THE BERGMAN KERNEL AND METRIC

PHD COURSE, JAGIELLONIAN UNIVERSITY, 2010

Zbigniew Błocki

1. Basic definitions and properties	1
2. The one dimensional case	7
3. Weierstrass elliptic functions	10
4. Suita conjecture	15
5. Hörmander's L^2-estimate for the $\bar\partial$-equation	19
6. Bergman completeness	23
7. Ohsawa-Takegoshi extension theorem	27
References	31

1. Basic definitions and properties

Bergman kernel. Let Ω be a bounded domain in $\mathbb{C}^n$ (we will assume it throughout, unless otherwise stated). By $H^2(\Omega)$ we will denote the space L^2-integrable holomorphic functions in Ω. For such an f the function $|f|^2$ is in particular subharmonic and thus for $B(z,r)\subset\Omega$

$$|f(z)|^2\le\frac{1}{\lambda(B(z,r))}\int_{B(z,r)}|f|^2d\lambda.$$

Therefore

$$|f(z)|\le\frac{c_n}{(\operatorname{dist}(z,\partial\Omega))^n}\,||f|| \tag{1.1}$$

and

$$\sup_K|f|\le C(K,\Omega)\,||f||,\quad K\Subset\Omega,$$

where by $||f||$ we denote the L^2-norm of f. It follows that the L^2-convergence in $H^2(\Omega)$ implies locally uniform convergence, and thus $H^2(\Omega)$ is a closed subspace of $L^2(\Omega)$.

Hence, $H^2(\Omega)$ is a separable Hilbert space with the scalar product

$$\langle f,g\rangle=\int_\Omega f\bar g\,d\lambda.$$

By (1.1), for a fixed $w\in\Omega$, the functional

$$H^2(\Omega)\ni f\longmapsto f(w)\in\mathbb{C}$$

is continuous. Therefore there is a unique element in $H^2(\Omega)$, which we denote by $K_\Omega(\cdot, w)$, such that

$$f(w) = \langle f, K_\Omega(\cdot, w)\rangle,$$

or equivalently

$$f(w) = \int_\Omega f(z)\overline{K_\Omega(z,w)}\, d\lambda(z),$$

for every $f \in H^2(\Omega)$. The function

$$K_\Omega : \Omega \times \Omega \to \mathbb{C}$$

is called the *Bergman kernel* for the domain Ω.

In particular, for $f = K_\Omega(\cdot, z)$ we get

$$K_\Omega(w, z) = \langle K_\Omega(\cdot, z), K_\Omega(\cdot, w)\rangle = \overline{K_\Omega(z, w)}.$$

It follows that $K_\Omega(z, w)$ is holomorphic in z and antiholomorphic in w. By the Hartogs theorem on separate analyticity the function $K_\Omega(\cdot, \bar{\cdot})$ is holomorphic (where it is defined) and therefore in particular $K_\Omega \in C^\infty(\Omega \times \Omega)$.

If $F : \Omega \to D$ is a biholomorphism then the mapping

$$H^2(D) \ni f \longmapsto f \circ F \operatorname{Jac} F \in H^2(\Omega)$$

is an isomorphism of the Hilbert spaces and

$$f(F(w)) = \int_D f\, \overline{K_D(\cdot, F(w))}\, d\lambda = \int_\Omega f \circ F\, \overline{K_D(\cdot, F(w)) \circ F}\, |\operatorname{Jac} F|^2\, d\lambda.$$

Therefore

$$K_\Omega(z, w) = K_D(F(z), F(w)) \operatorname{Jac} F(z)\, \overline{\operatorname{Jac} F(w)}. \tag{1.2}$$

Example. In the unit disc Δ we have

$$f(0) = \frac{1}{\pi r^2} \int_{\Delta(0,r)} f\, d\lambda, \quad f \in H^2(\Delta),\ r < 1.$$

Therefore

$$f(0) = \frac{1}{\pi} \int_\Delta f\, d\lambda,$$

that is

$$K_\Delta(\cdot, 0) = \frac{1}{\pi}.$$

For arbitrary $w \in \Delta$ we use automorphisms of Δ

$$T_w(z) = \frac{z - w}{1 - z\bar{w}},$$

so that $T_w^{-1} = T_{-w}$ and

$$T_w'(z) = \frac{1 - |w|^2}{(1 - z\bar{w})^2}.$$

Then by (1.2)

$$K_\Delta(z,w) = K_\Delta(z, T_{-w}(0)) = K_\Delta(T_w(z),0)\, T_w'(z)\, \overline{T_w'(w)} = \frac{1}{\pi(1-z\bar{w})^2}.$$

More generally, for the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, we similarly have

$$K_{\mathbb{B}}(\cdot,0) = \frac{1}{\lambda_n},$$

where $\lambda_n = \lambda(\mathbb{B}) = \pi^n/n!$. For $w \in \mathbb{B}$ we can use the automorphism of $\mathbb{B}$

$$T_w(z) = \frac{\left(\frac{\langle z,w\rangle}{1+s_w} - 1\right) w + s_w z}{1 - \langle z,w\rangle},$$

where $s_w = \sqrt{1-|w|^2}$ (see e.g. [Ru]). Then $T_w^{-1} = T_{-w}$ and

$$\operatorname{Jac} T_w(z) = \frac{(1-|w|^2)^{(n+1)/2}}{(1-\langle z,w\rangle)^{n+1}}.$$

Therefore

$$K_{\mathbb{B}}(z,w) = \frac{1}{\lambda_n} \operatorname{Jac} T_w(z)\, \overline{\operatorname{Jac} T_w(w)} = \frac{n!}{\pi^n(1-\langle z,w\rangle)^{n+1}}.$$

If $\{\phi_k\}$ is an orthonormal system in $H^2(\Omega)$ then

$$f = \sum_k \langle f, \phi_k\rangle \phi_k, \quad f \in H^2(\Omega),$$

and the convergence is also locally uniform. Therefore

$$K_\Omega(z,w) = \sum_k \langle K_\Omega(\cdot,w), \phi_k\rangle \phi_k(z) = \sum_k \phi_k(z)\overline{\phi_k(w)}$$

and

$$K_\Omega(z,z) = \sum_k |\phi_k(z)|^2.$$

Exercise 1. *Find an orthonormal system for $H^2(\mathbb{B})$ and use it to compute in another way the Bergman kernel for $\mathbb{B}$.*

Example. For the annulus $P = \{r < |\zeta| < 1\}$ we have for $j,k \in \mathbb{Z}$

$$\langle \zeta^j, \zeta^k\rangle = \int_0^{2\pi} e^{i(j-k)t}dt \int_r^1 \rho^{j+k+1} d\rho = \begin{cases} 0, & j \neq k \\ \frac{\pi}{j+1}(1-r^{2j+2}), & j = k \neq -1 \\ -2\pi \log r, & j = k = -1. \end{cases}$$

Therefore $\{\zeta^j\}_{j\in\mathbb{Z}}$ is an orthogonal system and we will get

$$K_P(z,w) = \frac{1}{\pi z\bar{w}} \left(\frac{1}{2\log(1/r)} + \sum_{j\in\mathbb{Z}} \frac{j(z\bar{w})^j}{1-r^{2j}} \right). \tag{1.3}$$

More examples can be obtained from the product formula:

$$K_{\Omega_1\times\Omega_2}\big((z^1,z^2),(w^1,w^2)\big)=K_{\Omega_1}(z^1,w^1)\,K_{\Omega_2}(z^2,w^2)$$

which easily follows directly from the definition (here $\Omega_1\subset\mathbb{C}^n$ and $\Omega_2\subset\mathbb{C}^m$).

On the diagonal we have

$$K_\Omega(z,z)=||K_\Omega(\cdot,z)||^2=\sup\{|f(z)|^2: f\in H^2(\Omega),\ ||f||\leq 1\}.$$

It follows that $\log K_\Omega(z,z)$ is a smooth plurisubharmonic function in Ω. We will show below that in fact it is strongly plurisubharmonic.

Bergman metric. By B^2_Ω we will denote the Levi form of $\log K_\Omega(z,z)$, that is

$$\begin{aligned}B^2_\Omega(z;X):&=\frac{\partial^2}{\partial\zeta\partial\bar\zeta}\log K_\Omega(z+\zeta X,z+\zeta X)\Big|_{\zeta=0}\\&=\sum_{j,k=1}^n\frac{\partial^2(\log K_\Omega(z,z))}{\partial z_j\partial\bar z_k}X_j\bar X_k,\quad z\in\Omega,\ X\in\mathbb{C}^n.\end{aligned}$$

Theorem 1.1. *We have*

$$B_\Omega(z;X)=\frac{1}{\sqrt{K_\Omega(z,z)}}\sup\{|f_X(z)|: f\in H^2(\Omega),\ ||f||\leq 1,\ f(z)=0\},$$

where

$$f_X=\sum_{j=1}^n\frac{\partial f}{\partial z_j}X_j.$$

Proof. Fix $z_0\in\Omega$, $X\in\mathbb{C}^n$ and set $H:=H^2(\Omega)$,

$$\begin{aligned}H':&=\{f\in H: f(z_0)=0\}\\H'':&=\{f\in H': f_X(z_0)=0\}.\end{aligned}$$

Then $H''\subset H'\subset H$ and in both cases the codimension is 1 (note in particular that $\langle\cdot-z_0,X\rangle\in H''\setminus H'$). Let $\phi_0,\phi_1,\dots$ be an orthonormal system in H such that $\phi_1\in H'$ and $\phi_k\in H''$ for $k\geq 2$. Since $k_\Omega=\sum_{p\geq 0}|\phi_k|^2$, we have

$$B^2_\Omega(\cdot,X)=(\sum_p|\phi_p|^2)^{-1}\sum_p|\phi_{p,X}|^2-(\sum_p|\phi_p|^2)^{-2}\left|\sum_p\phi_{p,X}\,\bar\phi_p\right|^2.$$

Therefore

$$K_\Omega(z_0,z_0)=|\phi_0(z_0)|^2,\quad B^2_\Omega(z_0,X)=\frac{|\phi_{1,X}(z_0)|^2}{|\phi_0(z_0)|^2}.$$

This gives $\leq$. For the reverse inequality take $f\in H'$ with $||f||\leq 1$. Then $\langle f,\phi_0\rangle=0$ and

$$f=\sum_{p\geq 1}\langle f,\phi_p\rangle\phi_p.$$

Therefore

$$|f_X(z_0)| = |\langle f, \phi_1\rangle \phi_{1,X}(z_0)| \le |\phi_{1,X}(z_0)|$$

and the result follows. □

It follows that $B_\Omega(z;X) > 0$ and hence $\log k_\Omega$ is strongly plurisubhamonic. It is thus a potential of a Kähler metric which we call the *Bergman metric*. Length of a curve $\gamma \in C^1([0,1],\Omega)$ in this metric is given by

$$l(\gamma) = \int_0^1 B_\Omega(\gamma(t), \gamma'(t))\, dt$$

and the *Bergman distance* by

$$\operatorname{dist}^B_\Omega(z,w) = \inf\{l(\gamma) :\ \gamma \in C^1([0,1],\Omega),\ \gamma(0) = z,\ \gamma(1) = w\}.$$

If $F : \Omega \to D$ is a biholomorphism then

$$B_\Omega(z;X) = B_D(F(z); F'(z).X)$$

and

$$\operatorname{dist}^B_\Omega(z,w) = \operatorname{dist}^B_D(F(z), F(w)),$$

that is the Bergman metric is biholomorphically invariant.

Kobayashi's construction. Define a mapping

$$\iota : \Omega \ni w \longmapsto [K_\Omega(\cdot, w)] \in \mathbb{P}(H^2(\Omega)).$$

It is well defined since $K_\Omega(\cdot,w) \not\equiv 0$. One can easily show that ι is one-to-one.

For any Hilbert space H one can define the Fubini-Study metric on $\mathbb{P}(H)$ as follows: $FS_{\mathbb{P}(H)} := \pi_* P$, where

$$\pi : H_* \ni f \longmapsto [f] \in \mathbb{P}(H),$$

$H_* = H \setminus \{0\}$ and

$$P^2(f;F) := \frac{\partial^2}{\partial\zeta\partial\bar\zeta} \log ||f + \zeta F||^2\Big|_{\zeta=0} = \frac{||F||^2}{||f||^2} - \frac{|\langle F, f\rangle|^2}{||f||^4}, \quad f \in H_*,\ F \in H.$$

One can show that $FS_{\mathbb{P}(H)}$ is well defined.

We have the following result of Kobayashi [K]:

Theorem 1.2. $B_\Omega = \iota^* FS_{\mathbb{P}(H^2(\Omega))}$.

