]).

Mahler [M1] conjectured that $M(K)$ is minimized by cubes, that is

$$\lambda_n(K)\lambda_n(K') \geq \lambda_n(\mathbb{B}_n^1)\lambda_n(\mathbb{B}_n^\infty) = \frac{4^n}{n!},$$

where $\mathbb{B}_n^\infty = [-1, 1]^n$. It can be easily proved for $n = 2$: if K is a polygon with k vertices and $\widetilde{K}$ is the polygon with $k - 1$ vertices obtained from K by moving one vertex as in the following picture



then $\lambda_2(\widetilde{K}) = \lambda_2(K)$ but one can show that $\lambda_2(\widetilde{K}') \geq \lambda_2(K')$.

Bourgain and Milman [BM] proved the following lower bound for the Mahler volume: there exists $c > 0$ such that

$$\lambda_n(K)\lambda_n(K') \geq c^n \frac{4^n}{n!}.$$

This is an important result in the theory of finitely-dimensional Banach spaces, it also has applications in number theory, see [BM]. We see that the Mahler conjecture is equivalent to this inequality with $c = 1$. The best known constant so far is $c = \pi/4$ and was obtained by Kuperberg [Ku].

One of possible difficulties with the Mahler conjecture is that if it is true then there would be more minimizers than cubes (and their linear images). We have $(\mathbb{B}_2^\infty)' = \mathbb{B}_2^1 \simeq \mathbb{B}_2^\infty$, where by $\simeq$ we denote the linear equivalence, and indeed for $n = 2$ the square is the only minimizer (up to linear transformations). However, for $n = 3$ the octahedron $\mathbb{B}_3^1 = (\mathbb{B}_3^\infty)'$ is not linearly equivalent to the cube $\mathbb{B}_3^\infty$. The conjecture for $n = 3$ is that the cube and octahedron are the only minimizers. For arbitrary n it should be so called Hansen-Lima bodies [HL]: these are intervals for $n = 1$ and in higher dimensions they are obtained by either taking products of lower-dimensional Hansen-Lima bodies or by taking their duals.

There is also a version of the Mahler conjecture for not necessarily symmetric bodies. Assuming that the origin is in the interior of K, it is expected that a centered simplex (that is the convex hull of affinely independent $v^1, \dots, v^{n+1} \in \mathbb{R}^n$ such that $v^1 + \cdots + v^{n+1} = 0$) is the only minimizer, that is

$$\lambda_n(K)\lambda_n(K') \geq \frac{(n+1)^{n+1}}{(n!)^2}.$$

Recently Nazarov [N1] proposed a complex analytic approach to the Bourgain-Milman inequality and Mahler conjecture. Considering the Bergman kernel on the tube domain $\Omega = intK + i\mathbb{R}^n$ at the origin

$$K_\Omega(0,0) = \sup\{\frac{|f(0)|^2}{||f||^2_{L^2(\Omega)}} : f \in \mathcal{O}(\Omega) \cap L^2(\Omega),\ f \not\equiv 0\}$$

and using the formula for the Bergman kernel in tube domains of Rothaus [R1], see also [Hs], he proved the upper bound

$$K_\Omega(0,0) \leq \frac{n!}{\pi^n}\frac{\lambda_n(K')}{\lambda_n(K)}. \tag{1}$$

The main part of his paper was devoted to the proof of the lower bound

$$K_\Omega(0,0) \geq \left(\frac{\pi}{4}\right)^{2n}\frac{1}{(\lambda_n(K))^2}. \tag{2}$$

As is usually the case with lower bounds for the Bergman kernel, the main tool was Hörmander's estimate [H1]. Combining (1) with (2) we immediately obtain the Bourgain-Milman inequality with $c = (\pi/4)^3$.

