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Cauchy–Riemann meet Monge–Ampère

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Abstract This is a relatively self-contained introduction to recent developments in the $\bar\partial$-equation, Ohsawa–Takegoshi extension theorem and applications of pluripotential theory to the Bergman kernel and metric. The main tools are the Hörmander L^2-estimate for $\bar\partial$ and Bedford–Taylor's theory of the complex Monge–Ampère operator.

Contents

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1 Introduction

Holomorphic functions of several variables are precisely solutions to the homogeneous Cauchy–Riemann equation (often called the $\bar\partial$-equation)

$$\bar\partial u = 0. \tag{1.1}$$

Here both sides are forms of type (0,1) which is a rather special case of the $\bar\partial$-equation because all solutions, even in the distributional sense, have to be smooth, in contrast to the general case of the equation for (p, q)-forms. The inhomogeneous $\bar\partial$-equation

$$\bar\partial u = \alpha, \tag{1.2}$$

where α is a $\bar\partial$-closed (0,1)-form, plays a fundamental role in the PDE approach to the theory of several complex variables: it is the main tool for constructing holomorphic functions. The basic idea is very simple: if $\alpha = \bar\partial\chi$ for some function χ and u is a solution to (1.2) then $u - \chi$ is holomorphic.

The famous L^2-estimate of Hörmander [65] asserts that for every smooth strongly plurisubharmonic function φ defined in a pseudoconvex open subset of $\mathbb{C}^n$ there exists a solution to (1.2) satisfying

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda. \tag{1.3}$$

The original Hörmander estimate was slightly weaker: the right-hand side depended on the minimal eigenvalue of the complex Hessian of φ but his method also gives this slightly stronger version (it was first formulated by Demailly [42]). This turns out to be an extremely powerful result as will be again demonstrated here. What makes this approach so useful is a big abundance of plurisubharmonic functions: they are usually much easier to construct than holomorphic functions and this is in fact where pluripotential theory comes into play.

As we will see, Hörmander's estimate (1.3) can also be formulated for non-smooth φ. In many cases an almost optimal choice for the weight φ in this and related estimates is

$$\varphi = 2nG_\Omega(\cdot, w),$$

where $G_\Omega(\cdot, w)$ is the pluricomplex Green function with pole at w. This is because it is essentially the largest negative plurisubharmonic function such that $e^{-\varphi}$ is not locally integrable near w. This is the main reason why pluripotential theory turned out to be so useful in the theory of the $\bar\partial$-equation.

The complex Monge–Ampère operator $(dd^c)^n$ plays the central role in pluripotential theory, it has been developed in this context by Bedford and Taylor [1,2]. For example, Demailly [43] characterized the pluricomplex Green function as a solution to the Monge–Ampère equation with point-mass on the right-hand side. This, together with standard techniques for the complex Monge–Ampère operator, e.g. integrating by parts, is often used to prove various properties of the Green function.

One of the most important results in several complex variables has been the Ohsawa–Takegoshi extension theorem [98]. It states that holomorphic functions can be extended from lower dimensional sections with L^2-estimates. It has found many applications in complex and algebraic geometry but it can be also very useful to study singularities of plurisubharmonic functions. For example, it turns out that two main results in this area, the theorem of Siu [107] on analyticity of level sets of Lelong numbers and the openness conjecture of Demailly and Kollár [46] follow relatively easily from the Ohsawa–Takegoshi theorem. The simple proof of the Siu theorem was found by Demailly [45] who devised a special approximation of an arbitrary plurisubharmonic function by smooth ones with possibly analytic singularities. The openness conjecture was first proved by Berndtsson [10] who subsequently simplified the proof in [11] using an approach of Guan and Zhou [58].

This survey is largely self-contained. It is organized as follows. In Sect. 2 we give proofs of all necessary L^2-estimates for $\bar\partial$ assuming Hörmander's estimate. It is mostly thanks to the method of Berndtsson from [5] that they are in fact formal consequences of (1.3) and one does not have to repeat Hörmander's arguments. Section 3 contains the simplest known proof of the Ohsawa–Takegoshi extension theorem. It is due to Chen [39] (see also [26]) and was the first one which used Hörmander's estimate directly. In Sect. 4 we present some applications of the Ohsawa–Takegoshi theorem to singularities of plurisubharmonic functions with simple proofs of the aforementioned openness conjecture and Siu's theorem, as well as basic results on Demailly's approximation. Section 5 is a brief introduction to the complex Monge–Ampère operator and the pluricomplex Green function. Section 6 discusses some applications of pluripotential theory and the $\bar\partial$-equation to the Bergman metric. In Sect. 7 we present the recently settled (see [27]) one-dimensional Suita conjecture from [110] and closely related versions of the Ohsawa–Takegoshi theorem with optimal constant. Another approach to the Suita conjecture from [28] and its multidimensional version from [31] are also discussed. The case of convex domains is analysed in greater detail in Sect. 8, following mostly [31], and it is used in Sect. 9 to present recent Nazarov's proof [93] of the Bourgain–Milman inequality [34] from convex analysis. Finally, in Sect. 10 we discuss a link between the lower bound for the Bergman kernel in terms of the pluricomplex Green function and possible symmetrization results for the complex Monge–Ampère equation and complex isoperimetric inequalities. The conclusion of this section is rather speculative in nature. Many open problems are mentioned throughout the whole paper.

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2 L^2-estimates for the $\bar\partial$-equation

We first recall the definition of the operator $\bar\partial$ for functions and (0, 1)-forms (this is all we will need). For a function u defined on an open subset of $\mathbb{C}^n$ set

$$\bar\partial u := \sum_j \frac{\partial u}{\partial \bar z_j} d\bar z_j$$

and for a (0,1)-form $\alpha = \sum_k \alpha_k d\bar z_k$

$$\bar\partial\alpha =: \sum_k \bar\partial\alpha_k \wedge d\bar z_k = \sum_{j<k}\left(\frac{\partial\alpha_k}{\partial\bar z_j} - \frac{\partial\alpha_j}{\partial\bar z_k}\right) d\bar z_j \wedge d\bar z_k.$$

We will consider the inhomogeneous $\bar\partial$-equation

$$\bar\partial u = \alpha \tag{2.1}$$

which is really a system of n equations with one unknown:

$$\frac{\partial u}{\partial \bar z_j} = \alpha_j, \quad j = 1, \dots, n.$$

Since $\bar\partial^2 = 0$, a necessary condition for (2.1) to have a solution is $\bar\partial\alpha = 0$, that is

$$\frac{\partial\alpha_j}{\partial\bar z_k} = \frac{\partial\alpha_k}{\partial\bar z_j}.$$

Recall that a function φ, defined on an open subset of $\mathbb{C}^n$ with values in $[-\infty, \infty)$, is called *plurisubharmonic* if locally it is upper semi-continuous, $\not\equiv -\infty$ and is either subharmonic or $\equiv -\infty$ on every complex line. Equivalently, the complex Hessian $(\partial^2\varphi/\partial z_j \bar\partial z_k)$ is positive semi-definite (in the distributional sense). It is in fact an open problem whether upper semi-continuity in the first definition follows from the other properties. We use the notation $PSH(\Omega)$ for the set of all plurisubharmonic functions in Ω and $PSH^-(\Omega)$ for the negative ones. The C^2 functions with positive definite complex Hessian at every point are called *strongly plurisubharmonic*. An open subset $\Omega \subset \mathbb{C}^n$ is called *pseudoconvex* if it admits a plurisubharmonic exhaustion function, that is there exists $\varphi \in PSH(\Omega)$ such that $\{\varphi \le t\} \Subset \Omega$ for all $t \in \mathbb{R}$. A C^2 smooth Ω is called *strongly pseudoconvex* if it admits a strongly plurisubharmonic defining function, that is strongly plurisubharmonic ρ defined in a neighbourhood of $\bar\Omega$ such that $\nabla\rho \neq 0$ on $\partial\Omega$ and $\Omega = \{\rho < 0\}$.

The notions of plurisubharmonic and strongly plurisubharmonic functions as well as pseudoconvex and strongly pseudoconvex sets in $\mathbb{C}^n$ correspond closely to that of convex and strongly convex functions and domains in $\mathbb{R}^m$. In this context, also the $\bar\partial$-operator can be treated as a counterpart of the d-operator, see [7].

We want to formulate Hörmander's estimate (1.3) also for non-smooth φ (see [23]). Notice that if φ is C^2 and strongly plurisubharmonic then

$$H := |\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k} \varphi^{j\bar k}\bar\alpha_j\alpha_k,$$

where $(\varphi^{j\bar{k}}) = (\partial^2\varphi/\partial z_j\bar{\partial}z_k)^{-1}$, is the smallest function satisfying

$$(\bar{\alpha}_j\alpha_k) \leq H(\partial^2\varphi/\partial z_j\bar{\partial}z_k).$$

This can be written as

$$i\bar{\alpha} \wedge \alpha \leq Hi\partial\bar{\partial}\varphi. \tag{2.2}$$

Note that if the coefficients of α are in L^2_{loc} and H is in L^∞_{loc} then both sides of (2.2) are well defined currents of order 0 (that is forms with complex measures as coefficients).

We can now state Hörmander's estimate as follows:

Theorem 2.1 *Assume that Ω is a pseudoconvex open subset of $\mathbb{C}^n$ and $\varphi \in PSH(\Omega)$. Let $\alpha \in L^2_{loc,(0,1)}(\Omega)$ be $\bar{\partial}$-closed and take non-negative $H \in L^\infty_{loc}(\Omega)$ satisfying (2.2). Then there exists $u \in L^2_{loc}(\Omega)$ solving (2.1) and such that*

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq \int_\Omega He^{-\varphi} d\lambda.$$

This estimate is easier to prove for $n = 1$, see [67]. As remarked by Berndtsson [9], it is therefore quite surprising that it had not been proved earlier in this case. Especially that it can lead to new nontrivial results in one dimensional complex analysis, see e.g. [27]. But of course it is especially powerful in higher dimensions. For example the solution of the Levi problem can be deduced quite easily from Theorem 2.1, see [67, Corollary 4.2.8].

Sometimes there is however an inconvenience with applying Hörmander's estimate directly: φ appears both as a weight as well as a Kähler potential on the right-hand side. The following estimate due to Donnelly and Fefferman [52] (formulated originally for $\varphi \equiv 0$) addressed this problem:

Theorem 2.2 *Let Ω, φ and α be as in Theorem 2.1. Assume in addition that $\psi \in PSH(\Omega)$ is such that*

$$i\partial\psi \wedge \bar{\partial}\psi \leq i\partial\bar{\partial}\psi. \tag{2.3}$$

Then there exists $u \in L^2_{loc}(\Omega)$ solving (2.1) and such that

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq C \int_\Omega |\alpha|^2_{i\partial\bar{\partial}\psi} e^{-\varphi} d\lambda$$

for some absolute constant C.