Proof. We have to show that $B_\Omega = A^* P$, where

$$A : \Omega \ni w \longmapsto K_\Omega(\cdot, w) \in H^2(\Omega),$$

that is that $B_\Omega(w;X) = P(f;F)$, where $f = K_\Omega(\cdot,w)$ and $F = D_X K_\Omega(\cdot, w)$ with D_X being the derivative in direction $X \in \mathbb{C}^n$ w.r.t. w. Let $\phi_0, \phi_1, \dots$ be an orthonormal system chosen as in the proof of Theorem 1.1. Then

$$f = \phi_0(w)\phi_0, \quad F = \phi_{0,X}(w)\phi_0 + \phi_{1,X}(w)\phi_1$$

and one can easily show that

$$P^2(f;F)=\frac{|\phi_{1,X}(z_0)|^2}{|\phi_0(z_0)|^2}=B^2_\Omega(w;X)$$

by the proof of Theorem 1.1. □

The mapping ι embeds Ω equipped with the Bergman metric into infinitely dimensional manifold $\mathbb{P}(H^2(\Omega))$ equipped with the Fubini-Study metric. In particular, it must be distance decreasing. Since the distance in $\mathbb{P}(H)$ is given by

$$d([f],[g])=\arccos\frac{|\langle f,g\rangle|}{||f||\,||g||},$$

we have thus obtained the following:

Theorem 1.3. $\operatorname{dist}^B_\Omega(z,w)\geq\arccos\dfrac{|K_\Omega(z,w)|}{\sqrt{K_\Omega(z,z)K_\Omega(w,w)}}$. □

Corollary 1.4. *If* $K_\Omega(z,w)=0$ *then* $\operatorname{dist}^B_\Omega(z,w)\geq\pi/2$.

The constant $\pi/2$ in Corollary 1.4 turns out to be optimal, it was shown for the annulus in [Di2].

Curvature. The sectional curvature of the Bergman metric is given by

$$R_\Omega(z;X):=-\frac{(\log B)_{\zeta\bar\zeta}}{B}\Big|_{\zeta=0},\quad z\in\Omega,\ X\in\mathbb{C}^n,$$

where $B(\zeta)=B^2_\Omega(z+\zeta X;X)$.

Theorem 1.5. *We have*

$$R_\Omega(z;X)=2-\frac{\sup\{|f_{XX}(z)|^2:f\in H^2(\Omega),\ ||f||\leq 1,\ f(z)=0,\ f_X(z)=0\}}{K_\Omega(z,z)\,B^4_\Omega(z;X)}.$$

Proof. Fix $z_0\in\Omega$, $X\in\mathbb{C}^n$ and let $\phi_0,\phi_1,\dots$ be as in the proof of Theorem 1.1, satisfying in addition that $\phi_k\in H'''$ for $k\geq 3$. Denoting $K(\zeta):=K_\Omega(z+\zeta X)$ we will get

$$-\frac{\big(\log(\log K)_{\zeta\bar\zeta}\big)_{\zeta\bar\zeta}}{(\log K)_{\zeta\bar\zeta}}=2-\frac{\big(\log(KK_{\zeta\bar\zeta}-|K_\zeta|^2)\big)_{\zeta\bar\zeta}}{(\log K)_{\zeta\bar\zeta}}$$
$$=2-\frac{KK_{\zeta\bar\zeta\zeta\bar\zeta}-|K_{\zeta\zeta}|^2}{K^2((\log K)_{\zeta\bar\zeta})^2}+\frac{|KK_{\zeta\bar\zeta\zeta}-K_{\bar\zeta}K_{\zeta\zeta}|^2}{K^4((\log K)_{\zeta\bar\zeta})^3}.$$

Denoting $\varphi_p(\zeta)=\phi_p(z+\zeta X)$ we have $K=\sum_{p\geq 0}|\varphi_p|^2$ and, for $\zeta=0$,

$$K=|\varphi_0|^2,\quad K_\zeta=\varphi_0'\overline{\varphi}_0,\quad K_{\zeta\bar\zeta}=|\varphi_0'|^2+|\varphi_1'|^2,\quad K_{\zeta\zeta}=\varphi_0''\overline{\varphi}_0,$$
$$K_{\zeta\bar\zeta\zeta}=\varphi_0''\overline{\varphi_0'}+\varphi_1''\overline{\varphi_1'},\quad K_{\zeta\bar\zeta\zeta\bar\zeta}=|\varphi_0'|^2+|\varphi_1'|^2+|\varphi_2'|^2.$$

We will get, for $\zeta=0$,

$$K_\Omega(z_0,z_0)=|\varphi_0|^2,\quad B^2_\Omega(z_0;X)=\frac{|\varphi_1'|^2}{|\varphi_0|^2},\quad R_\Omega(z_0;X)=2-\frac{|\varphi_0|^2|\varphi_2''|^2}{|\varphi_1'|^4}.$$

We thus obtain $\leq$ and the reverse inequality can be obtained the same way as in the proof of Theorem 1.1. □

We conclude in particular that always $R_\Omega(z;X)<2$. This estimate is in fact optimal, as can be shown for the annulus $\{r<|\zeta|<1\}$ with $r\to 0$, see [Di1] (and a simplification in [Z2]).

The following result will be useful:

Theorem 1.6. *Assume that Ω_j is a sequence of domains increasing to Ω (that is $\Omega_j \subset \Omega_{j+1}$ and $\sum_j \Omega_j = \Omega$). Then we have locally uniform convergences $K_{\Omega_j} \to K_\Omega$ (in $\Omega \times \Omega$), $B_{\Omega_j}(\cdot, X) \to B_\Omega(\cdot, X)$, $R_{\Omega_j}(\cdot, X) \to R_\Omega(\cdot, X)$ (in Ω), for every $X \in \mathbb{C}^n$.*

Proof. It is enough to prove the first convergence as the other will then be a consequence of it using the following elementary result: if h_j is a sequence of harmonic functions converging locally uniformly to h then $D^\alpha h_j \to D^\alpha h$ locally uniformly for any multi-index α.

For $\Omega' \Subset \Omega$ by the Schwarz inequality for j sufficiently big we have

$$|K_{\Omega_j}(z,w)|^2 \le K_{\Omega_j}(z,z) K_{\Omega_j}(w,w) \le K_{\Omega'}(z,z) K_{\Omega'}(w,w), \quad z, w \in \Omega',$$

and thus the sequence K_{Ω_j} is locally uniformly bounded in $\Omega \times \Omega$. By the Montel theorem (applied to holomorphic functions $K_{\Omega_j}(\cdot, \bar{\cdot})$) there is a subsequence of K_{Ω_j} converging locally uniformly. Therefore, to conclude the proof it is enough to show that if $K_\Omega \to K$ locally uniformly then $K = K_\Omega$.

Fix $w \in \Omega$. We have

$$\begin{aligned} ||K(\cdot, w)||^2_{L^2(\Omega')} &= \lim_{j\to\infty} ||K_{\Omega_j}(\cdot, w)||^2_{L^2(\Omega')} \\ &\le \liminf_{j\to\infty} ||K_{\Omega_j}(\cdot, w)||^2_{L^2(\Omega_j)} \\ &= \liminf_{j\to\infty} K_{\Omega_j}(w,w) \\ &= K(w,w). \end{aligned}$$

Therefore $||K(\cdot, w)||^2 \le K(w,w)$, in particular $K(\cdot, w) \in H^2(\Omega)$ and it remains to show that for any $f \in H^2(\Omega)$

$$f(w) = \int_\Omega f\, \overline{K(\cdot, w)} d\lambda.$$

For j big enough we have

$$\begin{aligned} f(w) - \int_\Omega f\, \overline{K(\cdot, w)} d\lambda &= \int_{\Omega_j} f\, \overline{K_{\Omega_j}(\cdot, w)} d\lambda - \int_\Omega f\, \overline{K(\cdot, w)} d\lambda \\ &= \int_{\Omega'} f \big(\overline{K_{\Omega_j}(\cdot, w)} - \overline{K(\cdot, w)} \big) d\lambda + \int_{\Omega_j \setminus \Omega'} f\, \overline{K_{\Omega_j}(\cdot, w)} d\lambda \\ &\quad - \int_{\Omega \setminus \Omega'} f\, \overline{K(\cdot, w)} d\lambda. \end{aligned}$$

The first integral converges to 0, whereas the other two are arbitrarily small if Ω' is chosen to be sufficiently close to Ω. □

2. The one dimensional case

We assume that Ω is a bounded domain in $\mathbb{C}$. We first show that in this case the Bergman kernel can be obtained as a solution of the Dirichlet problem:

Theorem 2.1. *Assume that Ω is regular. Then for $w \in \Omega$ we have*

$$K_\Omega(\cdot, w) = \frac{\partial v}{\partial z},$$

where v is a complex-valued harmonic function in Ω, continuous on $\bar{\Omega}$, such that

$$v(z) = \frac{1}{\pi\overline{(z-w)}}, \quad z \in \partial\Omega.$$

Proof. We have to show that for $f \in H^2(\Omega)$

$$f(w) = \int_\Omega f \bar{v}_{\bar{z}} d\lambda.$$

By Theorem 1.6 we may assume that $\partial\Omega$ is smooth and f is defined in a neighborhood of $\bar{\Omega}$. Then we have

$$\int_\Omega f \bar{v}_{\bar{z}}\, d\lambda = -\frac{i}{2}\int_\Omega d(f\bar{v}\, dz) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-w}\, dz = f(w). \qquad \square$$

The Green function of Ω with pole at $w \in \Omega$ can be defined as

$$G_\Omega(\cdot, w) := \sup\{v \in SH^-(\Omega) : \limsup_{\zeta \to w}(v(\zeta) - \log|\zeta - w|) < \infty\}.$$

Then $G_\Omega(\cdot, w)$ is a negative subharmonic function in Ω such that $G_\Omega(z,w) - \log|z-w|$ is harmonic in z. The Green function G_Ω is symmetric. If Ω is regular then $G_\Omega(\cdot, w)$ is continuous on $\bar{\Omega} \setminus \{w\}$ and vanishes on $\partial\Omega$.

We have the following relation due to Schiffer:

Theorem 2.2. *Away from the diagonal of $\Omega \times \Omega$ we have*

$$K_\Omega = \frac{2}{\pi}\frac{\partial^2 G_\Omega}{\partial z \partial \bar{w}}.$$

Proof. We may assume that $\partial\Omega$ is smooth. The function

$$\psi(z, w) := G_\Omega(z, w) - \log|z - w|$$

is then smooth in $\bar{\Omega} \times \Omega$. For a fixed $w_0 \in \Omega$ set

$$u := \frac{\partial \psi}{\partial \bar{w}}(\cdot, w_0).$$

Then u is harmonic in Ω, continuous on $\bar{\Omega}$ and

$$u(z) = \frac{1}{2\overline{(z-w)}}, \quad z \in \partial\Omega.$$

Therefore by Theorem 2.1

$$K_\Omega(\cdot, w_0) = \frac{2}{\pi}\frac{\partial u}{\partial z} = \frac{2}{\pi}\frac{\partial^2 G_\Omega}{\partial z \partial \bar{w}}(\cdot, w_0). \qquad \square$$

On the diagonal we have the following formula due to Suita [Su]:

Theorem 2.3. *We have*

$$K_\Omega(z,z) = \frac{1}{\pi}\frac{\partial^2 \rho_\Omega}{\partial z \partial \bar{z}},$$

where

$$\rho_\Omega(w) = \lim_{z\to w}(G_\Omega(z,w) - \log|z-w|)$$

is the Robin function for Ω*.*

Proof. This in fact follows easily from the previous result: we have

$$\rho_\Omega(\zeta) = \psi(\zeta,\zeta),$$

where ψ is as in the proof of Theorem 2.2. We will get

$$\frac{\partial^2 \rho_\Omega}{\partial\zeta\partial\bar{\zeta}} = \psi_{z\bar{z}} + 2\psi_{z\bar{w}} + \psi_{w\bar{w}}.$$

The result now follows from Theorem 2.2, since ψ is harmonic in both z and w. $\square$

Suita metric. Assume for a moment that M is a Riemann surface such that the Green function G_M exists. (This is equivalent to the existence of a nonconstant bounded subharmonic function on M.) Then for $w \in M$ the Robin function

$$\rho_M(w) = \lim_{z\to w}\big(G_M(z,w) - \log|z-w|\big)$$

is ambiguously defined: it depends on the choice of local coordinates. In fact, if change local coordinates by $z = f(\zeta)$, where f is a local biholomorphism with $f(w) = w$, then it is easy to check that

$$\rho_M(w) = \widetilde{\rho_M}(w) + \log|f'(w)|,$$

where $\widetilde{\rho_M}(w)$ is the Robin constant w.r.t. the new coordinates. It follows that the metric

$$e^{\rho_M}|dz|$$

is invariantly defined on M, we call it the *Suita metric.*

We will analyze the curvature of the Suita metric:

$$S_M := K_{e^{\rho_M}|dz|} = -2\frac{(\rho_M)_{z\bar{z}}}{e^{2\rho_M}},$$

which is of course also invariantly defined. Coming back to the case when Ω is a bounded domain in $\mathbb{C}$, by Theorem 2.3 we have

$$S_\Omega(z) = -2\pi\frac{K_\Omega(z,z)}{e^{2\rho_\Omega(z)}}.$$

Exercise 3. *i) Show that if $F:\Omega\to D$ is a biholomorphism then*

$$\rho_\Omega=\rho_D\circ F+\log|F'|.$$

ii) Prove that if Ω is simply connected then $S_\Omega\equiv -2$.

iii) Set $D:=\Delta\cap\Delta(1,r)$. For $w\in D$ let $F_w:D\to\Delta$ be biholomorphic and such that $F_w(w)=w$. Show that

$$\lim_{\substack{w\to 1\\ w\in D}}|F_w'(w)|=1.$$

iv) Prove that if Ω has a C^2 boundary then

$$\lim_{z\to\partial\Omega}S_\Omega(z)=-2.$$

The case of annulus is less trivial and we have the following result of Suita [Su]:

Theorem 2.4. *For the annulus $P=\{r<|\zeta|<1\}$ we have $S_P<-2$ in P.*

To prove this we will use the theory of elliptic functions.