In Sect. 2 we will present Nazarov's equivalent complex analytic formulation of the Mahler conjecture using the Paley-Wiener theorem. The upper bound (1) is explained in Sect. 3. We include the proof of Rothaus' [R1] integral formula for the Bergman kernel in tube domains, since it is not so well known. In Sect. 4 we discuss the lower bound using some simplifications from [Bln]. We also show that this approach cannot give the Mahler conjecture. We will see, although only numerically using *Mathematica*, that although the Bergman kernel for tube domains does behave well under taking products, it does not under taking duals.

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2 Equivalent SCV Formulation

Assume that K is a convex body in $\mathbb{R}^n$, not necessarily symmetric. For $u \in L^2(K')$ consider its Fourier transform

$$\widehat{u}(z) = \int_{K'} u(x) e^{-ix\cdot z} d\lambda(x), \quad z \in \mathbb{C}^n,$$

it is an entire holomorphic function. By the Schwarz inequality and the Parseval formula

$$|\widehat{u}(0)|^2 \le \lambda_n(K') \int_{K'} |u|^2 d\lambda_n = \frac{\lambda_n(K')}{(2\pi)^n} \int_{\mathbb{R}^n} |\widehat{u}(x)|^2 d\lambda_n(x)$$

and we have equality for $u \equiv 1$ on K'. It is clear that $f = \widehat{u}$ satisfies

$$|f(z)| \le C e^{q_K(\operatorname{Im} z)}, \quad z \in \mathbb{C}^n, \tag{3}$$

for some $C > 0$. On the other hand, if $f \in \mathscr{O}(\mathbb{C}^n)$ satisfies (3) and is such that

$$\int_{\mathbb{R}^n} |f(x)|^2 d\lambda_n(x) < \infty \tag{4}$$

then by the Plancherel theorem $f = \widehat{u}$ for some $u \in L^2(\mathbb{R}^n)$ and by the Paley-Wiener theorem $\mathrm{supp}\, \mathrm{u} \subset \mathrm{K}'$. Therefore

$$\lambda_n(K') = (2\pi)^n \sup_{f\in\mathscr{P}, f\not\equiv 0} \frac{|f(0)|^2}{||f||^2_{L^2(\mathbb{R}^n)}},$$

where $\mathscr{P}$ denotes the family of entire holomorphic functions satisfying (3) and (4).

This way we have obtained a formula for the volume of the dual K' which is expressed only in terms of K, and not K'. It means that the Mahler conjecture is equivalent to finding $f \in \mathscr{O}(\mathbb{C}^n)$ with $f(0) = 1$, satisfying (3) and such that

$$\int_{\mathbb{R}^n} |f(x)|^2 d\lambda_n(x) \le n! \left(\frac{\pi}{2}\right)^n \lambda_n(K)$$

in the symmetric case, and

$$\int_{\mathbb{R}^n} |f(x)|^2 d\lambda_n(x) \le \frac{(n!)^2 (2\pi)^n}{(n+1)^{n+1}} \lambda_n(K)$$

in the asymmetric one.

3 The Upper Bound

Nazarov [N1] showed that the upper bound (1) easily follows from the formula for the Bergman kernel in tube domains $\Omega = D + i\mathbb{R}^n$, where D is an arbitrary convex domain in $\mathbb{R}^n$:

$$K_\Omega(z, w) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{(z+\bar{w})\cdot y}}{J_D(y)} d\lambda_n(y), \tag{5}$$

where

$$J_D(y) = \int_K e^{2x\cdot y} d\lambda_n(x)$$

(see [R1] and [Hs]). Indeed, for $y \in \mathbb{R}^n$ and $x_0 \in K$ using the fact that $(x_0 + K)/2 \subset K$ and that K is symmetric we get

$$J_K(y) \geq \frac{1}{2^n} \int_K e^{(x_0+x)\cdot y} d\lambda_n(x) \geq \frac{\lambda_n(K)}{2^n} e^{x_0\cdot y}.$$