Theorem 2.2 is stated here somewhat imprecisely although it is rather clear what the right statement should be: if ψ is not smooth and strongly plurisubharmonic then $|\alpha|^2_{i\partial\bar{\partial}\psi}$ should be replaced by any non-negative locally bounded H such that $i\bar{\alpha} \wedge \alpha \leq Hi\partial\bar{\partial}\psi$. Plurisubharmonic functions satisfying (2.3) are precisely of the form

$$\psi = -\log(-v)$$

for some $v \in PSH^-(\Omega)$. It was shown by Berndtsson [5] that Theorem 2.2 is a formal consequence of Hörmander's estimate:

Proof of Theorem 2.2 By standard approximation we may assume that ψ is smooth, strongly plurisubharmonic and that Ω, φ, ψ are bounded. Let u be the solution to (2.1) which is minimal in the $L^2(\Omega, e^{-\varphi-\psi/2})$-norm. This means that u is perpendicular to $\ker\bar\partial$ in $L^2(\Omega, e^{-\varphi-\psi/2})$, that is

$$\int_\Omega u\bar f e^{-\varphi-\psi/2} d\lambda = 0, \quad f \in \ker\bar\partial,$$

and therefore

$$v := e^{\psi/2} u$$

is the minimal solution to

$$\bar\partial v = \beta,$$

where $\beta = e^{\psi/2}\big(\alpha + u\,\bar\partial\psi/2\big)$, in the $L^2(\Omega, e^{-\varphi-\psi})$-norm. (Note that by our regularity assumptions the spaces $L^2(\Omega, e^{-\varphi-\psi/2})$ and $L^2(\Omega, e^{-\varphi-\psi})$ are the same as sets and so is $\ker\bar\partial$ in both cases.) Theorem 2.1 implies that

$$\int_\Omega |v|^2 e^{-\varphi-\psi} d\lambda \le \int_\Omega |\beta|^2_{i\partial\bar\partial(\varphi+\psi)} e^{-\varphi-\psi} d\lambda \le \int_\Omega |\beta|^2_{i\partial\bar\partial\psi} e^{-\varphi-\psi} d\lambda,$$

that is

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega |\alpha + u\,\bar\partial\psi/2|^2_{i\partial\bar\partial\psi} e^{-\varphi} d\lambda.$$

By (2.3) for any $t > 0$

$$|\alpha + u\,\bar\partial\psi/2|^2_{i\partial\bar\partial\psi} \le \left(1 + \frac{t}{2}\right)|\alpha|^2_{i\partial\bar\partial\psi} + \left(\frac{1}{4} + \frac{1}{2t}\right)|u|^2$$

and we obtain the required estimate if we take any $t > 2/3$, with the optimal choice $t = 2$, we then get $C = 4$. □

The idea of *twisting* the $\bar\partial$-equation seen in the proof of Theorem 2.2 had been used before but Berndtsson [5] seems to have been the first to realize that it can be applied directly to Hörmander's estimate, without repeating the technical parts of its proof like the so called Bochner–Kodaira identity, integration by parts etc.

The constant $C = 4$ we got here was originally obtained in [21] and it was shown to be optimal in [29]. Take $\Omega = \Delta$, the unit disc in $\mathbb{C}$, $\varphi \equiv 0$ and $\psi(z) = -\log(-\log|z|)$. For smooth, compactly supported η on $(0, \infty)$ one can show that

$$u(z) = \frac{\eta(-\log|z|)}{z}$$

is the minimal solution to (2.1) in $L^2(\Delta)$, where

$$\alpha = -\frac{\eta'(-\log|z|)}{2|z|^2}d\bar{z}.$$

Then by Theorem 2.2

$$\int_0^\infty \eta^2 dt \le 4\int_0^\infty (\eta')^2 t^2 dt, \quad \eta \in W_0^{1,2}((0,\infty)),$$

and one can show the constant 4 cannot be improved here.

The Donnelly–Fefferman estimate was generalized by Berndtsson [4]: he showed that with the assumptions of Theorem 2.2 and with $0 \le \delta < 1$ one can obtain solution u satisfying

$$\int_\Omega |u|^2 e^{\delta\psi-\varphi} d\lambda \le \frac{4}{(1-\delta)^2}\int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\delta\psi-\varphi} d\lambda. \tag{2.4}$$

This particular constant was obtained in [21] and, similarly as above, it was shown in [29] to be optimal for every δ.

Berndtsson's estimate is closely related to the Ohsawa–Takegoshi extension theorem, see [4], but the latter cannot be deduced directly from it. If (2.4) were true for $\delta = 1$ (with some finite constant) then it would be sufficient. Building on an idea of Chen [40] in his remarkable proof of the extension theorem, this was overcome in [26]. The following is a counterpart of Berndtsson's estimate (2.4) for $\delta = 1$:

Theorem 2.3 *Let Ω, φ, ψ and α be as in Theorem 2.2. Assume in addition that $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le a < 1$ on* supp α *(note that (2.2) means that $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le 1$ in Ω). Then we can find a solution $u \in L^2_{loc}(\Omega)$ to (2.1) satisfying*

$$\int_\Omega \left(1 - |\bar\partial\psi|^2_{i\partial\bar\partial\psi}\right)|u|^2 e^{\psi-\varphi} d\lambda \le \frac{1+\sqrt{a}}{1-\sqrt{a}}\int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\psi-\varphi} d\lambda.$$

The trade-off compared with the previous estimates is the extra error term on the left-hand side. On the other hand, this estimate can be used to prove the Ohsawa–Takegoshi theorem directly as we will see in Sect. 3. It is however not sufficient to get the extension theorem with optimal constant. A more general one which is sufficient for that purpose is the following $\bar\partial$-estimate from [27] where only one weight has to be plurisubharmonic and the other one is essentially arbitrary:

Theorem 2.4 *Let Ω be pseudoconvex in $\mathbb{C}^n$ and $\alpha \in L^2_{loc,(0,1)}(\Omega)$ be $\bar\partial$-closed. Assume that $\varphi \in PSH(\Omega)$ and $\psi \in W^{1,2}_{loc}(\Omega)$ which is locally bounded from above satisfy*

$$|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \le \begin{cases} 1 & \text{in } \Omega \\ a < 1 & \text{on } \operatorname{supp}\alpha \end{cases}.$$

Then there exists $u \in L^2_{loc}(\Omega)$ *solving* (2.1) *and such that*

$$\int_\Omega \left(1 - |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}\right) |u|^2 e^{2\psi-\varphi} d\lambda \le \frac{1+\sqrt{a}}{1-\sqrt{a}} \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda.$$

Proof The proof will be similar to that of Theorem 2.2. Again by approximation we may assume that ψ is smooth, strongly plurisubharmonic and Ω, φ, ψ are bounded. Let u be the minimal solution to (2.1) in $L^2(\Omega, e^{\psi-\varphi})$. Since u is perpendicular to $\ker\bar\partial$ in $L^2(\Omega, e^{\psi-\varphi})$, it follows that $v := ue^\psi$ is perpendicular to $\ker\bar\partial$ in $L^2(\Omega, e^{-\varphi})$. Therefore v is the minimal solution to $\bar\partial v = \beta := e^\psi(\alpha + u\,\bar\partial\psi)$ in $L^2(\Omega, e^{-\varphi})$ and by Hörmander's estimate

$$\int_\Omega |v|^2 e^{-\varphi} d\lambda \le \int_\Omega |\beta|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda.$$

Therefore

$$\begin{aligned}\int_\Omega |u|^2 e^{2\psi-\varphi} d\lambda &\le \int_\Omega |\alpha + u\,\bar\partial\psi|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda \\ &\le \int_\Omega \left(|\alpha|^2_{i\partial\bar\partial\varphi} + 2|u|\sqrt{H}|\alpha|_{i\partial\bar\partial\varphi} + |u|^2 H\right) e^{2\psi-\varphi} d\lambda,\end{aligned}$$

where $H = |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}$. For $t > 0$ we will get

$$\begin{aligned}&\int_\Omega |u|^2(1-H)e^{2\psi-\varphi} d\lambda \\ &\le \int_\Omega \left[|\alpha|^2_{i\partial\bar\partial\varphi}\left(1 + t^{-1}\frac{H}{1-H}\right) + t|u|^2(1-H)\right] e^{2\psi-\varphi} d\lambda \\ &\le \left(1 + t^{-1}\frac{a}{1-a}\right)\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda \\ &\quad + t\int_\Omega |u|^2(1-H)e^{2\psi-\varphi} d\lambda.\end{aligned}$$

We will obtain the required estimate if we take $t = 1/(a^{-1/2}+1)$. □

This is the most general $\bar\partial$-estimate of all discussed so far. First of all note that, unlike the previous ones, it recovers Hörmander's estimate: it is enough to take $\psi \equiv 0$ and $a = 0$. It also easily implies all the previous results with optimal constants. To obtain Berndtsson's estimate (2.4) (and thus also Donnelly–Fefferman's for $\delta = 0$) for plurisubharmonic φ, ψ satisfying (2.2) and $\delta < 1$ set

$$\widetilde\varphi := \varphi + \psi, \quad \widetilde\psi := \frac{1+\delta}{2}\psi.$$

Then $2\widetilde\psi - \widetilde\varphi = \delta\psi - \varphi$ and

$$|\bar\partial\widetilde\psi|^2_{i\partial\bar\partial\widetilde\varphi} \le \frac{(1+\delta)^2}{4} =: a.$$

Theorem 2.4 will give (2.4) with the constant

$$\frac{1+\sqrt{a}}{(1-\sqrt{a})(1-a)} = \frac{4}{(1-\delta)^2}.$$

If φ, ψ and a are as in Theorem 2.3 and $\widetilde{\varphi} := \varphi + \psi$ then $|\bar\partial\psi|^2_{i\partial\bar\partial\widetilde{\varphi}} \leq |\bar\partial\psi|^2_{i\partial\bar\partial\psi}$ and Theorem 2.4 immediately gives Theorem 2.3.

3 Ohsawa–Takegoshi extension theorem

The following theorem proved by Ohsawa and Takegoshi [98] turned out to be one of the most important results in complex analysis and complex geometry.

Theorem 3.1 *Let Ω be a bounded pseudoconvex open set in $\mathbb{C}^n$ and let H be an affine complex subspace of $\mathbb{C}^n$. Then for any $\varphi \in PSH(\Omega)$ and f holomorphic in $\Omega' := \Omega \cap H$ there exists a holomorphic extension F of f in Ω satisfying*

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq C \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda',$$

where C is a constant depending only on n and the diameter of Ω.

The original proof from [98] was very complicated: it used abstract Kähler geometry and nontrivial Kähler identities. It was subsequently simplified by Siu [109] and Berndtsson [4]. The big breakthrough came recently with a very short proof by Chen [40] who was the first one to succeed in deducing the Ohsawa–Takegoshi theorem directly from Hörmander's estimate. In fact he proved even a slightly more general result, obtained earlier by McNeal and Varolin [91] with more complicated methods:

Theorem 3.2 *Assume that $\Omega \subset \mathbb{C}^{n-1} \times \Delta$ is pseudoconvex and let $H := \{z_n = 0\}$. Then for any $\varphi \in PSH(\Omega)$ and f holomorphic in $\Omega' := \Omega \cap H$ there exists a holomorphic extension F of f in Ω satisfying*

$$\int_\Omega \frac{|F|^2}{|z_n|^2 \log^2 |z_n|} e^{-\varphi} d\lambda \leq C \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda',$$

where C is an absolute constant.

Note that Theorem 3.2 clearly implies Theorem 3.1 by iteration and since $|\zeta|^2 \log^2 |\zeta|$ is bounded in Δ. Theorem 3.2 will easily follow from Theorem 2.3 and the following completely elementary lemma:

Lemma 3.3 *For $\zeta \in \mathbb{C}$ with $|\zeta| \leq (2e)^{-1/2}$ and $\varepsilon > 0$ sufficiently small set*

$$\psi(\zeta) := -\log\left[-\log\left(|\zeta|^2+\varepsilon^2\right) + \log\left(-\log\left(|\zeta|^2+\varepsilon^2\right)\right)\right].$$

Then ψ is subharmonic in $\{|\zeta| < (2e)^{-1/2}\}$ and there exist constants C_1, C_2, C_3 such that

(i) $\left(1-\dfrac{|\psi_\zeta|^2}{\psi_{\zeta\bar\zeta}}\right)e^\psi \geq \dfrac{1}{C_1\log^2(|\zeta|^2+\varepsilon^2)}$ *on* $\{|\zeta|\leq (2e)^{-1/2}\}$;

(ii) $\dfrac{|\psi_\zeta|^2}{\psi_{\zeta\bar\zeta}} \leq \dfrac{C_2}{-\log\varepsilon}$ *on* $\{|\zeta|\leq\varepsilon\}$;

(iii) $\dfrac{e^\psi}{|\zeta|^2\psi_{\zeta\bar\zeta}} \leq C_3$ *on* $\{\varepsilon/2\leq|\zeta|\leq\varepsilon\}$.