3. Weierstrass elliptic functions

For $\omega_1,\omega_2\in\mathbb{C}$, linearly independent over $\mathbb{R}$, let $\Lambda:=\{2j\omega_1+2k\omega_2:(j,k)\in\mathbb{Z}^2\}$ be the lattice in $\mathbb{C}$. We define the *Weierstrass elliptic function* $\mathcal{P}$ by

$$\mathcal{P}(z)=\mathcal{P}(z;\omega_1,\omega_2):=\frac{1}{z^2}+\sum_{\omega\in\Lambda_*}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right),$$

Since

$$\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}=\frac{-z^2+2\omega z}{\omega^2(z-\omega)^2}=O(|\omega|^{-3}),$$

it follows that $\mathcal{P}$ is holomorphic in $\mathbb{C}\setminus\Lambda$. From

$$\frac{1}{(z-\omega)^2}+\frac{1}{(z+\omega)^2}=2\frac{z^2+\omega^2}{(z^2-\omega^2)^2},$$

it follows that

$$\mathcal{P}(-z)=\mathcal{P}(z).$$

We further have

$$\mathcal{P}'(-z)=-\mathcal{P}'(z),$$

$$\mathcal{P}'(z)=-2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3},$$

so that

$$\mathcal{P}'(z+2\omega_1)=\mathcal{P}'(z)=\mathcal{P}'(z+2\omega_2).$$

It follows that $\mathcal{P}(z+2\omega_1)=\mathcal{P}(z)+A$ for some constant A, but since $\mathcal{P}(-\omega_1)=\mathcal{P}(\omega_1)$, we have in fact $A=0$, that is

$$\mathcal{P}(z+2\omega_1)=\mathcal{P}(z)=\mathcal{P}(z+2\omega_2).$$

The differential equation for $\mathcal{P}$. Write

$$\mathcal{P} = z^{-2} + az^2 + bz^4 + O(|z|^6)$$

and

$$\mathcal{P}' = -2z^{-3} + 2az + 4bz^3 + O(|z|^5).$$

Then

$$\mathcal{P}^3 = \left(z^{-2} + az^2 + bz^4\right)^3 + O(|z|^2) = z^{-6} + 3az^{-2} + 3b + O(|z|^2)$$

and

$$(\mathcal{P}')^2 = \left(-2z^{-3} + 2az + 4bz^3\right)^2 + O(|z|^2) = 4z^{-6} - 8az^{-2} - 16b + O(|z|^2).$$

Therefore

$$(\mathcal{P}')^2 - 4\mathcal{P}^3 + 20a\mathcal{P} + 28b = O(|z|^2).$$

The left-hand side is an entire holomorphic function with periods $2\omega_1$ and $2\omega_2$. It is thus bounded and hence, by the Liouville theorem, constant. We thus obtained the following result:

Theorem 3.1. *We have*

$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3,$$

where

$$g_2 = 60 \sum_{\omega\in\Lambda_*} \frac{1}{\omega^4}, \qquad g_3 = 140 \sum_{\omega\in\Lambda_*} \frac{1}{\omega^6}. \qquad \square$$

Remark. The function $\mathcal{P}$ can be also defined using the constants g_2, g_3 instead of the half-periods ω_1, ω_2 by the relation

$$z = \int_{\mathcal{P}(z)}^{\infty} \frac{1}{\sqrt{4t^3 - g_2 t - g_3}} dt.$$

The Weierstrass function ζ is determined by

$$\zeta' = -\mathcal{P}, \qquad \zeta(z) = \frac{1}{z} + O(|z|).$$

One can easily compute that

$$\zeta(z) = \frac{1}{z} - \sum_{\omega\in\Lambda_*} \left(\frac{1}{z-\omega} + \frac{z}{\omega^2} + \frac{1}{\omega} \right).$$

Again, adding any pair from Λ_* with opposite signs we easily get

$$\zeta(-z) = -\zeta(z).$$

Since $\zeta'(z+2\omega_1) = \zeta'(z) = \zeta'(z+2\omega_2)$, we have

$$\zeta(z+2\omega_1) = \zeta(z) + 2\eta_1, \quad \zeta(z+2\omega_2) = \zeta(z) + 2\eta_2 \tag{3.1}$$

where $\eta_1 = \zeta(\omega_1)$, $\eta_2 = \zeta(\omega_2)$.

Exercise 4. *Show that*

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}. \tag{3.2}$$

We can also define the Weierstrass elliptic function σ by

$$\sigma'/\sigma = \zeta, \qquad \sigma(z) = z + O(|z|^2).$$

One can easily show that

$$\sigma(z) = z \prod_{\omega\in\Lambda_*} \left[\left(1 - \frac{z}{\omega}\right)\exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)\right].$$

It follows that

$$\sigma(-z) = -\sigma(z).$$

From the definition of σ and from (3.1) we infer $\sigma(z+2\omega_1) = Be^{2\eta_1 z}\sigma(z)$ for some constant B. Substituting $z = -\omega_1$ we will get $B = -e^{2\eta_1\omega_1}$, so that

$$\sigma(z+2\omega_1) = -e^{2\eta_1(z+\omega_1)}\sigma(z),$$

and, similarly,

$$\sigma(z+2\omega_2) = -e^{2\eta_2(z+\omega_1)}\sigma(z).$$

The following formula will allow to express ρ_P, where P is an annulus, in terms of σ.

Theorem 3.2. *Assume that* $\mathrm{Im}\,(\omega_2/\omega_1) > 0$. *Then*

$$\sigma(z) = \frac{2\omega_1}{\pi}\exp\frac{\eta_1 z^2}{2\omega_1}\sin\frac{\pi z}{2\omega_1}\prod_{n=1}^{\infty}\frac{\cos(2n\pi\omega_2/\omega_1) - \cos(\pi z/\omega_1)}{\cos(2n\pi\omega_2/\omega_1) - 1} \tag{3.3}$$

and

$$\mathcal{P}(z) = -\frac{\eta_1}{\omega_1} + \frac{\pi^2}{4\omega_1^2}\sum_{j\in\mathbb{Z}}\sin^{-2}\frac{\pi(z+2j\omega_2)}{2\omega_1}. \tag{3.4}$$

Proof. On one hand we have

$$\frac{\cos(2n\pi\omega_2/\omega_1) - \cos(\pi z/\omega_1)}{\cos(2n\pi\omega_2/\omega_1) - 1} = \frac{1 - 2q^{2n}\cos\frac{\pi z}{\omega_1} + q^{4n}}{(1-q^{2n})^2}, \tag{3.5}$$

where $q := \exp(\pi i\omega_2/\omega_1)$. Since $|q| < 1$, it follows that the infinite product is convergent. On the other hand,

$$1 - 2q^{2n}\cos\frac{\pi z}{\omega_1} + q^{4n} = 4q^{2n}\sin\frac{\pi(z+2n\omega_2)}{2\omega_1}\sin\frac{\pi(z-2n\omega_2)}{2\omega_1}. \tag{3.6}$$

Denote the r.h.s. of (3.3) by $\widetilde{\sigma}$. We see that both σ and $\widetilde{\sigma}$ are entire holomorphic functions with simple zeros at Λ. It is straightforward that

$$\widetilde{\sigma}(z+2\omega_1) = -e^{2\eta_1(z+\omega_1)}\widetilde{\sigma}(z).$$

Since $\widetilde{\sigma}(z) = z + O(|z|^2)$, to finish the proof of (3.3) it is therefore enough to show that

$$\widetilde{\sigma}(z+2\omega_2) = -e^{2\eta_2(z+\omega_2)}\widetilde{\sigma}(z)$$

and use the Liouville theorem for the function $\sigma/\widetilde{\sigma}$. We have, denoting $A = \exp(\pi i z/2\omega_1)$ and using (3.2)

$$\begin{aligned}\frac{\widetilde{\sigma}(z+2\omega_2)}{\widetilde{\sigma}(z)} &= \exp\frac{2\eta_1\omega_2(z+\omega_2)}{\omega_1}\lim_{N\to\infty}\frac{\sin\frac{\pi(z+2(N+1)\omega_2)}{2\omega_1}}{\sin\frac{\pi(z-2N\omega_2)}{2\omega_1}}\\ &= A^2 q e^{2\eta_2(z+\omega_2)}\lim_{N\to\infty}\frac{A^2q^{2(N+1)}-1}{A^2q-q^{2N+1}}\end{aligned}$$

and thus (3.3) follows.

To prove (3.4) it is enough to combine (3.3) with (3.5) and (3.6) plus the fact that $\mathcal{P} = -(\log\sigma)''$. $\square$

Proof of Theorem 2.4. We first want to express ρ_P in terms of σ. By Myrberg's theorem we have

$$G_\Omega(z,w) = \sum_j \log\left|\frac{\varphi_0(w)-\varphi_j(z)}{1-\overline{\varphi_0(w)}\varphi_j(z)}\right|,$$

where $\varphi_j = (p|_{V_j})^{-1}$, $p : \Delta \to \Omega$ is a covering, $p^{-1}(U) = \bigcup_j V_j$, U is a small neighborhood of w, V_j are disjoint and $\varphi_0(w) \in V_0$. Then

$$\rho_\Omega = \log\frac{|\varphi_0'|}{1-|\varphi_0|^2} + \sum_{j\neq 0}\log\left|\frac{\varphi_j-\varphi_0}{1-\bar{\varphi}_0\varphi_j}\right|.$$

For $\Omega = P$ we can take a covering $\Delta \to P$ given by

$$p(\zeta) = \exp\left(\frac{\log r}{\pi i}\operatorname{Log}\left(i\frac{1+\zeta}{1-\zeta}\right)\right).$$

Its inverses defined in a neighborhood of the interval $(r,1)$ are given by

$$\varphi_j(z) = \frac{e^{\pi i(\operatorname{Log} z+2j\pi i)/\log r}-i}{e^{\pi i(\operatorname{Log} z+2j\pi i)/\log r}+i}, \quad j\in\mathbb{Z},$$

It is clear that $\rho_P(z)$ depends only on $|z|$. We will get

$$e^{-\rho_P(z)} = \frac{2|z|\log(1/r)}{\pi}\sin\frac{\pi\log|z|}{\log r}\prod_{n=1}^{\infty}\frac{\cosh\frac{2\pi^2 n}{\log r}-\cos\frac{2\pi\log|z|}{\log r}}{\cosh\frac{2\pi^2 n}{\log r}-1}. \tag{3.7}$$

Now choose $\omega_1 = -\log r$ and $\omega_2 = \pi i$. By Theorem 3.2 we will obtain

$$\rho_P(z) = \frac{t}{2} - \log\sigma(t) + \frac{c}{2}t^2 =: \gamma(t),$$

where $t = -2\log|z| \in (0, 2\omega_1)$ and $c = \eta_1/\omega_1$. By Theorem 2.3

$$K_P(z,z) = \frac{\gamma''}{\pi|z|^2} = \frac{1}{\pi}(\mathcal{P}+c)e^t. \tag{3.8}$$

Combining this with (1.3)

$$\mathcal{P}(t) = \frac{1}{2\omega_1} - c + \sum_{j=-\infty}^{\infty} \frac{je^{-jt}}{1-r^{2j}}.$$

One can easily check that $\mathcal{P}(0) = \infty$ and $\mathcal{P}$ decreases in $(0, \omega_1)$. We also have $\mathcal{P}(2\omega_1 - t) = \mathcal{P}(t)$ and $\mathcal{P}'(\omega_1) = 0$. Set

$$F := \log\frac{\pi K_P}{e^{2\rho_P}} = \log(\mathcal{P}+c) + 2\log\sigma - ct^2.$$

Then $F(2\omega_1 - t) = F(t)$ and

$$F' = \frac{\mathcal{P}'}{\mathcal{P}+c} + 2\zeta - 2ct.$$

Since $\mathcal{P} = t^{-2} + O(t^2)$, $\zeta = t^{-1} + O(t)$, we get $F'(0) = 0$. We also have $F'(\omega_1) = 0$. Theorem 3.1 gives $(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3$, and thus $\mathcal{P}'' = 6\mathcal{P}^2 - g_2/2$. Therefore

$$F'' = \frac{(g_2 - 12c^2)\mathcal{P} - cg_2 + 2g_3 - 4c^3}{2(\mathcal{P}+c)^2}. \tag{3.9}$$

By (3.8) $\mathcal{P} + c > 0$. We also have $F(0) = 0$ and we claim that

$$F(\omega_1) > 0. \tag{3.10}$$

This will finish the proof because from (3.9) and $F'(0) = F'(\omega_1) = 0$ we will conclude that F'' has precisely one zero in $(0, \omega_1)$ and thus $F' > 0$ there. It thus remains to show (3.10).