Therefore $J_K \geq 2^{-n} e^{q_{K'}}$ and to obtain (1) it is enough to observe that

$$\int_{\mathbb{R}^n} e^{-q_K} d\lambda_n = \int_0^\infty e^{-t} \lambda_n(\{q_K < t\}) dt = n! \lambda_n(K).$$

Proof (Proof of (5)) Take $\widetilde{x} \in D$ and $r > 0$ such that $C_r := \widetilde{x} + r(-1, 1)^n \subset D$. Then

$$J_D(y) \geq J_{C_r}(y) = e^{2\widetilde{x}\cdot y} \frac{\sinh(2ry_1)}{y_1} \cdots \frac{\sinh(2ry_n)}{y_n}$$

and thus

$$\int_{\mathbb{R}^n} \frac{e^{2\widetilde{x}\cdot y}}{J_D(y)} d\lambda(y) \leq \left(\frac{c}{r}\right)^{2n}, \tag{6}$$

where

$$c^2 = \frac{1}{2} \int_0^\infty \frac{t}{\sinh t} dt = \frac{\pi^2}{8}.$$

Since D is convex, we have $D + D = 2D$ and from (6) it follows in particular that the integral on the right-hand side of (5) is convergent.

For $u \in L^2(\mathbb{R}^n, J_D)$ and $z \in T_D$ set

$$\widetilde{u}(z) = \int_{\mathbb{R}^n} u(y) e^{z\cdot y} d\lambda(y).$$

By (6) the integral is convergent and thus $\widetilde{u}$ is holomorphic in T_D. It also follows that $h(y) := u(y) e^{\operatorname{Re} z\cdot y} \in L^2(\mathbb{R}^n)$ and we can write $\widetilde{u}(z) = \widehat{h}(-\operatorname{Im} z)$. By the Parseval formula and the Fubini theorem

$$||\widetilde{u}||^2_{L^2(T_D)} = (2\pi)^n \int_K \int_{\mathbb{R}^n} |u(y)|^2 e^{2x\cdot y} d\lambda(y) d\lambda(x) = (2\pi)^n ||u||^2_{L^2(\mathbb{R}^n, J_D)}. \quad (7)$$

We claim that in fact the mapping

$$L^2(\mathbb{R}^n, J_D) \ni u \longmapsto \widetilde{u} \in A^2(T_D) \quad (8)$$

is onto. For $f \in A^2(T_D)$ approximating D by relatively compact subsets from inside and using the fact that $|f|^2$ is subharmonic we may assume that f is bounded in T_D. Multiplying f by functions of the form $e^{\varepsilon z\cdot z}$ we may even assume that it satisfies the estimate

$$|f(z)| \le M e^{-\varepsilon |\mathrm{Im} z|^2} \quad (9)$$

for some positive constants M and ε. For a fixed $x \in D$ and $f_x(y) = f(x+iy)$ we have $f_x(y) = \widetilde{u}(x+iy)$ where $u(y) = (-2\pi)^{-n} \widehat{f_x}(y) e^{-x\cdot y}$. We have to prove that for a fixed y the definition of u is independent of x. From (9) it follows that we can differentiate under the sign of integration

$$\frac{\partial}{\partial x_j} \int_{\mathbb{R}^n} f(x+ia) e^{-(x+ia)\cdot y} d\lambda(a)$$
$$= \int_{\mathbb{R}^n} \left(\frac{\partial f}{\partial x_j}(x+ia) - y_j f(x+ia) \right) e^{-(x+ia)\cdot y} d\lambda(a).$$

We have $\partial f/\partial x_j = -i \partial f/\partial a_j$ and by (9) we can also integrate by parts. Therefore

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x+ia) e^{-(x+ia)\cdot y} d\lambda(a) = -i \int_{\mathbb{R}^n} \frac{\partial f}{\partial a_j}(x+ia) e^{-(x+ia)\cdot y} d\lambda(a)$$
$$= \int_{\mathbb{R}^n} y_j f(x+ia) e^{-(x+ia)\cdot y} d\lambda(a)$$

and therefore $u(y)$ is independent of x and the mapping (8) is onto.