Proof Write $t=2\log|\zeta|$ and let γ be such that $\psi=\gamma(t)$. That is

$$\gamma=-\log(-\delta+\log(-\delta)),$$

where $\delta=-\log(e^t+\varepsilon^2)$. We have $\psi_\zeta=\gamma'/\zeta$, $\psi_{\zeta\bar\zeta}=\gamma''/|\zeta|^2$ and thus

$$\frac{|\psi_\zeta|^2}{\psi_{\zeta\bar\zeta}}=\frac{(\gamma')^2}{\gamma''}.$$

We have to prove that

$$\left(1-\frac{(\gamma')^2}{\gamma''}\right)\geq\frac{-\delta+\log(-\delta)}{C_1\delta^2} \qquad \text{if } t\leq-\log(2e) \tag{3.1}$$

$$\frac{(\gamma')^2}{\gamma''}\leq\frac{C_2}{-\log\varepsilon} \qquad \text{if } t\leq 2\log\varepsilon \tag{3.2}$$

$$(-\delta+\log(-\delta))\gamma''\geq\frac{1}{C_3} \qquad \text{if } 2\log(\varepsilon/2)\leq t\leq 2\log\varepsilon. \tag{3.3}$$

We can compute that

$$\gamma'=\frac{1-\delta^{-1}}{-\delta+\log(-\delta)}\delta'$$

and

$$\gamma''\geq\frac{1-\delta^{-1}}{-\delta+\log(-\delta)}\delta''.$$

Therefore we get (3.3) and since

$$\frac{(\gamma')^2}{\gamma''}\leq\frac{1-\delta^{-1}}{-\delta+\log(-\delta)}\frac{(\delta')^2}{\delta''}=\frac{1-\delta^{-1}}{-\delta+\log(-\delta)}e^t,$$

we also obtain (3.1) and (3.2). □

Proof of Theorem 3.2 It will be no loss of generality to prove the result in a slightly smaller disc than Δ, say the same as in Lemma 3.3. By approximation we may assume that Ω is bounded, smooth, strongly pseudoconvex, φ is smooth up to the boundary, and f is holomorphic in a neighborhood of $\overline{\Omega'}$. Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) = 1$ for $t \le -2$, $\chi(t) = 0$ for $t \ge 0$, and $|\chi'| \le 1$. For $\varepsilon > 0$ sufficiently small the function fv, where

$$v = v_\varepsilon := \chi(2\log(|z_n|/\varepsilon)),$$

is defined in Ω. We will use Theorem 2.3 for

$$\alpha = \alpha_\varepsilon := \bar\partial(fv) = f\,\chi'(2\log(|z_n|/\varepsilon))\frac{d\bar z_n}{\bar z_n},$$

$\widetilde\varphi := \varphi + 2\log|z_n|$, and ψ as in Lemma 3.3. We will find $u = u_\varepsilon \in L^2_{loc}(\Omega)$ such that $\bar\partial u = \alpha$ (in fact u has to be continuous, since fv is) and

$$\int_\Omega (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\psi})|u|^2 e^{\psi-\widetilde\varphi}d\lambda \le \frac{1+\sqrt{a}}{1-\sqrt{a}}\int_\Omega |\alpha|^2_{i\partial\bar\partial\psi}e^{\psi-\widetilde\varphi}d\lambda, \qquad (3.4)$$

where $a = -C_2/\log\varepsilon$ by Lemma 3.3ii. For a given ε the function

$$\left(1 - |\bar\partial\psi|^2_{i\partial\bar\partial\psi}\right)e^{\psi-\widetilde\varphi}$$

is not integrable near H, and thus by (3.4) $u = 0$ on Ω'. This means that $F_\varepsilon := fv - u$ is a holomorphic extension of f to Ω. (3.4) together with Lemma 3.3i also give

$$\int_\Omega \frac{|u|^2}{|z_n|^2\log^2(|z_n|^2+\varepsilon^2)}e^{-\varphi}d\lambda \le C_1\frac{1+\sqrt{a}}{1-\sqrt{a}}\int_\Omega |\alpha|^2_{i\partial\bar\partial\psi}e^{\psi-\varphi}d\lambda'.$$

Using Lemma 3.3iii we will obtain

$$\limsup_{\varepsilon\to 0}\int_\Omega \frac{|F_\varepsilon|^2}{|z_n|^2\log^2|z_n|}|e^{-\varphi}d\lambda \le C\int_{\Omega'}|f|^2e^{-\varphi}d\lambda'$$

and it remains to apply the Banach–Alaouglu theorem. □

4 Singularities of plurisubharmonic functions

We will start with the following recent result of Berndtsson [10] (proved by Favre and Jonsson [54] in dimension 2) confirming the *openness conjecture* of Demailly–Kollár [46].

Theorem 4.1 *For a plurisubharmonic function φ defined in a neighbourhood of $z_0 \in \mathbb{C}^n$ the set of those $p \in \mathbb{R}$ such that $e^{-p\varphi}$ is integrable near z_0 is an open interval of the form $(-\infty, p_0)$.*

The whole point is that the limit p_0 does not belong to this set. First of all it is easy to see that this holds for $n = 1$. Then φ can be written as a sum of a harmonic function and the potential

$$U^\mu(z) = \int_{\mathbb{C}} \log|\zeta - z| d\mu(\zeta),$$

where μ is a positive measure with compact support in $\mathbb{C}$ such that $\mu = \Delta\varphi/2\pi$ near z_0. We may thus assume that $\varphi = U^\mu$ and then one can then easily prove that $e^{-p\varphi}$ is integrable near z_0 if and only if $p\,\mu(\{z_0\}) < 2$.

The original proof of Theorem 3.1 from [10] was more complicated but Berndtsson [11] extracted the following simple one from the method of Guan–Zhou [58] who showed a more general *strong openness conjecture*, where instead of $e^{-p\varphi}$ one is interested in local integrability of $|f|^2 e^{-p\varphi}$ for some fixed holomorphic f. The proof of this was simplified by Hiep [64].

Proof of Theorem 4.1 We may assume that z_0 is the origin, φ is defined in a neighbourhood of $\bar{\Delta}^n$ and $\varphi \le 0$. We first claim that if φ is not locally integrable near the origin then

$$\int_{\Delta^{n-1}} e^{-\varphi(\cdot, z_n)} d\lambda' \ge \frac{c_n}{|z_n|^2}, \quad |z_n| \le 1/2, \tag{4.1}$$

where c_n is a positive constant depending only on n. For a fixed z_n we may assume that the left-hand side of (4.1) is finite. By the Ohsawa–Takegoshi theorem there exists a holomorphic F in Δ^n such that $F(\cdot, z_n) = 1$ in Δ^{n-1} and

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda \le C_1 \int_{\Delta^{n-1}} e^{-\varphi(\cdot, z_n)} d\lambda' < \infty. \tag{4.2}$$

It is elementary that

$$|F(0,\zeta)|^2 \le C_2 \int_{\Delta^n} |F|^2 d\lambda \le C_2 \int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda, \quad |\zeta| \le 1/2. \tag{4.3}$$

Since $e^{-\varphi}$ is not locally integrable near the origin, by (4.2) we have $F(0,0) = 0$, and thus by (4.3) and the Schwarz lemma

$$|F(0,\zeta)|^2 \le C_3 |\zeta|^2 \int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda, \quad |\zeta| \le 1/2.$$

For $\zeta = z_n$ using (4.2) and the fact that $F(0, z_n) = 1$ we get (4.1).

Now assume that the result is true for functions of $n-1$ variables and suppose that

$$\int_{\Delta^n} e^{-p_0\varphi} d\lambda < \infty. \tag{4.4}$$

Since for $p > p_0$ we know that $e^{-p\varphi}$ is not locally integrable near the origin, by (4.1)

$$\int_{\Delta^{n-1}} e^{-p\varphi(\cdot,z_n)} d\lambda' \geq \frac{c_n}{|z_n|^2}, \quad |z_n| \leq 1/2. \tag{4.5}$$

From (4.4) it follows that for almost all $z_n \in \Delta$

$$\int_{\Delta^{n-1}} e^{-p_0\varphi(\cdot,z_n)} d\lambda' < \infty$$

and thus by the inductive assumption for p sufficiently close to p_0

$$\int_{\Delta^{n-1}} e^{-p\varphi(\cdot,z_n)} d\lambda' < \infty.$$

The Lebesgue dominated convergence theorem now implies that (4.5) also holds for $p = p_0$ which contradicts (4.4). □

It is quite remarkable that to prove a result on plurisubharmonic functions one has to use tools like holomorphic function and $\bar\partial$-equation.

For a plurisubharmonic φ defined in a neighborhood of z_0 its *Lelong number* at z_0 is defined by

$$\nu_\varphi(z_0) = \liminf_{z\to z_0} \frac{\varphi(z)}{\log|z-z_0|} = \lim_{r\to 0^+} \frac{\varphi^r(z_0)}{\log r},$$

where

$$\varphi^r(z) = \max_{|\zeta - z|\leq r} \varphi(\zeta). \tag{4.6}$$

(One can show that φ^r, defined in $\Omega_r := \{z \in \Omega\colon B(z,r) \subset \Omega\}$, is continuous, plurisubharmonic and decreases to φ as r decreases to 0.) In other words, $\nu_\varphi(z_0)$ is the maximal number $c \geq 0$ such that

$$\varphi(z) \leq c\log|z-z_0| + A$$

for some constant A and z in a neighbourhood of z_0. Lelong number measures the singularity of a plurisubharmonic function at a point.

The classical result on Lelong numbers is the following due to Siu [107]:

Theorem 4.2 *For any plurisubharmonic function φ and $c \in \mathbb{R}$ the superlevel set $\{\nu_\varphi \geq c\}$ is analytic.*

The original proof in [107] was very complicated. It was later simplified and generalized by Kiselman [75,77] (see also [66]) and Demailly [44]. It was Demailly [45] who found a surprisingly simple proof of the Siu theorem using the Ohsawa–Takegoshi theorem. It was done through the following approximation of plurisubharmonic functions:

Theorem 4.3 *Let φ be plurisubharmonic in a bounded pseudoconvex Ω in $\mathbb{C}^n$. For $m = 1, 2, \ldots$ define*

$$\varphi_m := \frac{1}{2m} \log \sup \left\{ |f|^2 \colon f \in \mathcal{O}(\Omega), \quad \int_\Omega |f|^2 e^{-2m\varphi} d\lambda \leq 1 \right\}.$$

Then there exist positive constants C_1 depending only on n and the diameter of Ω and C_2 depending only on n such that

$$\varphi - \frac{C_1}{m} \leq \varphi_m \leq \varphi^r + \frac{1}{m} \log \frac{C_2}{r^n} \quad \text{in} \quad \Omega_r \tag{4.7}$$

and

$$\nu_\varphi - \frac{n}{m} \leq \nu_{\varphi_m} \leq \nu_\varphi. \tag{4.8}$$

In particular, $\varphi_m \to \varphi$ pointwise and in L^1_{loc}.

Proof By the Ohsawa–Takegoshi theorem for every $z \in \Omega$ we can find $f \in \mathcal{O}(\Omega)$ such that

$$\int_\Omega |f|^2 e^{-2m\varphi} d\lambda \leq C |f(z)|^2 e^{-2m\varphi(z)} = 1.$$

This implies that

$$\varphi_m(z) \geq \frac{1}{2m} \log |f(z)|^2 = \varphi(z) - \frac{\log C}{2m}$$

and we obtain the first inequality in (4.7). The proof of the second one is completely elementary: $|f|^2$ is in particular subharmonic and thus for $r < \operatorname{dist}(z, \partial\Omega)$

$$|f(z)|^2 \leq \frac{1}{\lambda(B(z,r))} \int_{B(z,r)} |f|^2 d\lambda \leq \frac{n!}{\pi^n r^{2n}} e^{2m\varphi^r(z)} \int_\Omega |f|^2 e^{-2m\varphi} d\lambda$$

which gives the second inequality in (4.7).