Using (3.7) we may write

$$\gamma = \log\frac{\pi}{2\omega_1} + \frac{t}{2} - \log\sin\frac{\pi t}{2\omega_1} + \log\prod_{n=1}^{\infty}\frac{a_n - 1}{a_n - \cos(\pi t/\omega_1)},$$

where $a_n = \cosh(2\pi^2 n/\omega_1)$. Then

$$\gamma'' = \frac{\pi^2}{4\omega_1^2\sin^2(\pi t/2\omega_1)} + \frac{\pi^2}{\omega_1^2}\sum_{n=1}^{\infty}\frac{1 - a_n\cos(\pi t/\omega_1)}{(a_n - \cos(\pi t/\omega_1))^2} \tag{3.11}$$

and

$$\begin{aligned} F &= \log\gamma'' + t - 2\gamma \\ &= \log\left(1 + 4\sin^2\frac{\pi t}{2\omega_1}\sum_{n=1}^{\infty}\frac{1 - a_n\cos(\pi t/\omega_1)}{(a_n - \cos(\pi t/\omega_1))^2}\right) + 2\sum_{n=1}^{\infty}\log\frac{a_n - \cos(\pi t/\omega_1)}{a_n - 1}. \end{aligned}$$

We will obtain

$$F(\omega_1) = \log\left(1 + 4\sum_{n=1}^{\infty}\frac{1}{a_n+1}\right) + 2\sum_{n=1}^{\infty}\log\frac{a_n+1}{a_n-1} > 0. \qquad \square$$

In the proof of Theorem 2.4 we showed in particular that

$$K_P(z,z) = \frac{1}{\pi|z|^2}\big(\mathcal{P}(2\log|z|) + \frac{\eta_1}{\omega_1}\big),$$

where $\mathcal{P}$ is the Weierstrass function with half-periods $\omega_1 = -\log r$ and $\omega_2 = \pi i$. In fact, we can show a similar formula also away from the diagonal and characterize precisely the zeros of K_P (compare with [R] and [Sk]):

Theorem 3.4. *We have*

$$K_P(z,w) = \frac{h(z\bar{w})}{\pi z\bar{w}},$$

where

$$h(\lambda) = \mathcal{P}(\log\lambda) + \frac{\eta_1}{\omega_1}. \tag{3.12}$$

The function h has exactly two simple zeros in the annulus $\{r^2 < |\lambda| < 1\}$, both on the interval $(-r^2,-1)$.

Proof. Let φ_j be as in the proof of Theorem 2.4. After some calculations we will get

$$G_P(z,w) = \sum_{j\in\mathbb{Z}}\log\left|\frac{1-f_j(w/z)}{1-f_j(z\bar{w})}\right|,$$

where

$$f_j(\zeta) = \exp\frac{\pi i(\operatorname{Log}\zeta + 2j\pi i)}{\log r}.$$

By Theorem 2.2 we will get (also after some calculations)

$$K_P(z,w) = -\frac{\pi}{\lambda\log^2 r}\sum_{j\in\mathbb{Z}}\frac{f_j(\lambda)}{(1-f_j(\lambda))^2},$$

where $\lambda = z\bar{w}$. Since

$$\frac{e^\alpha}{(1-e^\alpha)^2} = -\frac{1}{4\sin^2(i\alpha/2)},$$

we will get

$$h(\lambda) = \frac{\pi^2}{4\log^2 r}\sum_{j\in\mathbb{Z}}\sin^{-2}\frac{\pi(\operatorname{Log}\lambda + 2j\pi i)}{2\log r} \tag{3.13}$$

and (3.12) follows from Theorem 3.2.

By (1.3) we have

$$h(\lambda) = \frac{1}{2\omega_1} + \sum_{j\in\mathbb{Z}}\frac{j\lambda^j}{1-r^{2j}}.$$

It follows in particular that h is real-valued for real λ and that $h(r^2/\lambda) = h(\lambda)$. We also have $f_j(-r) = -q^{-(2j+1)}$ and $f_j(-1) = q^{-(2j+1)}$, where $q = e^{\pi^2/\log r}$. Therefore by (3.13)

$$h(-r) = \frac{\pi^2}{\log^2 r}\sum_{j\in\mathbb{Z}}\frac{q^{2j+1}}{(1+q^{2j+1})^2} > 0,$$
$$h(-1) = h(-r^2) = -\frac{\pi^2}{\log^2 r}\sum_{j\in\mathbb{Z}}\frac{q^{2j+1}}{(1-q^{2j+1})^2} < 0.$$

This implies that there are two simple zeros on the interval $(-1, -r^2)$. The following result guarantees that there are no more than two in the annulus $\{r^2 < |\lambda| < 1\}$:

Proposition 3.5. *In the parallelogram $\{2t\omega_1 + 2s\omega_2 : s, t \in [0,1)\}$ the Weierstrass function $\mathcal{P}$ attains every value exactly twice (counting with multiplicities).*

Proof. For any complex number w let C be an oriented contour given by the boundary of this parallelogram moved slightly, so that it doesn't contain neither zeros nor poles of $\mathcal{P} - w$. Then

$$\frac{1}{2\pi i}\int_C \frac{\mathcal{P}'(z)}{\mathcal{P}(z) - w}dz = Z - P,$$

where Z is the number of zeros an P the number of poles of $\mathcal{P}$ inside C. We have $P = 2$ because $\mathcal{P}$ has precisely one double pole inside C. On the other hand, since the function under the sign of integration is doubly periodic with periods $2\omega_1$ and $2\omega_2$, it follows easily that the integral must vanish. □

4. Suita conjecture

The Suita conjecture [Su] asserts that $S_\Omega \le -2$, that is that

$$e^{2\rho_\Omega(z)} \le \pi K_\Omega(z,z).$$

By approximation it is enough to prove the estimate for domains with smooth boundary. The conjecture is still open. Ohsawa [O] showed, using the theory of the $\bar\partial$-equation, that

$$e^{2\rho_\Omega(z)} \le 750\pi K_\Omega(z,z).$$

We want to prove the following improvement from [Bł3]:

Theorem 4.1. *We have*

$$e^{2\rho_\Omega(z)} \le 2\pi K_\Omega(z,z),$$

that is $S_\Omega \le -1$.

We may assume that Ω has smooth boundary. We will use the weighted $\bar\partial$-Neumann operator and an approach of Berndtsson [B1]. Denote

$$\partial\alpha = \frac{\partial\alpha}{\partial z}, \quad \bar\partial\alpha = \frac{\partial\alpha}{\partial\bar z}.$$

If φ is smooth in $\bar\Omega$ then the formal adjoint to $\bar\partial$ with respect to the scalar product in $L^2(\Omega, e^{-\varphi})$ is given by

$$\bar\partial^*\alpha = -e^{\varphi}\partial(e^{-\varphi}\alpha) = -\partial\alpha + \alpha\partial\varphi.$$

The complex Laplacian in $L^2(\Omega, e^{-\varphi})$ is defined by

$$\Box\alpha = -\bar\partial\,\bar\partial^*\alpha = \partial\bar\partial\alpha - \partial\varphi\bar\partial\alpha - \alpha\partial\bar\partial\varphi.$$

The following formula relating $\Box$ to the standard Laplacian can be proved by direct computation:

Proposition 4.2.

$$\partial\bar\partial(|\alpha|^2 e^{-\varphi}) = \big(2\mathrm{Re}\,(\bar\alpha\,\Box\alpha) + |\bar\partial\alpha|^2 + |\bar\partial^*\alpha|^2 + |\alpha|^2\partial\bar\partial\varphi\big)\, e^{-\varphi}. \qquad \Box$$

We may assume that $0 \in \Omega$. If φ is subharmonic (which we assume from now on) then by PDEs we can find $N \in C^\infty(\bar\Omega \setminus \{0\}) \cap L^1(\Omega)$ such that

$$\Box N = \frac{\pi}{2} e^{\varphi(0)}\delta_0, \quad N = 0 \ \text{ on } \partial\Omega.$$

(The constant $\pi/2$ is chosen so that $N = G$, where $G = G_\Omega(\cdot, 0)$, if $\varphi \equiv 0$.)

The key in the proof of Theorem 4.1 will be the following estimate of Berndtsson [B1]:

Theorem 4.3. $|N|^2 \le e^{\varphi+\varphi(0)}G^2$.

Proof. Set

$$u := |\alpha|^2 e^{-\varphi} + \varepsilon.$$

Then

$$|\partial u| = \left|\alpha\overline{\bar\partial\alpha} + \bar\alpha\bar\partial^*\alpha\right| e^{-\varphi} \le |\alpha|(|\bar\partial\alpha| + |\bar\partial^*\alpha|)e^{-\varphi}$$

and by Proposition 4.2

$$\begin{aligned}
\partial\bar\partial(u^{1/2}) &= \frac{1}{2}u^{-1/2}\partial\bar\partial u - \frac{1}{4}u^{-3/2}|\partial u|^2 \\
&\ge \frac{1}{2}u^{-3/2}|\alpha|^2\big[2\mathrm{Re}\,(\bar\alpha\Box\alpha) + |\bar\partial\alpha|^2 + |\bar\partial^*\alpha|^2 - \frac{1}{2}(|\bar\partial\alpha| + |\bar\partial^*\alpha|)^2\big]e^{-2\varphi} \\
&\ge -u^{-3/2}|\alpha|^3 e^{-2\varphi}|\Box\alpha| \\
&\ge -|\Box\alpha|e^{-\varphi/2}.
\end{aligned}$$

Now approximating N by smooth functions and letting $\varepsilon \to 0$ we will get

$$\partial\bar\partial\left(-|N|e^{-(\varphi+\varphi(0))/2}\right) \le \frac{\pi}{2}\delta_0 = \partial\bar\partial G$$

and the theorem follows. $\Box$

Proof of Theorem 4.1. Set

$$\varphi := 2(\log|z| - G).$$

Then φ is harmonic in Ω, smooth on $\bar\Omega$ and

$$\varphi(0) = -2\rho_\Omega(0).$$

For harmonic weights the operators $\bar\partial$ and its adjoint commute

$$\Box = -\bar\partial\,\bar\partial^* = -\bar\partial^*\bar\partial.$$

Therefore

$$\bar\partial(e^{-\varphi}\partial\bar N) = \bar\partial(-e^{-\varphi(0)}\bar\partial^* N) = \frac{\pi}{2}\delta_0.$$

It follows that the function

$$f := ze^{-\varphi}\partial\bar N$$

is holomorphic in Ω, smooth on $\bar\Omega$, and, since $\bar\partial(2f/z - 1/z) = 0$, $f(0) = 1/2$.

Using the fact that both $|N|^2e^{-\varphi}$ and its derivative vanish on $\partial\Omega$, integration by parts and Proposition 4.1 give

$$\int_\Omega |N|^2 e^{-\varphi}\partial\bar\partial(|z|^2e^{-\varphi})d\lambda = \int_\Omega |z|^2(|\bar\partial N|^2 + |\bar\partial^* N|^2)e^{-2\varphi}d\lambda \ge \int_\Omega |f|^2 d\lambda.$$

On the other hand, we have $|z|^2e^{-\varphi} = e^{2G}$ and by Theorem 4.3

$$\int_\Omega |N|^2 e^{-\varphi}\partial\bar\partial(|z|^2e^{-\varphi})d\lambda \le e^{\varphi(0)}\int_\Omega G^2\partial\bar\partial e^{2G}d\lambda.$$

We need the following simple lemma.