By $K(z, w)$ denote the right-hand side of (5) and fix $w \in T_D$. Then $K(\cdot, w) = (2\pi)^{-n} \widetilde{v}$, where

$$v(y) = \frac{e^{\bar{w}\cdot y}}{J_D(y)} \in L^2(\mathbb{R}^n, J_D)$$

by (6). It follows from (7) that $K(\cdot, w) \in A^2(T_D)$ and to finish the proof we have to show that it has the reproducing property. For $f = \widetilde{u} \in A^2(T_D)$ where $u \in L^2(\mathbb{R}^n, J_D)$ by (7)

$$\langle f, K(\cdot, w)\rangle_{A^2(T_D)} = \frac{1}{(2\pi)^n} \langle \widetilde{u}, \widetilde{v}\rangle_{A^2(T_D)} = \langle u, v\rangle_{L^2(\mathbb{R}^n, J_D)} = \int_{\mathbb{R}^n} u(y) e^{w\cdot y} d\lambda(y) = f(w).$$

This finishes the proof of (5).

4 The Lower Bound

The lower bound (2) easily follows from a general lower bound for the Bergman kernel proved in [Bln]: if Ω is a pseudoconvex domain in $\mathbb{C}^n$ then for $w \in \Omega$ and $t \le 0$

$$K_\Omega(w, w) \ge \frac{1}{e^{-2nt}\lambda_{2n}(\{G_\Omega(\cdot, w) < t\})}, \tag{10}$$

where

$$G_\Omega(z, w) = \sup\{u(z)\colon u \in PSH^-(\Omega),\ \limsup_{z\to w}(u(z) - \log|z - w|) < \infty\}$$

is the pluricomplex Green function of Ω. It was proved in [Bln] using the Donnelly-Fefferman [DF] estimate for $\bar\partial$ (which can be easily deduced from Hörmander's estimate, see [Ber]) and the tensor-power trick. A simpler proof using subharmonicity of sections of the Bergman kernel from [Ber2] was later given by Lempert [L2] (see [Bms]).

The estimate (10) has various consequences when we let $t \to -\infty$. For example for $n = 1$ it gives the Suita conjecture

$$c_\Omega(w)^2 \le \pi K_\Omega(w, w),$$

where

$$c_\Omega(w) = \exp(\lim_{z\to w}(G_\Omega(z, w) - \log|z - w|))$$

is the logarithmic capacity of $\mathbb{C} \setminus \Omega$ with respect to w. It was originally proved in [Bin]. For arbitrary n if Ω is convex then using Lempert's theory [L1] one can obtain the estimate

$$K_\Omega(w, w) \ge \frac{1}{\lambda_{2n}(I_\Omega(w))}, \tag{11}$$

where

$$I_\Omega(w) = \{\varphi'(0)\colon \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$$

is the Kobayashi indicatrix (Δ is the unit disk in $\mathbb{C}$). This particular estimate for convex domains seems to be very accurate, see [BZ1, BZ2] for details.

Now let us come back to the case of the tube domain $\Omega = intK + i\mathbb{R}^n$ where K is a convex symmetric body in $\mathbb{R}^n$. Let $\varphi \in \mathcal{O}(\Delta, \Omega)$ be such that $\varphi(0) = 0$. By S denote the strip $\{|\mathrm{Re}\,\zeta| < 1\}$ in $\mathbb{C}$ and let $\Phi : S \to \Delta$ be biholomorphic with $\Phi(0) = 0$. By the Schwarz lemma for $u \in K'$

$$\left|\frac{\partial}{\partial\zeta}\bigg|_{\zeta=0} \Phi(\varphi(\zeta)\cdot u)\right| \le 1$$

and since $|\Phi'(0)| = \pi/4$ we obtain

$$|\varphi'(0) \cdot u| \leq \frac{4}{\pi}.$$

It follows that

$$I_\Omega(0) \subset \frac{4}{\pi}(K'' + iK'') = \frac{4}{\pi}(K + iK)$$

and

$$\lambda_{2n}(I_\Omega(0)) \leq \left(\frac{4}{\pi}\right)^{2n} (\lambda_n(K))^2.$$

The estimate (11) now gives the lower bound (2).