Now (4.8) easily follows from (4.7): the first inequality in (4.7) implies that $\nu_{\varphi_m} \leq \nu_{\varphi - C_1/m} = \nu_\varphi$ and the second one gives

$$\varphi_m^r \leq \varphi^{2r} + \frac{1}{m} \log \frac{C_2}{r^n},$$

hence $\nu_\varphi - n/m \leq \varphi_{n/m}$. □

Proof of Theorem 4.2 The result is local so we may assume that φ is defined in bounded pseudoconvex domain Ω. Then by (4.8)

$$\{\nu_\varphi \geq c\} = \bigcap_m \left\{ \nu_{\varphi_m} \geq c - \frac{n}{m} \right\}.$$

Let $\{\sigma_j\}$ be an orthonormal basis of $\mathcal{O}(\Omega) \cap L^2(\Omega, e^{-2m\varphi})$. Then

$$\varphi_m = \frac{1}{2m} \log \sum_j |\sigma_j|^2 \tag{4.9}$$

and one can show that

$$\left\{\nu_{\varphi_m} \geq c - \frac{n}{m}\right\} = \bigcap_{\substack{|\alpha|<mc-n \\ j}} \{\partial^\alpha \sigma_j = 0\}$$

which finishes the proof. □

It is interesting that the Ohsawa–Takegoshi theorem also gives the following subadditivity of the Demailly approximation from [47]:

Theorem 4.4 *Under the assumptions of Theorem 4.3 there exists a positive constant C_3 depending only on n and the diameter of Ω such that*

$$(m_1 + m_2)\varphi_{m_1+m_2} \leq m_1\varphi_{m_1} + m_2\varphi_{m_2} + C_3. \tag{4.10}$$

Proof By the Ohsawa–Takegoshi theorem for every $f \in \mathcal{O}(\Omega)$ with

$$\int_\Omega |f|^2 e^{-2(m_1+m_2)\varphi} d\lambda \leq 1$$

there exists $F \in \mathcal{O}(\Omega \times \Omega)$ such that $F(z, z) = f(z)$ for $z \in \Omega$ and

$$\iint_{\Omega\times\Omega} |F(z, w)|^2 e^{-2m_1\varphi(z) - m_2\varphi(w)} d\lambda(z) d\lambda(w) \leq C. \tag{4.11}$$

Let $\{\sigma_j\}$ be an orthonormal basis in $\mathcal{O}(\Omega) \cap L^2(\Omega, e^{-2m_1\varphi})$ and $\{\sigma'_k\}$ an orthonormal basis in $\mathcal{O}(\Omega) \cap L^2(\Omega, e^{-2m_2\varphi})$, then $\{\sigma_j(z)\sigma'_k(w)\}$ is an orthonormal basis in $\mathcal{O}(\Omega \times \Omega) \cap L^2(\Omega \times \Omega, e^{-2m_1\varphi(z)-2m_2\varphi(w)})$. If

$$F(z, w) = \sum_{j,k} c_{jk}\sigma_j(z)\sigma'_k(w)$$

then by (4.11) $\sum_{j,k} |c_{jk}|^2 \leq C$ and thus by the Schwarz inequality and (4.9)

$$|f(z)|^2 = |F(z, z)|^2 \leq C \sum_j |\sigma_j(z)|^2 \sum_k |\sigma'_k(z)|^2 = C e^{2m_1\varphi_{m_1}(z)} e^{2m_2\varphi_{m_2}(z)}.$$

This gives (4.10) with $C_3 = \log C/2$. □

Theorem 4.4 gives monotonicity of a subsequence of φ_m. More precisely, for example the sequence $\varphi_{2^k} + C_3/2^{k+1}$ is decreasing. It was recently showed by Kim [74] that in general one cannot expect monotonicity of the entire sequence φ_m, even after adding a sequence of constants converging to 0.

5 Pluricomplex Green function and the complex Monge–Ampère operator

If Ω is an open subset of $\mathbb{C}^n$ then for $z, w \in \Omega$ the *pluricomplex Green function* is defined as

$$G_\Omega(z, w) = \sup\{u(z) \colon u \in \mathcal{B}(\Omega, w)\},$$

where $\mathcal{B}(\Omega, w)$ is the family of negative plurisubharmonic functions in Ω that have a logarithmic pole at w, that is

$$\mathcal{B}(\Omega, w) = \left\{u \in PSH^-(\Omega) \colon \limsup_{z \to w}(u(z) - \log|z - w|) < \infty\right\}.$$

One can show that for a given $w \in \Omega$ we either have $G_\Omega(\cdot, w) \in \mathcal{B}(\Omega, w)$ or $\mathcal{B}(\Omega, w) = \emptyset$. This general definition of the pluricomplex Green function was first given independently by Klimek [78] and Zakharyuta [115]. The fundamental properties were proved by Demailly [43].

One of the big differences between one and higher dimensional cases is that for $n \geq 2$ the Green function is usually not symmetric. The first example of this kind is due to Bedford and Demailly [3]. The following simple one was given by Klimek [79]: for $\Omega = \{|z_1 z_2| < 1\} \subset \mathbb{C}^2$ one can show that

$$G_\Omega(z, w) = \begin{cases} \log\left|\frac{z_1 z_2 - w_1 w_2}{1 - \bar{w}_1 \bar{w}_2 z_1 z_2}\right| & w \neq 0, \\ \frac{1}{2}\log|z_1 z_2| & w = 0. \end{cases}$$

In particular, $G_\Omega(z, 0) = \frac{1}{2}\log|z_1 z_2|$ but $G_\Omega(0, z) = \log|z_1 z_2|$. On the other hand, it follows from Lempert's theory [85] that G_Ω is symmetric for convex Ω.

The main tool when dealing with the pluricomplex Green function is Bedford–Taylor's theory of the complex Monge–Ampère operator [1,2]. It is convenient to consider the operators $d = \partial + \bar\partial$ and $d^c := i(\bar\partial - \partial)$, so that $dd^c = 2i\partial\bar\partial$. For smooth u we then have

$$(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u = 4^n n! \det(\partial^2 u/\partial z_j \partial \bar{z}_k)\, d\lambda$$

and one would like to define $(dd^c u)^n$ as a positive regular measure for arbitrary plurisubharmonic u. This turned out to be impossible in general. First example was found by Shiffman and Taylor, see [108]. This was later simplified by Kiselman [76]: for $n \geq 2$ the function

$$u(z) = (-\log|z_1|)^{1/n}(|z_2|^2 + \cdots + |z_n|^2 - 1)$$

is plurisubharmonic near the origin, smooth away from $\{z_1 = 0\}$ but $(dd^c u)^n$ is not locally integrable near $\{z_1 = 0\}$.

Bedford and Taylor [2] proved however that it is possible to define $(dd^c u)^n$ for locally bounded plurisubharmonic u and Demailly [43] extended this to plurisubharmonic functions that are possibly unbounded on a compact subset. In both cases the operator $(dd^c)^n$ is continuous in the weak* topology of measures for monotone sequences. In fact, the domain of definition of the complex Monge–Ampère operator, defined as the maximal subclass of the class of plurisubharmonic functions where the operator can be defined as a positive measure in such a way that it is continuous for decreasing sequences, was characterized in [22] and [24]. In particular, for $n = 2$ these are precisely the plurisubharmonic functions which belong to the Sobolev space $W^{1,2}_{loc}$.

A plurisubharmonic function u in Ω is called *maximal* if for any other $v \in PSH(\Omega)$ such that $v \le u$ in $\Omega \setminus K$ for some $K \Subset \Omega$ we have $v \le u$ in Ω. For $n = 1$ these are precisely harmonic functions but they may be completely irregular in higher dimensions: for example if a plurisubharmonic function is independent of one of the variables then it is maximal. One of the main points of Bedford–Taylor's pluripotential theory [1,2] is that for locally bounded plurisubharmonic functions u we have

$$u \text{ is maximal } \Leftrightarrow (dd^c u)^n = 0. \tag{5.1}$$

The same characterization remains true for functions from the domain of definition of $(dd^c)^n$ (see [22]) but there are maximal plurisubharmonic functions which do not belong to the domain of definition, for example $\log|z_1|$ in $\mathbb{C}^n$ for $n \ge 2$. It remains an open problem whether maximality is a local property in general. By the above characterization as a solution to the homogeneous complex Monge–Ampère equation, it is true for locally bounded plurisubharmonic functions, or more generally functions from the domain of definition.

One can show that

$$G_{B(w,R)}(z, w) = \log \frac{|z - w|}{R}$$

and thus if $B(w, r) \subset \Omega \subset B(w, R)$ then

$$\log \frac{|z - w|}{R} \le G_\Omega(z, w) \le \log \frac{|z - w|}{r}.$$

Therefore, if Ω is bounded then for $w \in \Omega$ the function $G_\Omega(\cdot, w)$ is plurisubharmonic and locally bounded in $\Omega \setminus \{w\}$. We can then define the Monge–Ampère operator and Demailly [43] proved that

$$(dd^c G_\Omega(\cdot, w))^n = (2\pi)^n \delta_w \tag{5.2}$$

(see also [20]).

A domain Ω in $\mathbb{C}^n$ is called *hyperconvex* if it admits a negative plurisubharmonic exhaustion function, that is there exists $u \in PSH^-(\Omega)$ such that $\{u < t\} \Subset \Omega$ for $t < 0$. For $n = 1$ this equivalent to Ω being regular with respect to classical potential theory. In general, Kerzman and Rosay [73] proved that hyperconvexity is a local property of the boundary and Demailly [43] showed that pseudoconvex domains with Lipschitz boundary are hyperconvex. It is an open problem whether pseudoconvex domains with continuous boundary have to be hyperconvex.

Demailly [43] showed that if Ω is bounded and hyperconvex then G_Ω is continuous on $\bar{\Omega}\times\Omega$ away from the diagonal of Ω, where we extend the definition of G_Ω to vanish on $\partial\Omega\times\Omega$ (see also [19] for a slightly different proof). It is an open problem whether in this case G_Ω is continuous on $\bar{\Omega}\times\bar{\Omega}$ away from the diagonal of $\bar{\Omega}$. Equivalently, we ask whether for bounded hyperconvex Ω if $w_j\in\Omega$ is a sequence of poles converging to $\partial\Omega$ then $G_\Omega(\cdot, w_j)$ converge locally uniformly to 0. We have the following weaker result from [30]:

Proposition 5.1 *Assume that Ω is bounded and hyperconvex. Then for any $p<\infty$*

$$\lim_{w\to\partial\Omega}||G_\Omega(\cdot,w)||_{L^p(\Omega)}=0.$$

Proof By [15] there exists unique $u\in PSH(\Omega)\cap C(\bar{\Omega})$ such that $u=0$ on $\partial\Omega$ and $(dd^cu)^n=d\lambda$. Write $G_w=G_\Omega(\cdot,w)$. Integrating by parts as in [14] we will get using (5.2)

$$\int_\Omega|G_w|^n d\lambda=\int_\Omega|G_w|^n(dd^cu)^n\le n!||u||^{n-1}_{L^\infty(\Omega)}\int_\Omega|u|(dd^cG_w)^n\le C|u(w)|,$$

where C depends only on n and the volume of Ω. This gives the result for $p=n$ and for other p it follows easily from it. □

The conjecture on locally uniform convergence of the Green function for poles converging to the boundary was confirmed by Herbort [63] for pseudoconvex domains with C^2 boundary (see also [23] for a slightly simplified proof). As in Proposition 5.1, the inequality for the complex Monge–Ampère operator from [14] is one of the tools. In fact, the only additional regularity of Ω used to prove this result is an existence of $u\in PSH(\Omega)$ such that

$$\frac{1}{A}\delta_\Omega(z)^a\le|u(z)|\le A\delta_\Omega(z)^b \tag{5.3}$$

for some positive constants A, a, b, where δ_Ω is the Euclidean distance to the boundary. For domains with C^2 boundary this is guaranteed by a theorem of Diederich and Fornæss [48], even with $a=b$. Since Harrington [61] generalized this Diederich–Fornæss result to pseudoconvex domains with Lipschitz boundary, the conjecture also holds in this case.