Lemma 4.4. *For every integrable* $\gamma : (-\infty, 0) \to \mathbb{R}$ *we have*

$$\int_\Omega \gamma\circ G\,|\nabla G|^2 d\lambda = 2\pi\int_{-\infty}^0 \gamma(t)dt.$$

Proof. Let $\chi : (-\infty, 0) \to \mathbb{R}$ be such that $\chi' = \gamma$ and $\chi(-\infty) = 0$. Then

$$\int_\Omega \gamma\circ G\,|\nabla G|^2 d\lambda = \int_\Omega \langle\nabla(\chi\circ G), \nabla G\rangle d\lambda = \int_{\partial\Omega} \chi(0)\frac{\partial G}{\partial n}d\sigma = 2\pi\chi(0). \qquad \Box$$

End of proof of Theorem 4.1. It follows that

$$\int_\Omega G^2\partial\bar\partial e^{2G}d\lambda = \int_\Omega G^2e^{2G}|\nabla G|^2 d\lambda = \frac{\pi}{2}$$

and thus

$$\int_\Omega |f|^2 d\lambda \le \frac{\pi}{2}e^{\varphi(0)},$$

from which the required estimate immediately follows. $\Box$

5. Hörmander's L^2-estimate for the $\bar\partial$-equation

We will first sketch the classical theory of the $\bar\partial$-equation from [Hö] in the special case $p = q = 0$, namely we consider the equation

$$\bar\partial u = \alpha,$$

where

$$\alpha = \sum_{j=1}^{n} \alpha_j d\bar z_j$$

is a $(0,1)$-form satisfying the necessary condition

$$\bar\partial \alpha = 0.$$

We will first show how to slightly modify the proof of Lemma 4.4.1 in [Hö] to obtain the following slight improvement:

Theorem 5.1. *Assume that Ω is a pseudoconvex domain in $\mathbb{C}^n$ (not necessarily bounded). Let φ be a C^2 strongly plurisubharmonic function in Ω and $\alpha \in L^2_{loc,(0,1)}(\Omega)$ with $\bar\partial\alpha = 0$. Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and such that*

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda, \tag{5.1}$$

where

$$|\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k=1}^{n} \varphi^{j\bar k} \bar\alpha_j \alpha_k$$

is the length of the form α w.r.t. the Kähler metric $i\partial\bar\partial\varphi$ (here $(\varphi^{j\bar k})$ is the inverse transposed of $(\partial^2\varphi/\partial z_j \partial \bar z_k)$).

Sketch of proof. If the right hand-side of (5.1) is not finite it is enough to apply Theorem 4.2.2. in [Hö], we may thus assume that it is finite and even equal to 1. We follow the proof of Lemma 4.4.1 in [Hö] and its notation: the function s is smooth, strongly plurisubharmonic in Ω and such that $\Omega_a := \{s < a\} \Subset \Omega$ for every $a \in \mathbb{R}$. We fix $a > 0$ and choose $\eta_\nu \in C_0^\infty(\Omega)$, $\nu = 1, 2, \dots$, such that $0 \le \eta_\nu \le 1$ and $\Omega_{a+1} \subset \{\eta_\nu = 1\} \uparrow \Omega$ as $\nu \uparrow \infty$. Let $\psi \in C^\infty(\Omega)$ vanish in Ω_a and satisfy $|\partial\eta_\nu|^2 \le e^\psi$, $\nu = 1, 2, \dots$, and let $\chi \in C^\infty(\mathbb{R})$ be convex and such that $\chi = 0$ on $(-\infty, a)$, $\chi \circ s \ge 2\psi$ and $\chi' \circ s\, i\partial\bar\partial s \ge (1 + a)|\partial\psi|^2 i\partial\bar\partial|z|^2$. This implies that with $\varphi' := \varphi + \chi \circ s$ we have in particular

$$i\partial\overline{\partial}\varphi' \ge i\partial\overline{\partial}\varphi + (1 + a)|\partial\psi|^2 i\partial\overline{\partial}|z|^2. \tag{5.2}$$

The $\overline{\partial}$-operator gives the densely defined operators T and S between Hilbert spaces:

$$L^2(\Omega, \varphi_1) \xrightarrow{T} L^2_{(0,1)}(\Omega, \varphi_2) \xrightarrow{S} L^2_{(0,2)}(\Omega, \varphi_3),$$

where $\varphi_j := \varphi' + (j - 3)\psi$, $j = 1, 2, 3$. (Recall that, if

$$F = \sum_{\substack{|J|=p\\|K|=q}}{}' F_{JK} dz_J \wedge d\bar z_K \in L^2_{loc,(p,q)}(\Omega),$$

then

$$|F|^2 = {\sum_{J,K}}' |F_{JK}|^2,$$

$$L^2_{(p,q)}(\Omega,\varphi) = \{F \in L^2_{loc,(p,q)}(\Omega) : ||F||^2_\varphi := \int_\Omega |F|^2 e^{-\varphi} d\lambda < \infty\},$$

$$\langle F, G\rangle_\varphi := \int_\Omega {\sum_{J,K}}' F_{JK}\overline{G}_{JK} e^{-\varphi} d\lambda, \quad F, G \in L^2_{(p,q)}(\Omega,\varphi).)$$

For $f = \sum_j f_j d\overline{z}_j \in C^\infty_{0,(0,1)}(\Omega)$ one can then compute

$$|Sf|^2 = \sum_{j<k} \left| \frac{\partial f_j}{\partial \overline{z}_k} - \frac{\partial f_k}{\partial \overline{z}_j} \right|^2 = \sum_{j,k} \left| \frac{\partial f_j}{\partial \overline{z}_k} \right|^2 - \sum_{j,k} \frac{\partial f_j}{\partial \overline{z}_k} \frac{\partial \overline{f}_k}{\partial z_j} \tag{5.3}$$

and

$$e^\psi T^* f = -\sum_j \delta_j f_j - \sum_j f_j \frac{\partial \psi}{\partial z_j},$$

where

$$\delta_j w := e^{\varphi'} \frac{\partial}{\partial z_j}(w e^{-\varphi'}) = \frac{\partial w}{\partial z_j} - w \frac{\partial \varphi'}{\partial z_j}.$$

Therefore

$$|\sum_j \delta_j f_j|^2 \le (1 + a^{-1}) e^{2\psi} |T^* f|^2 + (1 + a)|f|^2 |\partial\psi|^2. \tag{5.4}$$

Integrating by parts we get

$$\int_\Omega |\sum_j \delta_j f_j|^2 e^{-\varphi'} d\lambda = \int_\Omega \sum_{j,k} \left(\frac{\partial^2 \varphi'}{\partial z_j \bar\partial z_k} f_j \overline{f}_k + \frac{\partial f_j}{\partial \overline{z}_k} \frac{\partial \overline{f}_k}{\partial z_j} \right) e^{-\varphi'} d\lambda.$$

Combining this with (5.2)-(5.4) we arrive at

$$\int_\Omega \sum_{j,k} \frac{\partial^2 \varphi'}{\partial z_j \bar\partial z_k} f_j \overline{f}_k e^{-\varphi'} d\lambda \le (1 + a^{-1}) ||T^* f||^2_{\varphi_1} + ||Sf||^2_{\varphi_3}. \tag{5.5}$$

We have

$$|\sum_j \overline{\alpha}_j f_j|^2 \le |\alpha|^2_{i\partial\bar\partial\varphi} \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \bar\partial z_k} f_j \overline{f}_k. \tag{5.6}$$

Hence, from the Schwarz inequality, (5.5) and from the fact that $\varphi - 2\varphi_2 \le -\varphi'$ we obtain

$$|\langle \alpha, f\rangle_{\varphi_2}|^2 \le (1 + a^{-1}) ||T^* f||^2_{\varphi_1} + ||Sf||^2_{\varphi_3} \tag{5.7}$$

for all $f \in C^\infty_{0,(0,1)}(\Omega)$ and thus also for all $f \in D_{T^*} \cap D_S$ (recall that we have assumed that the right hand-side of (5.1) is 1).

If $f' \in L^2_{(0,1)}(\Omega, \varphi_2)$ is orthogonal to the kernel of S then it is also orthogonal to the range of T and thus $T^* f' = 0$. Moreover, since $S\alpha = 0$, we then also have $\langle \alpha, f' \rangle_{\varphi_2} = 0$. Therefore by (5.7)

$$|\langle \alpha, f \rangle_{\varphi_2}| \le \sqrt{1 + a^{-1}} ||T^* f||_{\varphi_1}, \quad f \in D_{T^*}.$$

By the Hahn-Banach theorem there exists $u_a \in L^2(\Omega, \varphi_1)$ with $||u_a||_{\varphi_1} \le \sqrt{1 + a^{-1}}$ and

$$\langle \alpha, f \rangle_{\varphi_2} = \langle u_a, T^* f \rangle_{\varphi_1}, \quad f \in D_{T^*}.$$

This means that $Tu_a = \alpha$ and, since $\varphi_1 \ge \varphi$ with equality in Ω_a, we have

$$\int_{\Omega_a} |u_a|^2 e^{-\varphi} d\lambda \le 1 + a^{-1}.$$

We may thus find a sequence $a_j \uparrow \infty$ and $u \in L^2_{loc}(\Omega)$ such that u_{a_j} converges weakly to u in $L^2(\Omega_a, \varphi) = L^2(\Omega_a)$ for every a. $\square$

It will be convenient to have a version of Theorem 5.1 for nonsmooth φ. Note that (5.6) holds pointwise for every f precisely when

$$i\bar{\alpha} \wedge \alpha \le |\alpha|^2_{i\partial\bar{\partial}\varphi} i\partial\bar{\partial}\varphi.$$

This observation allows to formulate the following generalization of Theorem 5.1:

Theorem 5.1'. *Assume that Ω is pseudoconvex and φ plurisubharmonic in Ω. Let $\alpha \in L^2_{loc,(0,1)}(\Omega)$ be such that $\overline{\partial}\alpha = 0$ and*

$$i\alpha \wedge \overline{\alpha} \le h i\partial\overline{\partial}\varphi \tag{5.8}$$

for some nonnegative function $h \in L^1_{loc}(\Omega)$ such that the right hand-side of (5.8) makes sense as a current of order 0 (that is the coefficients of $hi\partial\overline{\partial}\varphi$ are complex measures; this is always the case if h is locally bounded). Then there exists $u \in L^2_{loc}(\Omega)$ with $\overline{\partial}u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega h e^{-\varphi} d\lambda.$$

Proof. First assume that φ is strongly plurisubharmonic (but otherwise arbitrary, that is possibly even not locally bounded). By the Radon-Nikodym theorem there exists $\beta = \sum_{j,k} \beta_{jk} i dz_j \wedge d\bar{z}_k \in L^1_{loc,(1,1)}(\Omega)$ such that $0 < \beta \le i\partial\overline{\partial}\varphi$ and $i\bar{\alpha} \wedge \alpha \le h\beta$. For $\varepsilon > 0$ let $a(\varepsilon)$ be such that $\varphi_\varepsilon := \varphi * \rho_\varepsilon \in C^\infty(\overline{\Omega}_{a(\varepsilon)})$ (where Ω_a is as in the proof of Theorem 5.1). Set $h_\varepsilon := |\alpha|^2_{i\partial\bar{\partial}\varphi_\varepsilon}$, so that h_ε is the least function satisfying $i\bar{\alpha} \wedge \alpha \le h_\varepsilon i\partial\overline{\partial}\varphi_\varepsilon$. By Theorem 5.1 we can find $u_\varepsilon \in L^2_{loc}(\Omega_{a(\varepsilon)})$ such that $\overline{\partial}u_\varepsilon = \alpha$ in $\Omega_{a(\varepsilon)}$ and

$$\int_{\Omega_{a(\varepsilon)}} |u_\varepsilon|^2 e^{-\varphi_\varepsilon} d\lambda \le \int_{\Omega_{a(\varepsilon)}} h_\varepsilon e^{-\varphi_\varepsilon} d\lambda \le \int_{\Omega_{a(\varepsilon)}} h_\varepsilon e^{-\varphi} d\lambda.$$

We have $\beta_\varepsilon := \beta * \rho_\varepsilon \le i\partial\overline{\partial}\varphi_\varepsilon$ and the coefficients of β_ε converge pointwise almost everywhere to the respective coefficients of β. Therefore

$$\overline{\lim_{\varepsilon\to 0}}\, h_\varepsilon \le \overline{\lim_{\varepsilon\to 0}} \sum_{j,k} \beta_\varepsilon^{jk}\overline{\alpha}_j\alpha_k = \sum_{j,k} \beta^{jk}\overline{\alpha}_j\alpha_k \le h,$$

where (β^{jk}) and (β_ε^{jk}) denote the inverse matrices of (β_{jk}) and $(\beta_{jk} * \rho_\varepsilon)$, respectively. By the Fatou lemma we thus have

$$\overline{\lim_{\varepsilon\to 0}} \int_{\Omega_{a(\varepsilon)}} |u_\varepsilon|^2 e^{-\varphi_\varepsilon} d\lambda \le \int_\Omega h e^{-\varphi} d\lambda.$$

Since φ_ε is decreasing as decreases to 0, we see that the $L^2(\Omega_a, \varphi_{\widetilde{\varepsilon}})$ norm of u_ε is bounded for every $\varepsilon \le \widetilde{\varepsilon}$ and fixed a and $\widetilde{\varepsilon}$. Therefore, we can find a subsequence u_{ε_l} converging weakly in Ω_a for every a to $u \in L^2_{loc}(\Omega)$. Moreover, for every $\delta > 0$, and l sufficiently big we then have

$$\int_{\Omega_a} |u|^2 e^{-\varphi_{\varepsilon_l}} d\lambda \le \delta + \int_\Omega h e^{-\varphi} d\lambda$$

and thus by the Lebesgue monotone convergence theorem we can conclude the proof for strongly plurisubharmonic φ.