It was conjectured in [Bln] that the following lower bound holds in tube domains

$$K_\Omega(0,0) \geq \left(\frac{\pi}{4}\right)^n \frac{1}{(\lambda_n(K))^2}. \tag{12}$$

It would be optimal because one can easily check using the product formula for the Bergman kernel that one has equality in (12) for the unit cube $K = [-1, 1]^n$.

We will show however that we do not have equality in (12) for all Hansen-Lima bodies. Take the octahedron

$$K = \mathbb{B}_3^1 = \{x \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \leq 1\}.$$

One can then compute that

$$J_K(y) = \frac{y_1 \sinh(2y_1)}{(y_1^2 - y_2^2)(y_1^2 - y_3^2)} + \frac{y_2 \sinh(2y_2)}{(y_2^2 - y_1^2)(y_2^2 - y_3^2)} + \frac{y_3 \sinh(2y_3)}{(y_3^2 - y_1^2)(y_3^2 - y_2^2)}$$

when all coordinates y_j are different and that it extends to a positive smooth function in $\mathbb{R}^3$. One can then compute numerically using (5) that

$$K_\Omega(0,0) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{d\lambda_3}{J_K} = 0.2758\ldots \tag{13}$$

However, since $\lambda_n(\mathbb{B}_n^1) = 2^n/n!$, the right-hand side of (12) is equal to

$$\frac{9\pi^3}{1024} = 0.2725\ldots$$

This shows (although only numerically) that the Bergman kernel for tube domains does not behave well under taking duals. It is also clear that even proving optimal versions of the estimates (2) and (1) cannot give an optimal lower bound for the

Mahler volume and thus this Nazarov's approach to the Bourgain-Milman inequality cannot give its expected optimal form, that is the Mahler conjecture.

To make this argument precise and get rid of the numerical computation in (13), one could try to consider the n-dimensional octahedron

$$K_n = \mathbb{B}^1_n = \{x \in \mathbb{R}^n : |x_1| + \cdots + |x_n| \le 1\}.$$

One can compute that

$$J_{K_n}(y) = \begin{cases} \displaystyle\sum_{j=1}^{n} \frac{y_j^{n-2}\cosh(2y_j)}{(y_j^2 - y_1^2)\dots(y_j^2 - y_{j-1}^2)(y_j^2 - y_{j+1}^2)\dots(y_j^2 - y_n^2)}, & n \text{ even} \\ \displaystyle\sum_{j=1}^{n} \frac{y_j^{n-2}\sinh(2y_j)}{(y_j^2 - y_1^2)\dots(y_j^2 - y_{j-1}^2)(y_j^2 - y_{j+1}^2)\dots(y_j^2 - y_n^2)}, & n \text{ odd} \end{cases}.$$

One could perhaps estimate J_{K_n} from above in such a way that it would imply that

$$\limsup_{n\to\infty} \left(\frac{1}{(n!)^2} \int_{\mathbb{R}^n} \frac{d\lambda_n}{J_{K_n}} \right)^{1/n} > \frac{\pi^2}{8}.$$

Another possibility would be to apply (11): it would be enough to show that there exists n such that if I_n is the Kobayashi indicatrix of the tube domain $int K_n + i\mathbb{R}^n$ at the origin then

$$\lambda_{2n}(I_n) < \frac{16^n}{(n!)^2 \pi^n}.$$

This could perhaps be possible using Lempert's theory for tube domains developed by Zając [Z1].

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