Further regularity of the pluricomplex Green function was established in [56] and [18] (see also [19]): if Ω is $C^{2,1}$-smooth and strongly pseudoconvex then for a fixed $w\in\Omega$ we have $G_\Omega(\cdot,w)\in C^{1,1}(\bar{\Omega}\setminus\{w\})$. This is the highest regularity we can expect, Bedford and Demailly showed that $G_\Omega(\cdot,w)$ does not have to be C^2-smooth up to the boundary even if Ω is C^∞-smooth and strongly pseudoconvex. Lempert [85] proved that $G_\Omega(\cdot,w)\in C^\infty(\bar{\Omega}\setminus\{w\})$ if Ω is C^∞-smooth and strongly convex.

The following result from [15] was used in the proof of Proposition 5.1: for any bounded hyperconvex Ω in $\mathbb{C}^n$ and nonnegative $F\in C(\bar{\Omega})$ there exists unique solution to the following Dirichlet problem:

$$\begin{cases} u \in PSH(\Omega) \cap C(\bar{\Omega}) \\ (dd^c u)^n = F\,d\lambda \\ u = 0 \quad \text{on } \partial\Omega \end{cases} . \tag{5.4}$$

It is an open problem whether the following interior regularity holds here: does $F \in C^\infty(\bar{\Omega})$ imply $u \in C^\infty(\Omega)$ (without any additional assumption on the regularity of Ω)? Of course when Ω is smooth and strongly pseudoconvex then it follows from the seminal work of Krylov [83] and Caffarelli et al. [37] that $u \in C^\infty(\bar{\Omega})$. In general however we cannot expect u to be smooth up to the boundary. The only case so far of a non-smooth domain where this problem was solved is a polydisk, see [16]. The main tool was transitivity of the group of holomorphic automorphisms used to show interior $C^{1,1}$-regularity, as in the classical result of Bedford and Taylor [1] for a ball. The corresponding result for the real Monge–Ampère equation in arbitrary bounded convex domain in $\mathbb{R}^n$ holds by the famous interior estimate of Pogorelov [99].

In Sect. 7 we will need the following product property of the pluricomplex Green function proved by Jarnicki and Pflug [69]:

Theorem 5.2 *Assume that $\Omega_j \subset \mathbb{C}^{n_j}$, $j = 1, 2$, are pseudoconvex. Then*

$$G_{\Omega_1 \times \Omega_2}\left(\left(z^1, z^2\right), \left(w^1, w^2\right)\right) = \max\left\{G_{\Omega_1}\left(z^1, w^1\right), G_{\Omega_2}\left(z^2, w^2\right)\right\}. \tag{5.5}$$

Proof Directly from the definition we have $\geq$. To show $\leq$ we may assume that Ω_j are bounded hyperconvex. Then it is enough to show that for fixed $w^j \in \Omega_j$ the right-hand side od (5.5), as a function of (z^1, z^2), is maximal in $\Omega_1 \times \Omega_2 \backslash \{(w^1, w^2)\}$. By (5.1) we have to prove that it solves the homogeneous complex Monge–Ampère equation. This follows from the following result of Zeriahi [116]:

$$(dd^c u_j)^{n_j} = 0 \;\Rightarrow\; \left(dd^c \max\left\{u_1\left(z^1\right), u_2\left(z^2\right)\right\}\right)^{n_1+n_2} = 0$$

which can be easily deduced from the following formula originally proved in [17]:

Theorem 5.3 *Let u, v be locally bounded plurisubharmonic functions defined on an open subset of $\mathbb{C}^n$ and $2 \leq p \leq n$. Then*

$$\begin{aligned}(dd^c \max\{u, v\})^p = dd^c \max\{u, v\} \wedge \sum_{k=0}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-1-k} \\ - \sum_{k=1}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-k}.\end{aligned}$$

Proof By approximation we may assume that u, v are smooth. A simple inductive argument reduces the proof to the case $p = 2$. Set $w := \max\{u, v\}$ and, for $\varepsilon > 0$, $w_\varepsilon := \max\{u+\varepsilon, v\}$. In an open set $\{u+\varepsilon > v\}$ we have $w_\varepsilon - u = \varepsilon$, whereas $w - v = 0$ in $\{u < v\}$. It follows that for every $\varepsilon > 0$ one has $dd^c(w_\varepsilon - u) \wedge dd^c(w - v) = 0$ and taking the limit we conclude that $dd^c(w - u) \wedge dd^c(w - v) = 0$. □

Edigarian [53] showed Theorem 5.2 without assuming pseudoconvexity. His proof however is much more complicated, it uses Poletsky's theory of analytic disks [100].

6 Bergman completeness

For a domain Ω in $\mathbb{C}^n$ we set $A^2(\Omega) := \mathcal{O}(\Omega) \cap L^2(\Omega)$. It is a closed subspace of $L^2(\Omega)$ and thus a Hilbert space. It is conjectured that when Ω is pseudoconvex then either $A^2(\Omega) = \{0\}$ or $A^2(\Omega)$ is infinitely dimensional. Wiegerinck [114] showed this for $n = 1$ and found non-pseudoconvex Ω with $A^2(\Omega)$ of arbitrary dimension.

For $w \in \Omega$ the functional

$$A^2(\Omega) \ni f \longmapsto f(w) \in \mathbb{C}$$

is bounded and thus $f(w) = \langle f, K_w \rangle$ for some $K_w \in A^2(\Omega)$ and all f. The Bergman kernel is characterized by the reproducing formula

$$f(w) = \int_\Omega f(z)\overline{K_\Omega(z, w)}d\lambda(z), \quad f \in A^2(\Omega), \ w \in \Omega.$$

Applying this for $f = K_\Omega(\cdot, z)$ we see that K_Ω is antisymmetric:

$$K_\Omega(w, z) = \overline{K_\Omega(z, w)}$$

and

$$K_\Omega(z, z) = ||K_\Omega(\cdot, z)||^2 = \sup\{|f(z)|^2 \colon f \in A^2(\Omega), \ ||f|| \le 1\}, \tag{6.1}$$

where $|| \cdot ||$ is the L^2-norm in Ω. By Hartogs' theorem on separate holomorphic functions K_Ω is smooth on $\Omega \times \Omega$. If $\{\sigma_j\}$ is an orthonormal system in $A^2(\Omega)$ then

$$K_\Omega(z, w) = \sum_j \sigma_j(z)\overline{\sigma_j(w)}$$

and on the diagonal

$$K_\Omega(z, z) = \sum_j |\sigma_j(z)|^2. \tag{6.2}$$

For other basic properties of K_Ω we refer to [70].

For a big class of domains, e.g. bounded ones, on the diagonal we have $K_\Omega > 0$ and thus $\log K_\Omega(z, z)$ is a smooth plurisubharmonic function in Ω. If it is also strongly plurisubharmonic then we say that Ω *admits* the Bergman metric and the Kähler metric defined by the potential $\log K_\Omega(z, z)$ is called the *Bergman metric* of Ω. One can show that the Levi form is given by the following extremal formula

$$\sum_{p,q=1}^{n} \frac{\partial^2(\log K_\Omega(z,z))}{\partial z_p \partial \bar{z}_q} X_p \bar{X}_q$$
$$= \frac{1}{K_\Omega(z,z)} \sup\left\{|D_X f(z)|^2 \colon\ f \in A^2(\Omega),\ f(z)=0,\ ||f|| \le 1\right\},$$

where $D_X = \sum_p X_p \partial/\partial z_p$, and it follows easily that for example all bounded domains admit the Bergman metric.

If Ω is complete with respect to the geodesic distance defined by the Bergman metric then we say that Ω is *Bergman complete*. The main tool in studying Bergman completeness is the following embedding of Kobayashi [80]:

$$\kappa \colon \Omega \ni z \longmapsto [K_\Omega(\cdot, z)] \in \mathbb{P}(A^2(\Omega)).$$

One can easily show that if Ω admits the Bergman metric then κ is an immersion and if Ω is bounded then it is an embedding. The main point is that the pull-back of the Fubini–Study metric on the (infinitely dimensional) projective space $\mathbb{P}(A^2(\Omega))$ by κ is precisely the Bergman metric of Ω. This is sometimes called Kobayashi's alternative definition of the Bergman metric. An immediate consequence of this is that κ is distance decreasing which means that

$$\operatorname{dist}_\Omega^B(z,w) \ge \arccos \frac{|K_\Omega(z,w)|}{\sqrt{K_\Omega(z,z)K_\Omega(w,w)}}, \tag{6.3}$$

where $\operatorname{dist}_\Omega^B$ is the distance defined by the Bergman metric. In particular,

$$K_\Omega(z,w) = 0 \ \Rightarrow\ \operatorname{dist}_\Omega^B(z,w) \ge \frac{\pi}{2}$$

and Dinew [50] showed that $\pi/2$ is an optimal constant here.

We have the following criterion of Kobayashi [80] for Bergman completeness:

Theorem 6.1 *Assume that Ω admits the Bergman metric and is such that for any sequence $z_j \in \Omega$ without accumulation point in Ω we have*

$$\lim_{j\to\infty} \frac{|f(z_j)|^2}{K_\Omega(z_j,z_j)} = 0, \qquad f \in A^2(\Omega). \tag{6.4}$$

Then Ω is Bergman complete.

Proof Assume that $z_j \in \Omega$ is a Cauchy sequence with respect to $\operatorname{dist}_\Omega^B$. If it has an accumulation point in Ω then it has a limit, since locally the Bergman metric is equivalent to the Euclidean metric. We may thus assume that it has no accumulation point in Ω. Since κ is distance decreasing, it follows that $\kappa(z_j)$ is a Cauchy sequence in $\mathbb{P}(A^2(\Omega))$ and thus has a limit there, say $[f]$ for some $f \in A^2(\Omega)$, $f \not\equiv 0$. This means that there exist $a_j \in \mathbb{C}$ such that

$$a_j K_\Omega(\cdot, z_j) \to f.$$

This gives $|a_j|\sqrt{K_\Omega(z_j,z_j)} \to ||f||$ and $|a_j|\,|f(z_j)| \to ||f||^2$ which imply that

$$\frac{|f(z_j)|^2}{K_\Omega(z_j,z_j)} \to ||f||^2,$$

a contradiction. □

We say that a bounded Ω is *Bergman exhaustive* if

$$\lim_{z\to\partial\Omega} K_\Omega(z,z) = \infty.$$

Note that bounded domains satisfying (6.4) must be Bergman exhaustive, simply take $f \equiv 1$. The Hartogs triangle

$$\{z \in \mathbb{C}^2 \colon |z_2| < |z_1| < 1\}$$

is an example of a domain which is Bergman exhaustive but not Bergman complete. This can be shown using the fact that the Hartogs triangle is biholomorphic to $\Delta \times \Delta_*$. This example also shows that Bergman exhaustiveness is not a biholomorphic invariant, contrary to Bergman completeness. On the other hand, Chen [39] proved that for $n = 1$ Bergman exhaustiveness does imply Bergman completeness.