If φ is not necessarily strongly plurisubharmonic then we may approximate it by functions of the form $\varphi + \varepsilon|z|^2$. Note that $i\bar{\alpha}\wedge\alpha \le h\, i\partial\overline{\partial}(\varphi + \varepsilon|z|^2)$ and the general case easily follows along the same lines as before. $\square$

The next result is due to Berndtsson [B2] (see also [B3]).

Theorem 5.2. *Let Ω, φ, α and h be as in Theorem 5.1'. Fix $r \in (0,1)$ and assume in addition that $-e^{-\varphi/r} \in PSH(\Omega)$. Then for any $\psi \in PSH(\Omega)$ we can find $u \in L^2_{loc}(\Omega)$ with $\overline{\partial}u = \alpha$ and*

$$\int_\Omega |u|^2 e^{\varphi-\psi} d\lambda \le \frac{1}{(1-\sqrt{r})^2} \int_\Omega h e^{\varphi-\psi} d\lambda.$$

Proof. Approximating $-e^{-\varphi/r}$ and ψ in the same way as in the proof of Theorem 5.1' we may assume that φ and ψ are smooth up to the boundary. Then we have in particular $L^2(\Omega) = L^2(\Omega, a\varphi + b\psi)$ for real a, b and $-e^{-\varphi/r} \in PSH(\Omega)$ means precisely that

$$i\partial\varphi \wedge \overline{\partial}\varphi \le r\, i\partial\overline{\partial}\varphi.$$

Let u be the solution to $\overline{\partial}u = \alpha$ which is minimal in the $L^2(\Omega, \psi)$ norm. This means that

$$\int_\Omega u\overline{f} e^{-\psi} d\lambda = 0, \quad f \in H^2(\Omega).$$

Set $v := e^\varphi u$. Then

$$\int_\Omega v\overline{f} e^{-\varphi-\psi} d\lambda = 0, \quad f \in H^2(\Omega),$$

thus v is the minimal solution in the $L^2(\Omega, \varphi + \psi)$ norm to $\overline{\partial}v = \beta$, where

$$\beta = \overline{\partial}(e^\varphi u) = e^\varphi(\alpha + u\overline{\partial}\varphi).$$

For every $t > 0$ we have

$$\begin{aligned} i\beta \wedge \overline{\beta} &\le e^{2\varphi}[(1+t^{-1})i\alpha \wedge \overline{\alpha} + (1+t)|u|^2 i\partial\overline{\partial}\varphi] \\ &\le e^{2\varphi}[(1+t^{-1})h + (1+t)r|u|^2] i\partial\overline{\partial}\varphi \\ &\le e^{2\varphi}[(1+t^{-1})h + (1+t)r|u|^2] i\partial\overline{\partial}(\varphi+\psi). \end{aligned}$$

Therefore by Theorem 5.1

$$\int_\Omega |u|^2 e^{\varphi-\psi} d\lambda = \int_\Omega |v|^2 e^{-\varphi-\psi} d\lambda \le (1+t^{-1}) \int_\Omega h e^{\varphi-\psi} d\lambda + (1+t) r \int_\Omega |u|^2 e^{\varphi-\psi} d\lambda.$$

For $t = r^{-1/2} - 1$ we obtain the required result. □

Applying Theorem 5.2 with $r = 1/4$ and φ, ψ replaced with $\varphi/4, \psi + \varphi/4$, respectively, we obtain the following estimate essentially due to Donnelly and Fefferman [DF].

Theorem 5.3. *Let Ω, φ, α and h satisfy the assumptions of Theorem 5.1'. Assume moreover that $-e^{-\varphi} \in PSH(\Omega)$. Then for any $\psi \in PSH(\Omega)$ we can find $u \in L^2_{loc}(\Omega)$ with $\overline{\partial} u = \alpha$ and*

$$\int_\Omega |u|^2 e^{-\psi} d\lambda \le 16 \int_\Omega h e^{-\psi} d\lambda. \quad \square$$

One can improve the constants in Theorems 5.2 and 5.3 to $4r/(1-r)^2$ and 4, respectively (see [Bł1]).

Exercise 5. *Let $n = 1$ and $\varphi = -\log(-\log|z|)$. Show that $u = \bar{z}$ is the minimal solution in $L^2(\Delta, \varphi)$ of the equation $\bar{\partial} u = d\bar{z}$. Prove that*

$$\int_\Delta |u|^2 d\lambda = 2 \int_\Delta |\bar{\partial} u|^2_{i\partial\bar{\partial}\varphi} d\lambda$$

and conclude that the constant in Theorem 5.3 cannot be better than 2.

6. Bergman completeness

Domains complete w.r.t. the Bergman metric are called *Bergman complete*.

Proposition 6.1. *Every Bergman complete domain is pseudoconvex.*

Proof. If Ω is not pseudoconvex then by the definition of a domain of holomorhpy there are domains Ω_1, Ω_2 such that $\emptyset \neq \Omega_1 \subset \Omega \cap \Omega_2$, $\Omega_2 \not\subset \Omega$ and for every f holomorphic in Ω there exists $\widetilde{f}$ holomorphic in Ω_2 such that $f = \widetilde{f}$ on Ω_1. We may assume that Ω_1 is a connected component of $\Omega \cap \Omega_2$ such that the set $\Omega_2 \cap \partial\Omega \cap \partial\Omega_1$ is nonempty. Since $K_\Omega(\cdot, \bar{\cdot})$ is holomorphic in $\Omega \times \Omega^*$, it follows that there exists $\widetilde{K} \in C^\infty(\Omega_2 \times \Omega_2)$ such that $\widetilde{K}(\cdot, \bar{\cdot})$ is holomorphic in $\Omega_2 \times \Omega_2^*$ and $\widetilde{K} = K_\Omega$ in $\Omega_1 \times \Omega_1$. This means that every sequence $z_k \to \Omega_2 \cap \partial\Omega \cap \partial\Omega_1$ is a Cauchy sequence with respect to $\operatorname{dist}_\Omega$, which contradicts the completeness of Ω. □

The converse is not true as the following exercise shows:

Exercise 6. *Show that every function from $H^2(\Delta \setminus \{0\})$ extends to a function in $H^2(\Delta)$. Conclude that $\Delta \setminus \{0\}$ is not Bergman complete.*

The main tool for the Bergman completeness is the following criterion of Kobayashi [K] (from now on we again assume that Ω is a bounded domain in $\mathbb{C}^n$):

Theorem 6.2. *Assume that*

$$\lim_{z \to \partial\Omega} \frac{|f(z)|^2}{K_\Omega(z,z)} = 0, \quad f \in H^2(\Omega). \tag{6.1}$$

Then Ω is Bergman complete.

Proof. Let z_k be a Cauchy sequence in Ω (with respect to the Bergman metric). Suppose that z_k has no accumulation point in Ω. It is easy to check that this is equivalent to the fact that $z_k \to \partial\Omega$. By Theorem 1.2 $\iota(z_k)$ is a Cauchy sequence in $\mathbb{P}(H^2(\Omega))$ which is a complete metric space. It follows that there is $f \in H^2(\Omega) \setminus \{0\}$ such that $\iota(z_k) \to \langle f \rangle$. Therefore

$$\frac{|f(z_k)|^2}{K_\Omega(z_k,z_k)} = \big|\langle f, \frac{K_\Omega(\cdot, z_k)}{\sqrt{K_\Omega(z_k,z_k)}}\rangle\big|^2 \to ||f||^2$$

as $k \to \infty$, which contradicts the assumption of the theorem. □

Zwonek [Z1] (see also [J]) showed that there exists a Bergman complete domain in $\mathbb{C}$ which does not satisfy (6.1). On the other hand, from the above proof it is clear that one can weaken (6.1) to

$$\limsup_{z \to \partial\Omega} \frac{|f(z)|^2}{K_\Omega(z,z)} < ||f||^2, \quad f \in H^2(\Omega) \setminus \{0\}. \tag{6.1'}$$

It is not known if there exists a Bergman complete domain not satisfying (6.1').

Similarly as in the one-dimensional case one defines the pluricomplex Green function of Ω with pole at $w \in \Omega$ as

$$G_\Omega(\cdot, w) := \sup \mathcal{F}_w,$$

where

$$\mathcal{F}_w := \{v \in PSH^-(\Omega) : \limsup_{\zeta \to w}(v(\zeta) - \log|\zeta - w|) < \infty\}.$$

Then $G_\Omega(\cdot, w) \in \mathcal{F}_w$ but G_Ω is not symmetric in general. We have the following estimate due to Herbort [H]:

Theorem 6.3. *For $f \in H^2(\Omega)$ and $w \in \Omega$, where Ω is pseudoconvex, we have*

$$\frac{|f(w)|^2}{K_\Omega(w,w)} \le c_n \int_{\{G_\Omega(\cdot,w) < -1\}} |f|^2 d\lambda.$$

Proof. We will use Theorem 5.3 with $\varphi := -\log(-g)$ and $\psi := 2ng$, where $g := G_{\Omega,w}$. Since g is a locally bounded plurisubharmonic function in $\Omega \setminus \{w\}$, it follows that $\bar\partial g \in L^2_{loc,(0,1)}(\Omega \setminus \{w\})$. Set

$$\alpha := \bar\partial(f \cdot \gamma \circ g) = f \cdot \gamma' \circ g\, \bar\partial g \in L^2_{loc,(0,1)}(\Omega),$$

where $\gamma \in C^\infty(\mathbb{R})$ is such that $\gamma(t) = 0$ for $t \geq -1$, $\gamma(t) = 1$ for $t \leq -3$ and $-1 \leq \gamma' \leq 0$. We have

$$i\bar{\alpha} \wedge \alpha = |f|^2(\gamma' \circ g)^2 i\partial g \wedge \overline{\partial} g \leq |f|^2(\gamma' \circ g)^2 g^2 i\partial\overline{\partial}\psi.$$

By Theorem 5.3 we can find $u \in L^2_{loc}(\Omega)$ with $\bar{\partial} u = \alpha$ and

$$\int_\Omega |u|^2 e^{-2ng} d\lambda \leq 16 \int_\Omega |f|^2(\gamma' \circ g)^2 g^2 e^{-2ng} d\lambda.$$

Therefore

$$||u||_{L^2(\Omega)} \leq 12e^{3n}||f||_{L^2(\{g<-1\})}.$$

The function $f \cdot \gamma \circ g - u$ is equal almost everywhere to a holomorphic $\widetilde{f}$. Moreover, since $e^{-\varphi}$ is not locally integrable near w it follows that $\widetilde{f}(w) = f(w)$. Therefore

$$\frac{|f(w)|}{\sqrt{K_\Omega(w,w)}} \leq ||\widetilde{f}|| \leq (1 + 12e^{3n})||f||_{L^2(\{g<-1\})}. \quad \square$$

From Theorems 6.2 and 6.3 we easily deduce the following (see [C1], [BP], [H]):

Corollary 6.4. *If pseudoconvex Ω satisfies*

$$\lim_{w \to \partial\Omega} \lambda(\{G_\Omega(\cdot, w) < -1\}) = 0 \tag{6.1}$$

then it is Bergman complete. $\square$

One can show that hyperconvex domains (that is domains admitting bounded plurisubharmonic exhaustion function) satisfy (6.1), and thus are Bergman complete (see [C1], [BP] and [H]).

For $f \equiv 1$ Theorem 6.3 gives

$$K_\Omega(w,w) \geq \frac{1}{c_n \lambda(\{G_\Omega(\cdot, w) < -1\})}$$

and thus in particular

$$\lim_{w \to \partial\Omega} K_\Omega(w,w) = \infty \tag{6.2}$$

for hyperconvex Ω (this is originally due to Ohsawa [O]).