Zwonek [118] showed that the converse to Theorem 6.1 does not hold: he gave an example of a bounded domain in $\mathbb{C}$ which is Bergman complete but not Bergman exhaustive. This example was simplified by Jucha [72]: he showed that

$$\Omega := \Delta_* \setminus \left(\bigcup_{k=1}^{\infty} \bar{\Delta}(2^{-k}, r_k) \right),$$

where $r_k > 0$ are such that $\bar{\Delta}(2^{-k}, r_k) \cap \bar{\Delta}(2^{-l}, r_l) = \emptyset$ for $k \neq l$, is Bergman complete if and only if

$$\sum_{k=1}^{\infty} \frac{2^k}{\sqrt{-\log r_k}} = \infty$$

and Bergman exhaustive if and only if

$$\sum_{k=1}^{\infty} \frac{4^k}{-\log r_k} = \infty.$$

Therefore, if for example $r_k = e^{-k^2 4^k}$ then Ω is Bergman complete but not Bergman exhaustive.

The proof of Theorem 6.1 really shows something slightly stronger: instead of (6.4) it is enough to assume that

$$\overline{\lim_{j\to\infty}} \frac{|f(z_j)|^2}{K_\Omega(z_j,z_j)} < ||f||^2, \quad f \in A^2(\Omega), \;\; f \not\equiv 0.$$

It is not known if this condition is equivalent to Bergman completeness or not. Another open problem is whether Bergman exhaustiveness is a biholomorphically invariant notion for $n = 1$. In view of Chen's result, an example showing that it is not would be another one showing that (6.4) is not equivalent to Bergman completeness.

It turns out that pluripotential theory gives a lot of examples of Bergman complete domains. The main result is due to Chen [38] in dimension one and independently to Herbort [62] and Pflug et al. [30] in arbitrary dimension:

Theorem 6.2 *Bounded hyperconvex domains are Bergman complete.*

We will prove this using the following estimate of Herbort [62]:

Theorem 6.3 *Assume that Ω is pseudoconvex. Then for every $f \in A^2(\Omega)$ and $w \in \Omega$ one has*

$$\frac{|f(w)|^2}{K_\Omega(w,w)} \leq c_n \int_{\{G_\Omega(\cdot,w)<-1\}} |f|^2 d\lambda. \tag{6.5}$$

Proof Approximating Ω from inside we may assume that it is bounded and hyperconvex. We will use Theorem 2.2 with

$$\varphi = 2nG, \quad \psi = -\log(-G),$$

and

$$\alpha = \bar\partial(f\chi\circ G) = f\chi'\circ G\,\bar\partial G,$$

where $G = G_\Omega(\cdot, w)$ and $\chi \in C^\infty((-\infty, 0))$ is such that $\chi(t) = 0$ for $t \geq -1/2$ and $\chi(t) = -1$ for $t \leq -2$. We have

$$i\bar\alpha\wedge\alpha \leq |f|^2 G^2(\chi'\circ G)^2 i\partial\bar\partial\psi$$

and thus by Theorem 2.2 there exists $u \in L^2_{loc}(\Omega)$ (in fact it has to be continuous) such that $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 d\lambda \leq \int_\Omega |u|^2 e^{-\varphi} d\lambda \leq C\int_\Omega |f|^2 G^2(\chi'\circ G)^2 e^{-2nG} d\lambda. \tag{6.6}$$

Since $e^{-\varphi}$ is not locally integrable near w, it follows that for $F := f\chi\circ G - u$ is holomorphic in Ω, $F(w) = f(w)$ and

$$\int_\Omega |F|^2 \leq c_n \int_{\{G<-1\}} |f|^2 d\lambda.$$

$\square$

Proof of Theorem 6.2 By Proposition 5.1

$$\lim_{w\to\partial\Omega}\lambda(\{G_\Omega(\cdot,w)<-1\})=0$$

and thus by Theorem 6.3

$$\lim_{w\to\partial\Omega}\frac{|f(w)|^2}{K_\Omega(w,w)}=0.$$

The result now follows from Kobayashi's criterion Theorem 6.1. □

Taking $f\equiv 1$ in Herbort's estimate (6.5) we get

$$K_\Omega(w,w)\geq\frac{1}{c_n\lambda(\{G_\Omega(\cdot,w)<-1\})}. \tag{6.7}$$

The proof of Proposition 5.1 now gives for bounded hyperconvex domains

$$K_\Omega(w,w)\geq\frac{1}{C(n,\lambda(\Omega))|u(w)|},$$

where u is the solution to (5.4) with $F\equiv 1$. This is an interesting lower bound for the Bergman kernel in terms of a solution to the complex Monge–Ampère equation and is in fact a quantitative version of the following result of Ohsawa [96]:

Theorem 6.4 *Bounded hyperconvex domains are Bergman exhaustive.*

It turns out that getting optimal constant in Herbort's estimate (6.5) and especially in (6.7) can be extremely useful. Herbort originally obtained the constant

$$c_n=1+4e^{4n+3+R^2},$$

so it depended in addition on the diameter R of Ω. If we look at the proof of Theorem 6.3 closer and choose χ a bit more carefully then we can improve the constant obtained there considerably. Take $\chi\in C^{0,1}((-\infty,0))$ such that $\chi(t)=0$ for $t\geq -1$ and for $t<-1$ choose it in such a way that $t\chi'(t)e^{-nt}=-1$, that is

$$\chi(t)=\begin{cases}0 & t\geq -1\\ \int_1^{-t}\frac{ds}{se^{ns}} & t<-1\end{cases}. \tag{6.8}$$

Then $F(w)=\chi(-\infty)f(w)$ and as in [23] we will get

$$c_n=\left(1+\frac{C}{\mathrm{Ei}\,(n)}\right)^2, \tag{6.9}$$

where

$$\mathrm{Ei}\,(a) = \int_a^\infty \frac{ds}{se^s}$$

and C is the constant from Theorem 2.2 (we know that $C = 4$ is optimal there). We will determine the optimal c_n in Sect. 7.

Bergman completeness of a bounded domain is equivalent to the fact that $\mathrm{dist}\,^B_\Omega(z, w) \to \infty$ as $z \to \partial\Omega$ and w is fixed. Theorem 6.2 does not give any quantitative version of this, even in terms of pluripotential theory. Diederich and Ohsawa [49] showed a lower bound for the Bergman distance for bounded pseudoconvex domains with C^2 boundary implying in particular completeness, this was later improved in [23]:

$$\mathrm{dist}\,^B_\Omega(z, w) \geq \frac{-\log \delta_\Omega(z)}{C \log(-\log \delta_\Omega(z))}, \tag{6.10}$$

where C is a positive constant depending only on Ω and w. The proof used the following estimate from [23] for the Bergman distance in terms of pluripotential theory:

Theorem 6.5 *Let Ω be pseudoconvex in $\mathbb{C}^n$ and assume that $z, w \in \Omega$ are such that*

$$\{G_\Omega(\cdot, z) < -1\} \cap \{G_\Omega(\cdot, w) < -1\} = \emptyset. \tag{6.11}$$

Then

$$\cdot\frac{|K_\Omega(z, w)|}{\sqrt{K_\Omega(z, z)K_\Omega(w, w)}} \leq \frac{1}{\sqrt{1 + a_n^2}}, \tag{6.12}$$

where

$$a_n = \left(1 + \frac{2e^n}{\mathrm{Ei}\,(n)}\right)^{-1},$$

and

$$\mathrm{dist}\,^B_\Omega(z, w) \geq \arctan a_n. \tag{6.13}$$

Proof First note that (6.13) follows directly from (6.12) and (6.3). The proof of (6.12) will be similar to that of Theorem 6.3. We may assume that Ω is bounded and hyperconvex. We will use Theorem 2.2 with

$$\varphi = 2n(G_z + G_w), \quad \psi = -\log(-G_z),$$

where $G_z = G_\Omega(\cdot, z)$. Set

$$f := \frac{K_\Omega(\cdot, w)}{\sqrt{K_\Omega(w, w)}} \in A^2(\Omega),$$

so that $||f|| = 1$, and

$$\alpha := \bar\partial(f\,\chi\circ G_z) = f\,\chi'\circ G_z\,\bar\partial G_z,$$

where χ is given by (6.8). We can find continuous u in Ω solving $\bar\partial u = \alpha$ and such that

$$\begin{aligned}\int_\Omega |u|^2 d\lambda \le \int_\Omega |u|^2 e^{-\varphi} d\lambda &\le 4\int_\Omega |f|^2 G_z^2 (\chi'\circ G_z)^2 e^{-2n(G_z+G_w)} d\lambda\\ &\le 4e^{2n}\int_{\{G_z<-1\}} |f|^2 d\lambda,\end{aligned}$$

where the last inequality follows from (6.11). We thus get $||u|| \le 2e^n$ (because $||f|| = 1$) and, since $e^{-\varphi}$ is not locally integrable near both z and w, that $u(z) = u(w) = 0$. The function $F = f\chi\circ G_z - u$ is thus holomorphic and such that $F(z) = \mathrm{Ei}\,(n) f(z)$, $F(w) = 0$ (the latter by (6.11) again). We also have $||F|| \le \mathrm{Ei}\,(n) + 2e^n$.

By the definition of f

$$\langle F, f\rangle = \frac{F(w)}{\sqrt{K_\Omega(w,w)}} = 0.$$

Therefore by (6.2)

$$K_\Omega(z,z) \ge |f(z)|^2 + \frac{|F(z)|^2}{||F||^2} \ge |f(z)|^2\left(1 + a_n^2\right),$$

and (6.12) follows. □

Theorem 6.5 reduced the proof of (6.10) in [23] to right estimates for the pluricomplex Green function, as in [63]. Since the only information really needed is (5.3) with $a = b$, by [61] the estimate (6.10) also holds for pseudoconvex domains with Lipschitz boundary. It is an open problem whether (6.10) can be improved to

$$\mathrm{dist}\,_\Omega^B(z,w) \ge \frac{1}{C}(-\log\delta_\Omega(z)),$$

which would be optimal. This estimate is known to hold for smooth strongly pseudoconvex domains and also for convex ones (without any regularity assumption, see [23]).

Lu Qi-Keng [87] showed that if the Bergman metric has constant sectional curvature then it is biholmorphic to a ball. A conjecture of Cheng asserts that this assumption can be weakened for smooth strongly pseudoconvex domains. It states that such a domain is biholomorphic to a ball if and only if its Bergman metric is Kähler–Einstein, that is its Ricci curvature is proportional to the metric. For $n = 1$ it follows from [87] and for $n = 2$ it was shown by Nemirovskii and Shafikov [94]. It remains open in higher dimensions.