The following result was proved in [C2]:

Theorem 6.5. *If $n = 1$ and Ω satisfies (6.2) then it is Bergman complete.*

Exercise 7. *Using the Hartogs triangle $\{(z,w) \in \mathbb{C}^2 : 0 < |z| < |w| < 1\}$ show that Theorem 6.5 does not hold for $n > 1$.*

For the proof of Theorem 6.5 we will need the following:

Lemma 6.6. *Assume that $f \in H^2(\Omega)$ and let $U \subset B(z_0, r)$ be such that $\Omega \cup U$ is a pseudoconvex domain contained in $B(z_0, R)$. Then there exists $F \in H^2(\Omega \cup U)$ such that*

$$||F - f||_{L^2(\Omega)} \leq \big(1 + \frac{4}{\log 2}\big)||f||_{L^2(\Omega \cap B(z_0, R\sqrt{r/R})}. \tag{6.3}$$

Proof. Assume for simplicity that $z_0 = 0$. We will use Theorem 5.3 with $\varphi = -\log(-\log(|z|/R))$, $\psi = 0$ and

$$\alpha = \bar\partial(f\,\gamma\circ\varphi) = f\,\gamma'\circ\varphi\,\bar\partial\varphi,$$

where

$$\gamma(t) = \begin{cases} 0, & t \le -\log(-\log(r/R)) \\ \frac{t+\log(-\log(r/R))}{\log 2}, & -\log(-\log(r/R)) < t < -\log(-\log(r/R)) + \log 2 \\ 1, & t \ge -\log(-\log(r/R)) + \log 2. \end{cases}$$

Then $\gamma\circ\varphi = 0$ in $B(0,r)$ and thus α is well defined in $\Omega\cup U$. We also have

$$i\bar\alpha\wedge\alpha = |f|^2(\gamma'\circ\varphi)^2 i\partial\varphi\wedge\bar\partial\varphi \le |f|^2(\gamma'\circ\varphi)^2 i\partial\bar\partial\varphi.$$

From Theorem 5.3 we obtain u with $\bar\partial u = \alpha$ and

$$\int_{\Omega\cup U}|u|^2 d\lambda \le 16\int_\Omega |f|^2(\gamma'\circ\varphi)^2 d\lambda.$$

For $F := f\,\gamma\circ\varphi - u$ the desired estimate now easily follows. □

The point in Lemma 6.6 is that $\Omega\cup U$ is pseudoconvex and that the r.h.s. converges to 0 as $r\to 0$. For $z_0\in\partial\Omega$ one can always find an appropriate neighborhood basis provided that $n = 1$.

Proof of Theorem 6.5. Fix $f\in H^2(\Omega)$, $z_0\in\partial\Omega$ and $\varepsilon > 0$. By Lemma 6.6 we can find $\widetilde f\in H^2(\Omega)$ which is bounded near z_0 and such that $||\widetilde f - f|| \le \varepsilon$. For $z\in\Omega$ we have

$$\frac{|f(z)|}{\sqrt{K_\Omega(z,z)}} \le ||\widetilde f - f|| + \frac{|\widetilde f(z)|}{\sqrt{K_\Omega(z,z)}}$$

and thus by (6.2)

$$\limsup_{z\to z_0}\frac{|f(z)|}{\sqrt{K_\Omega(z,z)}} \le \varepsilon.$$

It is now enough to use Theorem 6.2. □

Our next goal is to prove the following relation between the Bergman distance and the Green function from [Bł2]:

Theorem 6.7. *Assume that $w_1, w_2\in\Omega$, where Ω is pseudoconvex, are such that $\{G_\Omega(\cdot,w_1) < -1\}\cap\{G_\Omega(\cdot,w_2) < -1\} = \emptyset$. Then $\operatorname{dist}^B_\Omega(w_1,w_2)\ge b_n > 0$.*

Proof. Set $f := K_\Omega(\cdot,w_2)/\sqrt{K_\Omega(w_2,w_2)}$ (so that $||f|| = 1$), $\varphi := -\log(-G_\Omega(\cdot,w_1))$ and $\psi := 2n(G_\Omega(\cdot,w_1) + G_\Omega(\cdot,w_2))$. Let $\gamma\in C^\infty(\mathbb{R})$ be such that $\gamma = 0$ for $t\ge 0$, $\gamma = 1$ for $t\le -2$ and $-1\le\gamma'\le 0$. Then by Theorem 5.3 we can find u with $\bar\partial u = \bar\partial(f\,\gamma\circ\varphi)$ and

$$\int_\Omega |u|^2 e^{-\psi}d\lambda \le 16\int_\Omega |f|^2(\gamma'\circ\varphi)^2 e^{-\psi}d\lambda \le 16e^{20n},$$

where the last inequality follows from the assumption, since on $\{\gamma'\circ\varphi\ne 0\}\subset\{-2\le\varphi\le 0\}$ we have $\psi\ge -2n(e^2+1)$. Therefore $u(w_1) = u(w_2) = 0$ and

$F := f\,\gamma\circ\varphi - u$ is holomorphic with $F(w_1)=f(w_1)$, $F(w_2)=0$. We also have $||F||\leq 1+4e^{10n}$.

Note that $\langle F,f\rangle = F(w_2)/\sqrt{K_\Omega(w_2,w_2)}=0$. We can therefore find an orthonormal basis $\varphi_0,\varphi_1,\dots$ such that $\varphi_0=f$ and $\varphi_1=F/||F||$. It follows that

$$K_\Omega(z,z)\geq |f(z)|^2+\frac{|F(z)|^2}{||F||^2}.$$

Now by Theorem 1.3

$$\operatorname{dist}{}^B_\Omega(w_1,w_2)\geq \arccos\frac{|F(w_1)|}{\sqrt{K_\Omega(w_1,w_1)}}\geq \arccos\frac{||F||}{\sqrt{1+||F||^2}}. \qquad \square$$

7. Ohsawa-Takegoshi extension theorem

The Ohsawa-Takegoshi extension theorem [OT] turned out to be one of the main tools in complex analysis:

Theorem 7.1. *Let Ω be a bounded pseudoconvex domain and H a complex hyperplane in $\mathbb{C}^n$. Set $\Omega' := \Omega\cap H$ and assume that φ is a plurisubharmonic function in Ω. Then for every holomorphic f in Ω' there exists a holomorphic F in Ω such that $F|_{\Omega'}=f$ and*

$$\int_\Omega |F|^2e^{-\varphi}d\lambda\leq C\int_{\Omega'}|f|^2e^{-\varphi'}d\lambda',$$

where $\varphi'=\varphi|_{\Omega'}$, $d\lambda'$ is the Lebesgue measure on Ω' and C depends only on n and the diameter of Ω.

Sketch of proof. We follow Berndtsson [B4] (see also [B2]). Without loss of generality we may assume that $H=\{z_1=0\}$ and $\Omega\subset\{|z_1|<1\}$. By approximating Ω from inside and φ from above we may assume that Ω is a strongly pseudoconvex domain with smooth boundary, φ is smooth up to the boundary, and f is defined in a neighborhood of $\overline{\Omega'}$ in H. Then it follows that f extends to some holomorphic function in Ω (we may use Hörmander's estimate with $\alpha=\overline\partial(\chi(z_1)f(z'))$, $\chi=1$ near 0 but with support sufficiently close to 0, $\varphi=2\log|z_1|$ will ensure that $u=0$ on H).

Let $F\in H^2(\Omega,e^{-\varphi}):=\mathcal{O}(\Omega)\cap L^2(\Omega,e^{-\varphi})$ be the function satisfying $F=f$ on H with minimal norm in $L^2(\Omega,e^{-\varphi})$. Then F is perpendicular to functions from $H^2(\Omega,e^{-\varphi})$ vanishing on H, and it is thus perpendicular to the space $z_1H^2(\Omega,e^{-\varphi})$. This means that $\overline{z}_1F$ is perpendicular to $H^2(\Omega,e^{-\varphi})$. Since $(H^2(\Omega,e^{-\varphi}))^\perp=(\ker\overline\partial)^\perp$ is equal to the range of $\overline\partial^*$, we have $\overline\partial^*\alpha=\overline{z}_1F$ for some $\alpha\in L^2_{(0,1)}(\Omega,e^{-\varphi})$. Choose such α with the minimal norm. Then α is perpendicular to $\ker\overline\partial^*$, and thus $\overline\partial\alpha=0$. We have

$$\int_\Omega|F|^2e^{-\varphi}d\lambda=\left\langle F/z_1,\overline\partial^*\alpha\right\rangle_{e^{-\varphi}}=\left\langle\overline\partial(F/z_1),\alpha\right\rangle_{e^{-\varphi}}=\left\langle F\overline\partial(1/z_1),\alpha\right\rangle_{e^{-\varphi}}$$

$$=\pi\int_{\Omega'}f\overline{\alpha}_1e^{-\varphi}d\lambda'\leq\pi\left(\int_{\Omega'}|f|^2e^{-\varphi}d\lambda'\right)^{1/2}\left(\int_{\Omega'}|\alpha_1|^2e^{-\varphi}d\lambda'\right)^{1/2}.$$

It is thus enough to estimate $\int_{\Omega'}|\alpha_1|^2 e^{-\varphi}d\lambda'$. We will use the Bochner-Kodaira technique (terminology of Siu [S2], see [B2] for details). One may compute that

$$\sum(\alpha_j\overline{\alpha}_k e^{-\varphi})_{j\bar{k}} = \left(-2\mathrm{Re}\,(\overline{\partial}\,\overline{\partial}^*\alpha\cdot\alpha)+|\overline{\partial}^*\alpha|^2+\sum|\alpha_{j,\bar{k}}|^2-|\overline{\partial}\alpha|^2+\sum\varphi_{j\bar{k}}\alpha_j\overline{\alpha}_k\right)e^{-\varphi}.$$

Integrating by parts and computing further one can show that for any (sufficiently regular) function w

$$\int_\Omega\sum w_{j\bar{k}}\alpha_j\overline{\alpha}_k e^{-\varphi}\,d\lambda-\int_{\partial\Omega}\sum\rho_{j\bar{k}}\alpha_j\overline{\alpha}_k e^{-\varphi}w\,\frac{d\sigma}{|\partial\rho|} = \int_\Omega\left(-2\mathrm{Re}\,(\overline{\partial}\,\overline{\partial}^*\alpha\cdot\alpha)+|\overline{\partial}^*\alpha|^2+\sum|\alpha_{j,\bar{k}}|^2-|\overline{\partial}\alpha|^2+\sum\varphi_{j\bar{k}}\alpha_j\overline{\alpha}_k\right)e^{-\varphi}w\,d\lambda,$$

where ρ is a defining function or Ω. In our case we have $\overline{\partial}\alpha=0$, $\overline{\partial}^*\alpha=\bar{z}_1F$, and if we take negative w depending only on z_1, then

$$\int_\Omega w_{1\bar{1}}|\alpha_1|^2e^{-\varphi}d\lambda\le -2\mathrm{Re}\int_\Omega F\overline{\alpha}_1e^{-\varphi}w\,d\lambda \tag{7.1}$$

(since we may choose plurisubharmonic ρ). Set

$$w:=2\log|z_1|+|z_1|^{2\delta}-1,$$

where $0<\delta<1$. Then $w_{1\bar{1}}=\pi\delta_0'+\delta^2|z_1|^{2\delta-2}$ and for $t>0$

$$\pi\int_{\Omega'}|\alpha_1|^2e^{-\varphi}d\lambda'+\delta^2\int_\Omega|\alpha_1|^2|z_1|^{2\delta-2}e^{-\varphi}d\lambda\le t\int_\Omega|F|^2e^{-\varphi}d\lambda+\frac{1}{t}\int_\Omega|\alpha_1|^2w^2e^{-\varphi}d\lambda.$$

Choosing t with $w^2\le\delta^2t|z_1|^{2\delta-2}$ in $\{|z_1|\le 1\}$ and combining this with (7.1) we arrive at

$$\int_\Omega|F|^2e^{-\varphi}d\lambda\le t\pi\int_{\Omega'}|f|^2e^{-\varphi}d\lambda'. \qquad \square$$

It is clear that iterating Theorem 7.1 we may take H to be an arbitrary complex affine subspace in $\mathbb{C}^n$, even a point.

The original motivation behind [OT] was the following estimate:

Theorem 7.2. *Assume that Ω is a bounded pseudoconvex domain with C^2 boundary. Then*

$$K_\Omega\ge\frac{1}{C\mathrm{dist}\,(z,\partial\Omega)^2}, \tag{7.2}$$

where C is a constant depending on Ω.