7 Suita conjecture

Let D be a domain in $\mathbb{C}$ admitting the Green function which means exactly that the complement of D is not polar. For $z \in D$ set

$$c_D(z) := \exp \lim_{\zeta \to z} (G_D(\zeta, z) - \log|\zeta - z|).$$

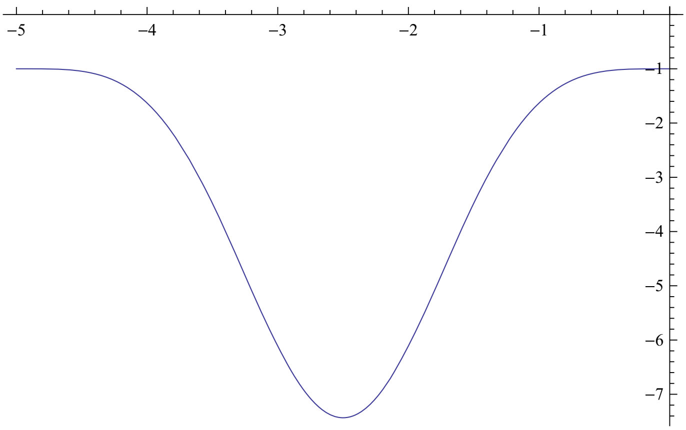
It is in fact the logarithmic capacity of the complement of D with respect to z and the function under the exponent is called the *Robin function* for G_D. The function c_D is not biholomorphically invariant but one can easily check that the metric $c_D|dz|$ does not depend on a local holomorphic change of variables and thus is an invariant metric even for Riemann surfaces. It is called the *Suita metric* of D. Its curvature is given by

$$Curv_{c_D|dz|} = -\frac{\partial^2(\log c_D)/\partial z \partial \bar{z}}{c_D^2}.$$

Suita [110] conjectured that

$$Curv_{c_D|dz|} \leq -1. \tag{7.1}$$

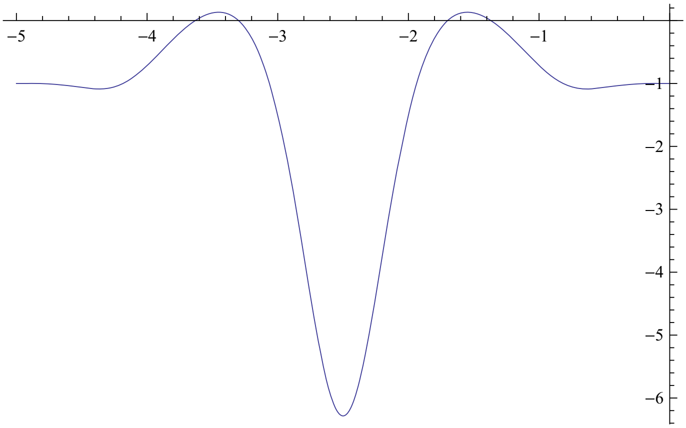
It is easy to see that we have equality for a disk and thus for simply connected domains. Using elliptic functions Suita showed that one has strict inequality in (7.1) if D is an annulus, and thus also any regular doubly connected domain. In fact, for $D = \{e^{-5} < |z| < 1\}$ the graph of $Curv_{c_D|dz|}$ as a function of $\log|z|$ looks as follows:[1]



By approximation it is enough to verify (7.1) for bounded smooth D and then one can show that we have equality in (7.1) on the boundary. Therefore the Suita conjecture essentially asks whether the curvature of the Suita metric satisfies the maximum principle. This is in fact a rather rare situation for invariant metrics in

[1] Figures were obtained using *Mathematica*.

complex analysis, for example it is not satisfied for the Bergman metric. For the same annulus as before we will then have the following picture:



See [51] and [119] for specific results on the curvature of the Bergman metric on an annulus.

Surprisingly, it turned out that only the methods of several complex variables have given any real progress in this one-dimensional problem. It was the breakthrough of Ohsawa [97] who noticed that it is really an extension problem closely related to the Ohsawa–Takegoshi theorem. It was proved already by Suita [110] that

$$\frac{\partial^2(\log c_D(z))}{\partial z\partial\bar{z}} = \pi K_D(z,z),$$

this in fact follows easily from the Schiffer formula

$$K_D(z,w) = \frac{2}{\pi}\frac{\partial^2 G_D(z,w)}{\partial z\partial\bar{w}}, \quad z \neq w,$$

and therefore (7.1) is equivalent to

$$c_D(z)^2 \le \pi K_D(z,z). \tag{7.2}$$

But this is in turn equivalent to the following extension problem: for a given $z \in D$ find $f \in \mathcal{O}(D)$ such that $f(z) = 1$ and

$$\int_D |f(z)|^2 d\lambda \le \frac{\pi}{c_D(z)^2}.$$

Ohsawa [97], using the same methods as in the original proof of the Ohsawa–Takegoshi theorem, proved the estimate

$$c_D(z)^2 \le C K_D(z,z).$$

for some large absolute constant C. It was later improved in [25] and [59].

The optimal constant was eventually obtained in [27] where the following version of the Ohsawa–Takegoshi theorem also with optimal constant was proved:

Theorem 7.1 *Assume that D is a domain in $\mathbb{C}$ containing the origin. Let $\Omega \subset \mathbb{C}^{n-1} \times D$ be pseudoconvex, $\varphi \in PSH(\Omega)$, and set $\Omega' := \Omega \cap \{z_n = 0\}$. Then for any $f \in \mathcal{O}(\Omega')$ there exists $F \in \mathcal{O}(\Omega)$ such that $F = f$ on Ω' and*

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{c_D(0)^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

For $n = 1$ we obtain the Suita conjecture (7.2).

The proof of Theorem 7.1 was similar to that of Theorem 3.2 but Theorem 2.4 was used instead of Theorem 2.3 and the weights were chosen more carefully. Theorem 2.4 was used in [27] with weights of the form

$$\widetilde{\varphi} = \varphi + 2G + \eta(-2G), \quad \psi = \gamma(-2G),$$

where $G = G_D(\cdot, 0)$. It was rather straightforward, although technical, how to define $\eta(t)$ and $\gamma(t)$ for $t \geq -2 \log \varepsilon$ (that is $\widetilde{\varphi}$ and ψ near $\{z_n = 0\}$). The main problem was to construct h, g on $(0, \infty)$ behaving like $-\log t$ near ∞ and such that

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1. \tag{7.3}$$

Eventually it turned out that solutions can be written explicitly:

$$\begin{aligned} h(t) &:= -\log(t + e^{-t} - 1) \\ g(t) &:= -\log(t + e^{-t} - 1) + \log(1 - e^{-t}) \end{aligned}$$

and we even have equality in (7.3). In fact, when a similar method was used earlier in [26] but with Theorem 2.3 instead of 2.4, it lead to an ODE with only one unknown:

$$\left(1 - \frac{(g')^2}{g''}\right) e^{g+t} \geq 1$$

and the best constant one can get this way is 1.95388..., the same as the one obtained earlier in [59].

After [27], Guan and Zhou [57] proved various generalizations of Theorem 7.1 but used essentially the same ODE with two unknowns as (7.3) and got essentially the same solutions. They also characterized precisely the case when there is equality in (7.2) answering a more precise question posed by Suita [110]:

Theorem 7.2 *Let M be a Riemann surface admitting the Green function (which is equivalent to the fact that there exists a bounded nonconstant subharmonic function on M). Then (7.2) holds and if we have equality for some $z \in M$ then M is biholomorphic to $\Delta \setminus F$ where F is a closed polar subset of Δ.*

Another approach to the Suita conjecture was presented in [28]. The idea was to obtain optimal constants in (6.7) for arbitrary sublevel sets. It turned out that the constant obtained already can be improved to the optimal one quite easily using the tensor power trick. The following general lower bound for the Bergman kernel on the diagonal was obtained:

Theorem 7.3 *Let Ω be pseudoconvex, $w \in \Omega$ and $t \leq 0$. Then*

$$K_\Omega(w, w) \geq \frac{1}{e^{-2nt}\lambda(\{G_\Omega(\cdot, w) < t\})}. \tag{7.4}$$

Proof Repeating the argument of the proof of Theorem 6.3 with the constant given by (6.9) for $f \equiv 1$ and arbitrary t we will obtain

$$K_\Omega(w, w) \geq \frac{c(n, t)}{\lambda(\{G_\Omega(\cdot, w) < t\})}, \tag{7.5}$$

where

$$c(n, t) = \left(1 + \frac{C}{\mathrm{Ei}\,(nt)}\right)^2$$

and C is the constant from Theorem 2.2. We now use the tensor power trick: for a positive integer m take $\widetilde{\Omega} = \Omega^m \subset \mathbb{C}^{nm}$ and $\widetilde{w} = (w, \dots, w)$. Then by the product properties for the Bergman kernel (see e.g. [70]) and for the pluricomplex Green function, Theorem 5.2, we have

$$K_{\widetilde{\Omega}}(\widetilde{w}, \widetilde{w}) = (K_\Omega(w))^m, \quad \{G_{\widetilde{\Omega}}(\cdot, \widetilde{w}) < t\} = \{G_\Omega(\cdot, w) < t\}^m,$$

and thus by (7.5)

$$K_\Omega(w, w) \geq \frac{c(nm, t)^{1/m}}{\lambda(\{G_\Omega(\cdot, w) < t\})}.$$

We can however easily check that

$$\lim_{m\to\infty} c(nm, t)^{1/m} = e^{2nt}$$

and the theorem follows. □

Of course the same method gives the optimal version of the Herbort estimate (6.5) for arbitrary sublevel set:

$$\frac{|f(w)|^2}{K_\Omega(w, w)} \leq e^{-2nt} \int_{\{G_\Omega(\cdot, w) < t\}} |f|^2 d\lambda.$$

It is now the most interesting what happens with the right-hand side of (7.4) as $t \to -\infty$. For $n = 1$ we can write

$$G_\Omega(z, w) = \log|z - w| + \varphi(z),$$

where φ is harmonic in Ω. Denoting by M_t and m_t the supremum and infimum of φ over $\{G_\Omega(\cdot, w) < t\}$, respectively, we see that

$$\Delta(w, e^{t-M_t}) \subset \{G_\Omega(\cdot, w) < t\} \subset \Delta(w, e^{t-m_t})$$

and therefore

$$\lim_{t\to-\infty} e^{-2t}\lambda(\{G_\Omega(\cdot, w) < t\}) = \pi e^{-2\varphi(w)} = \frac{\pi}{c_\Omega(w)^2}.$$

We have thus obtained another proof of the Suita conjecture (7.2). Unlike the previous one which could have been presented entirely in dimension one, this one makes direct use of arbitrarily many complex variables to prove a one-dimensional result—the tensor power trick is crucial in this approach. Observe that this trick does not seem to work in another bound for the Bergman kernel (6.12)—there the constant

$$\left(\frac{1}{\sqrt{1+a_{nm}^2}}\right)^{1/m}$$

increases to 1 as m increases to ∞, so in fact we get worse estimate than the original one.

In higher dimensions we have the following recent result from [31]:

Theorem 7.4 *Let Ω be bounded and hyperconvex. Then*

$$\lim_{t\to-\infty} e^{-2nt}\lambda(\{G_\Omega(\cdot, w) < t\}) = \lambda(I^A_\Omega(w)),$$

where

$$I^A_\Omega(w) = \left\{X \in \mathbb{C}^n : \overline{\lim_{\zeta\to 0}}(G_\Omega(w+\zeta X, w) - \log|\zeta|) < 0\right\}$$

is the Azukawa indicatrix of Ω at w.

Proof We may assume that $w = 0$. Write $G := G_{\Omega,0}$, $I_t := e^{-t}\{G < t\}$. By Zwonek [117] the function

$$A(X) = \overline{\lim_{\zeta\to 0}}(G(\zeta X) - \log|\zeta|)$$

is continuous on $\mathbb{C}^n$ and $\overline{\lim}$ is equal to lim. Therefore

$$A(X) = \lim_{t\to-\infty}(G(e^t X) - t)$$

and by the Lebesgue bounded convergence theorem

$$\lim_{t\to-\infty} \lambda(I_t) = \lambda(\{A < 0\})$$

(if Ω is contained in $B(0, R)$ then so is I_t). $\square$

Combining this with Theorem 7.3 by approximation we thus obtain the following multidimensional version of the Suita conjecture:

Theorem 7.5 *For a pseudoconvex Ω and $w \in \Omega$ one has*

$$K_\Omega(w, w) \geq \frac{1}{\lambda(I_\Omega^A(w))}. \qquad \square$$

It should be mentioned that recently Lempert [86] gave another proof of Theorem 7.3. He observed that considering the following pseudoconvex domain in $\mathbb{C}^{n+1}$

$$\{(z, \zeta) \in \Omega \colon G_\Omega(z, w) + \operatorname{Re} \zeta < 0\}$$

and using the result on log-plurisubharmoncity of sections of the Bergman kernel due to Maitani and Yamaguchi [89] for $n = 1$ and Berndtsson [6] for arbitrary n, one can get that the function $\log K_{\{G_\Omega(\cdot,w)<t\}}(w, w)$ is convex in t. For $r > 0$ with $B(w, r) \subset \Omega$ we have

$$\log K_{\{G_\Omega(\cdot,w)<t\}}(w, w) \leq -\log \lambda(B(w, re^t)),$$

and therefore the function

$$2nt + \log K_{\{G_\Omega(\cdot,w)<t\}}(w, w)$$

is convex and bounded from above on $(-\infty, 0]$, hence non-decreasing. We get

$$K_\Omega(w, w) \geq e^{2nt} K_{\{G_\Omega(\cdot,w)<t\}}(w, w) \geq \frac{e^{2nt}}{\lambda(\{G_\Omega(\cdot, w) < t\})},$$

since we can always take $f \equiv 1$ in (6.1). This gives another proof of the one-dimensional Suita conjecture, this time making crucial use of two complex variables.