Proof. It follows almost immediately from Theorem 1.1. For let $r>0$ be such that for any $w\in\partial\Omega$ there exists $w^*\in\mathbb{C}^n\setminus\overline{\Omega}$ such that $\overline{\Omega}\cap\overline{B}(w^*,r)=\{w\}$. If $z\in\Omega$,

$w \in \partial\Omega$ is such that $\operatorname{dist}(z, \partial\Omega) = |z - w|$, and w^* is as above then z, w, and w^* lie on the same line (normal to $\partial\Omega$ at w). For the corresponding complex line H and $\Omega' = \Omega \cap H$ we obtain

$$\begin{aligned} K_\Omega(z) \geq \frac{1}{C_\Omega} K_{\Omega'}(z) &\geq \frac{1}{C_\Omega} K_{\mathbb{C}\setminus\overline{\Delta}(0,r)}(r + |z - w|) \\ &= \frac{r^2}{\pi C_\Omega \operatorname{dist}(z, \partial\Omega)^2 (2r + \operatorname{dist}(z, \partial\Omega))^2}. \qquad \square \end{aligned}$$

The exponent 2 in (7.2) is optimal (for example it cannot be improved for a domain whose boundary near the origin is given by $|z_1 - 1| = 0$). Previously a weaker form of (7.2) was proved by Pflug [P] using Hörmander's estimate (with arbitrary exponent lower than 2).

Demailly approximation. In the proof of Theorem 7.2 we used Theorem 7.1 only with $\varphi \equiv 0$. The fact that the weight may be an arbitrary plurisubharmonic function was used by Demailly [D] to introduce a new type of regularization of plurisubharmonic functions: by smooth plurisubharmonic functions with analytic singularities (that is functions that locally can be written in the form $\log(|f_1|^2 + \dots + |f_k|^2) + u$, where $f_1, \dots, f_k$ are holomorphic and u is C^∞ smooth) which have very similar singularities to the initial function. The Demailly approximation turned out to be an important tool in complex geometry, see e.g. [D], [DPS] or [DP]. Demailly [D] presented also a simple proof of the Siu theorem on analyticity of level sets of Lelong numbers of plurisubharmonic functions ([S1], see also [Hö]). As we will see below, the Demailly approximation shows that the Siu theorem follows rather easily from Theorem 7.1 applied when H is just a point.

Recall that the Lelong number of $\varphi \in PSH(\Omega)$ at $z_0 \in \Omega$ is defined by

$$\nu_\varphi(z_0) = \varliminf_{z \to z_0} \frac{\varphi(z)}{\log|z - z_0|} = \lim_{r \to 0^+} \frac{\varphi^r(z_0)}{\log r},$$

where for $r > 0$ we use the notation

$$\varphi^r(z) := \max_{\overline{B}(z,r)} \varphi, \quad z \in \Omega_r := \{\delta_\Omega > r\}.$$

One can show that φ^r is a plurisubharmonic continuous function in Ω_r, decreasing to φ as r decreases to 0. Now we are in position to prove a result from [D]:

Theorem 7.3. *For a plurisubharmonic function φ in a bounded pseudoconvex domain Ω in $\mathbb{C}^n$ and $m = 1, 2 \dots$ set*

$$\varphi_m := \frac{1}{2m} \log K_{\Omega, e^{-2m\varphi}} = \frac{1}{2m} \log \sup\{|f|^2 : f \in \mathcal{O}(\Omega), \int_\Omega |f|^2 e^{-2m\varphi} \leq 1\}.$$

Then there exist $C_1, C_2 > 0$ depending only on Ω such that

$$\varphi - \frac{C_1}{m} \leq \varphi_m \leq \varphi^r + \frac{1}{m} \log \frac{C_2}{r^n} \quad \text{in } \Omega_r. \tag{7.3}$$

In particular, $\varphi_m \to \varphi$ pointwise and in $L^1_{loc}(\Omega)$. Moreover,

$$\nu_\varphi - \frac{n}{m} \leq \nu_{\varphi_m} \leq \nu_\varphi \quad \text{in } \Omega. \tag{7.4}$$

Proof. First note that (7.4) is an easy consequence of (7.3): by the first inequality in (7.3) we get $\nu_{\varphi_m} \le \nu_{\varphi - C_1/m} = \nu_\varphi$, and by the second one

$$\varphi_m^r \le \varphi^{2r} + \frac{1}{m} \log \frac{C_2}{r^n},$$

thus $\nu_\varphi - n/m \le \nu_{\varphi_m}$.

By Theorem 7.1 for every $z \in \Omega$ there exists $f \in \mathcal{O}(\Omega)$ with $f(z) \neq 0$ and

$$\int_\Omega |f|^2 e^{-2m\varphi} d\lambda \le C_\Omega |f(z)|^2 e^{-2m\varphi(z)}.$$

We may choose f so that the right-hand side is equal to 1. Then

$$\varphi_m(z) \ge \frac{1}{m} \log |f(z)| = \varphi(z) - \frac{1}{2m} \log C_\Omega$$

and we get the first inequality in (7.3).

To get the second one we observe that for any holomorphic f the function $|f|^2$ is in particular subharmonic and thus for $z \in \Omega_r$

$$|f(z)|^2 \le \frac{1}{\lambda(B(z,r))} \int_{B(z,r)} |f|^2 d\lambda \le \frac{n!}{\pi^n r^{2n}} e^{2m\varphi^r(z)} \int_{B(z,r)} |f|^2 e^{-2m\varphi} d\lambda.$$

Taking the logarithm and multiplying by $1/(2m)$ we will easily get the second inequality in (7.3). □

By (7.4) for any real c we have

$$\{\nu_\varphi \ge c\} = \bigcap_m \{\nu_{\varphi_m} \ge c - \frac{n}{m}\}. \tag{7.5}$$

If $\{\sigma_j\}$ is an orthonormal basis in $H^2(\Omega, e^{-2m\varphi})$ then

$$K_{\Omega, e^{-2m\varphi}} = \sum_j |\sigma_j|^2 \tag{7.6}$$

and one can show that

$$\{\nu_{\varphi_m} \ge c - \frac{n}{m}\} = \bigcap_{|\alpha| < mc-n} \bigcap_j \{\partial^\alpha \sigma_j = 0\}.$$

Therefore (7.5) is an analytic subset of Ω, which gives the Siu theorem [S1]:

Theorem 7.4. *For any plurisubharmonic function φ and a real number c the set $\{\nu_\varphi \ge c\}$ is analytic.* □

The following sub-additivity property was proved in [DPS]. It also relies on the extension theorem, here however we will be using it for the diagonal of $\Omega \times \Omega$.

Theorem 7.5. *With the notation of Theorem 4.1 there exists $C_3 > 0$, depending only on Ω, such that*

$$(m_1 + m_2)\varphi_{m_1+m_2} \le C_3 + m_1 \varphi_{m_1} + m_2 \varphi_{m_2}.$$

Proof. Take $f \in H^2(\Omega, e^{-2(m_1+m_2)\varphi})$ with norm ≤ 1. If we embed Ω in $\Omega \times \Omega$ as the diagonal then by Theorem 7.1 there exists F holomorphic in $\Omega \times \Omega$ such that $F(z,z) = f(z)$, $z \in \Omega$, and

$$\int_{\Omega\times\Omega} |F(z,w)|^2 e^{-2m_1\varphi(z)-2m_2\varphi(w)} d\lambda(z)d\lambda(w) \le C(= C_{\Omega\times\Omega}). \tag{7.7}$$

If $\{\sigma_j\}$ is an orthonormal basis in $H^2(\Omega, e^{-2m_1\varphi_{m_1}})$ and $\{\sigma'_k\}$ an orthonormal basis in $H^2(\Omega, e^{-2m_1\varphi_{m_2}})$ then one can easily check that $\{\sigma_j(z)\sigma'_k(w)\}$ is an orthonormal basis in $H^2(\Omega \times \Omega, e^{-2m_1\varphi_{m_1}(z)-2m_2\varphi_{m_2}(w)})$. We may write

$$F(z,w) = \sum_{j,k} c_{jk}\sigma_j(z)\sigma'_k(w)$$

and by (7.7)

$$\sum_{j,k} |c_{jk}|^2 \le C.$$

Therefore by the Schwarz inequality

$$|f(z)|^2 = |F(z,z)|^2 \le C \sum_j |\sigma_j(z)|^2 \sum_k |\sigma'_k(z)|^2 = Ce^{2m_1\varphi_{m_1}(z)}e^{2m_2\varphi_{m_2}(z)}$$

(using (7.6)). Since f was arbitrary, the theorem follows with $C_3 = (\log C)/2$. □

Corollary 7.6. *The sequence $\varphi_{2^k} + C_3/2^{k+1}$ is decreasing.* □

It is an open problem if the whole sequence φ_m from Theorem 7.3 (perhaps modified by constants as in Corollary 7.6) is decreasing.

References

[B1] B. Berndtsson, *Weighted estimates for $\overline{\partial}$ in domains in $\mathbb{C}$*, Duke Math. J. **66** (1992), 239-255.

[B2] B. Berndtsson, *The extension theorem of Ohsawa-Takegoshi and the theorem of Donnelly-Fefferman*, Ann. Inst. Fourier **46** (1996), 1083-1094.

[B3] B. Berndtsson, *Weighted estimates for the $\overline{\partial}$-equation*, Complex Analysis and Geometry, Columbus, Ohio, 1999, Ohio State Univ. Math. Res. Inst. Publ., vol. 9, Walter de Gruyter, 2001, pp. 43-57.

[B4] B. Berndtsson, *Integral formulas and the Ohsawa-Takegoshi extension theorem*, Science in China Ser. A Math. **48 suppl.** (2005), 61-73.

[Bł1] Z. Błocki, *A note on the Hörmander, Donnelly-Fefferman, and Berndtsson L^2-estimates for the $\overline{\partial}$-operator*, Ann. Pol. Math. **84** (2004), 87-91.

[Bł2] Z. Błocki, *The Bergman metric and the pluricomplex Green function*, Trans. Amer. Math. Soc. **357** (2005), 2613-2625.

[Bł3] Z. Błocki, *Some estimates for the Bergman kernel and metric in terms of logarithmic capacity*, Nagoya Math. J. **185** (2007), 143-150.

[BP] Z. Błocki, P. Pflug, *Hyperconvexity and Bergman completeness*, Nagoya Math. J. **151** (1998), 221-225.

[C1] B.-Y. Chen, *Completeness of the Bergman metric on non-smooth pseudoconvex domains*, Ann. Pol. Math. **71** (1999), 241-251.

[C2] B.-Y. Chen, *A remark on the Bergman completeness*, Complex Variables Theory Appl. **42** (2000), 11-15.

[D] J.-P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Alg. Geom. **1** (1992), 361-409.

[DP] J.-P. Demailly, M. Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. **159** (2004), 1247-1274.

[DPS] J.-P. Demailly, T, Peternell, M. Schneider, *Pseudo-effective line bundles on compact Kähler manifolds*, Internat. J. Math. **12** (2001), 689-741.

[Di1] Ż. Dinew, *An example for the holomorphic sectional curvature of the Bergman metric*, Ann. Polon. Math. **98** (2010), 147167.

[Di2] Ż. Dinew, *On the Bergman representative coordinates*, arXiv:1001.4206.

[DF] H. Donnelly, C. Fefferman, *L^2-cohomology and index theorem for the Bergman metric*, Ann. of Math. **118** (1983), 593-618.

[H] L. Hörmander, *An introduction to complex analysis in several variables*, D. van Nostrand, Princeton, 1966.

[JP] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis*, Walter de Gruyter, 1993.

[J] P. Jucha, *Bergman completeness of Zalcman type domains*, Studia Math. **163** (2004), 71-83.

[K] S. Kobayashi, *Geometry of bounded domains*, Trans. Amer. Math. Soc. **92** (1959), 267-290.

[O] T. Ohsawa, *Addendum to "On the Bergman kernel of hyperconvex domains"*, Nagoya Math. J. **137** (1995), 145-148.

[OT] T. Ohsawa, K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Z. **195** (1987), 197-204.

[P] P. Pflug, *Quadratintergrable holomorphe Funktionen und die Serre-Vermutung*, Math. Ann. **216** (1975), 285-288.

[R] P. Rosenthal, *On the zeros of the Bergman function in doubly-connected domains*, Proc. Amer. Math. Soc. **21** (1969), 33-35.

[Ru] W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Springer-Verlag, 1980.

[S1] Y.-T. Siu, *Analyticity of sets associated to Lelong numbers and extension of closed positive currents*, Invent. Math. **27** (1974), 53-156.

[S2] Y.-T. Siu, *Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems*, J. Diff. Geom. **17** (1982), 55-138.

[Sk] M. Skwarczyński, *The invariant distance in the theory of pseudoconformal transformations and the Lu Qi-keng conjecture*, Proc. Amer. Math. Soc. **22** (1969), 305-310.

[Su] N. Suita, *Capacities and kernels on Riemann surfaces*, Arch. Ration. Mech. Anal. **46** (1972), 212-217.

[Z1] W. Zwonek, *An example concerning Bergman completeness*, Nagoya Math. J. **164** (2001), 89-101.

[Z2] W. Zwonek, *Asymptotic behavior of the sectional curvature of the Bergman metric for annuli*, Ann. Polon. Math. **98** (2010), 291-299.