Berndtsson and Lempert [12] very recently improved this method to obtain the Ohsawa–Takegoshi theorem with optimal constant as well. They use a stronger tool than log-plurisubharmoncity of sections of the Bergman kernel, namely Berndtsson's positivity of direct image bundles [8].

8 Suita conjecture for convex domains in $\mathbb{C}^n$

Theorems 7.3 and 7.5 seem to be especially interesting when Ω is convex. Then it is known, see [70], that the Lempert theory [85] implies that the Azukawa indicatrix $I_\Omega^A(w)$ is equal to the Kobayashi indicatrix

$$I_\Omega^K(w) = \{\varphi'(0) \colon \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}.$$

We thus have the following estimate from [28]:

Theorem 8.1 *For $w \in \Omega \subset \mathbb{C}^n$, where Ω is a convex domain, we have*

$$K_\Omega(w, w) \geq \frac{1}{\lambda(I_\Omega^K(w))}.$$

□

In this case, it turns out that a similar upper bound for the Bergman kernel also holds. We have the following result from [31]:

Theorem 8.2 *Under the assumptions of Theorem 8.1 we have*

$$K_\Omega(w, w) \leq \frac{4^n}{\lambda(I_\Omega^K(w))}.$$

If Ω is in addition symmetric with respect to w than the constant 4 above can be replaced with $16/\pi^2 = 1.621\ldots$

Proof Assume that $w = 0$ and let I be the interior of $I_\Omega^K(0)$. We will show that $I \subset 2\Omega$, then since I is balanced (that is $z \in I$ implies $\zeta z \in I$ for $\zeta \in \bar{\Delta}$) we will have

$$K_\Omega(0, 0) \leq K_{I/2}(0, 0) = \frac{1}{\lambda(I/2)} = \frac{4^n}{\lambda(I)}.$$

The proof that $I \subset 2\Omega$ will be similar to the proof of Proposition 1 in [95]. For $X = \varphi'(0) \in \bar{I}$ by L denote the complex line generated by X. Let a be the point from $L \cap \partial\Omega$ with the smallest distance to the origin, write it as $a = \zeta_0 X$. We want to show that $|X| \leq 2|a|$, that is that $|\zeta_0| \geq 1/2$.

Let H be the complex supporting hyperplane in $\mathbb{C}^n$ to Ω at a, that is $H \cap \Omega = \emptyset$ and $a \in H$. Without loss of generality we may assume that $H = \{z_n = a_n\}$. Let D be a half-plane in $\mathbb{C}$ containing the image of the projection of Ω to the nth variable and such that $a_n \in \partial D$. Then φ_n, the nth component of φ, belongs to $\mathcal{O}(\Delta, D)$ and $\varphi_n(0) = 0$. By the Schwarz lemma $|X_n| = |\varphi_n'(0)| \leq 2|a_n|$ which implies that $|\zeta_0| \geq 1/2$.

If Ω is in addition symmetric then as D we may take a strip instead of a half-plane and then $|\varphi_n'(0)| \leq (4/\pi)|a_n|$. □

We have thus seen that for convex Ω the biholomorphically invariant function

$$F_\Omega(w) := (K_\Omega(w, w)\lambda(I_\Omega^K(w)))^{1/n}$$

satisfies

$$1 \leq F_\Omega \leq 4.$$

The lower bound was obtained using the $\bar{\partial}$-equation whereas the proof of the upper bound was relatively elementary. The lower bound is optimal—for example if Ω is balanced with respect to w then we have equality—and it would be interesting to find an optimal upper bound. It is in fact not so trivial to prove that we may at all have

$F_\Omega(w) > 1$. This was done in [31] and [32] where F_Ω was computed for certain complex convex ellipsoids and some w. Here are two results:

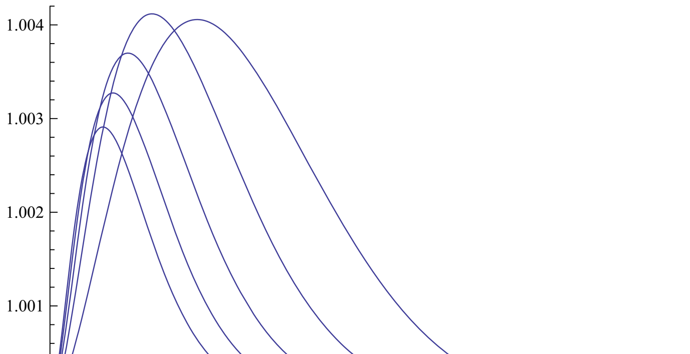
Theorem 8.3 *For* $\Omega = \{z \in \mathbb{C}^n : |z_1| + \cdots + |z_n| < 1\}$ *and* $w = (b, 0, \ldots, 0)$, *where* $0 \le b < 1$, *one has*

$$K_\Omega(w)\lambda(I_\Omega^K(w)) = 1 + (1-b)^{2n}\frac{(1+b)^{2n} - (1-b)^{2n} - 4nb}{4nb(1+b)^{2n}}$$
$$= 1 + \frac{(1-b)^{2n}}{(1+b)^{2n}}\sum_{j=1}^{n-1}\frac{1}{2j+1}\binom{2n-1}{2j}b^{2j}.$$

The proof uses the formula for the Bergman kernel for this ellipsoid

$$K_\Omega((b, 0, \ldots, 0)) = \frac{2n-1}{4\pi\omega b}((1-b)^{-2n} - (1+b)^{-2n}),$$

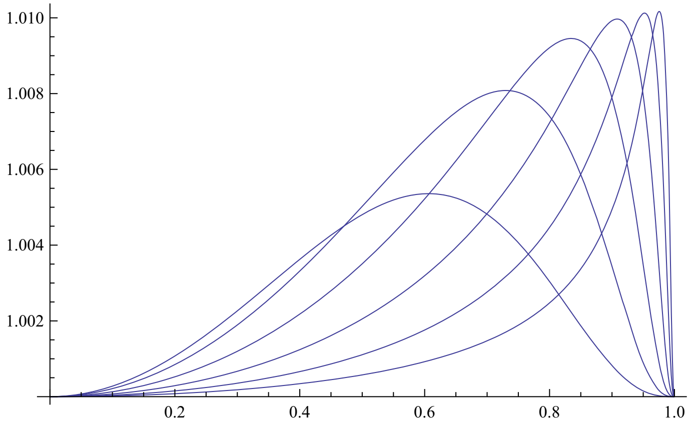
where $\omega = \lambda(\{z \in \mathbb{C}^{n-1} : |z_1| + \cdots + |z_{n-1}| < 1\})$, obtained from the deflation method of Boas–Fu–Straube [33]. The main part of the proof was to compute $\lambda(I_\Omega^K(w))$. For that the formula of Jarnicki–Pflug–Zeinstra [71] for geodesics in convex complex ellipsoids (which is based on Lempert's theory [85]) was used. Here are the resulting graphs of $F_\Omega(b, 0, \ldots, 0)$ for $n = 2, 3, \ldots, 6$:



Theorem 8.4 *For* $m \ge 1/2$ *set* $\Omega_m := \{|z_1|^{2m} + |z_2|^2 < 1\}$ *and* $w = (b, 0)$ *where* $0 \le b < 1$. *Then*

$$\lambda(I_{\Omega_m}((b,0))) = \pi^2\Bigg[-\frac{m-1}{2m(3m-2)(3m-1)}b^{6m+2} - \frac{3(m-1)}{2m(m-2)(m+1)}b^{2m+2}$$
$$+\frac{m}{2(m-2)(3m-2)}b^6 + \frac{3m}{3m-1}b^4 - \frac{4m-1}{2m}b^2 + \frac{m}{m+1}\Bigg].$$

Some computations leading to this formula were done with the help of *Mathematica*. The Kobayashi function for this ellipsoid was computed implicitly by Blank–Fan–Klein–Krantz–Ma–Pang [13] (explicitly up to solving a real equation which is a polynomial one of degree $2m$ if m is an integer) and this had only sufficed for numerical computations of $\lambda(I^K_{\Omega_m}(w))$. It turns out however that just the indicatrix $I^K_{\Omega_m}(w)$ and its volume can be described with explicit although rather complicated formulas. Here is the graph of $F_{\Omega_m}(b,0)$ for $m = 4, 8, 16, 32, 64$ and 128:



In this particular case all values of $F_{\Omega_m}(w)$ are attained for $w = (b,0)$, $0 < b < 1$. One can compute numerically that

$$\sup_{m\geq 1/2}\ \sup_{\Omega_m} F_{\Omega_m} = 1.010182\ldots$$

and this is the highest value of F_Ω for convex Ω in any dimension we have been able to obtain so far. It seems that the lower bound given by Theorem 8.1 is very accurate.

9 Mahler conjecture and Bourgain–Milman inequality

Let K be a convex symmetric (that is $K = -K$) body (that is K is compact and has non-empty interior) in $\mathbb{R}^n$. Its *dual* is defined by

$$K' := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } x \in K\},$$

where $\cdot$ denotes the inner product in $\mathbb{R}^n$. The number

$$\lambda(K)\lambda(K')$$

is called the *Mahler volume* of K. One can show that it is independent of linear transformations of $\mathbb{R}^n$ and of the choice of the inner product. It is thus an invariant of the n-dimensional real Banach space whose unit ball is K. Santaló [105] showed that the Mahler volume is maximized for balls (it was earlier proved by Blaschke in dimensions 2 and 3) and by Saint–Raymond [104] these are the only maximizers (up to linear transformations). For a proof of the Blaschke–Santaló inequality using the $\bar\partial$-equation see [41].

Mahler [88] conjectured that the Mahler volume is maximized by cubes. He proved it in dimension 2 and the problem still remains open in higher dimensions. This, together with the Blaschke–Santaló inequality, would mean that the Mahler volume is biggest for the roundest convex symmetric bodies and smallest for the least round. One of the difficulties with the Mahler conjecture is that, if true, cubes cannot be the only minimizers, even up to linear transformations. The other candidates are the so called *Hansen–Lima bodies* [60]: in $\mathbb{R}$ these are symmetric closed intervals and in higher dimensions they are produced by taking either products of lower dimensional Hansen–Lima bodies or a dual. This way we do not get anything new in $\mathbb{R}^2$, since the dual of $[-1,1]^2$ is the linearly equivalent rhombus $\{|x_1|+|x_2|\le 1\}$. However, already in dimension 3 the dual of the unit cube $[-1,1]^3$ is the octahedron $\{|x_1|+|x_2|+|x_3|<1\}$ and they are not linearly equivalent. These two are the only Hansen–Lima bodies in $\mathbb{R}^3$ and there are more in higher dimensions. It is conjectured that Hansen–Lima bodies are the only minimizers of the Mahler volume (up to linear transformations).

An important lower bound for the Mahler volume is the Bourgain–Milman inequality [34]:

Theorem 9.1 *There exists an absolute constant $c>0$ such that for a symmetric convex body K in $\mathbb{R}^n$ one has*

$$\lambda(K)\lambda(K')\ge c^n\frac{4^n}{n!}. \tag{9.1}$$

Since the Mahler volume of a cube in $\mathbb{R}^n$ is equal to $4^n/n!$, the Mahler conjecture is equivalent to (9.1) with $c=1$. The original proof from [34] was qualitative, it did not give any particular value of c. So far the best constant in (9.1) was obtained by Kuperberg [84] who proved it with $c=\pi/4$. Recently Nazarov [93] gave a different proof of the Bourgain–Milman inequality and although he obtained a worse constant than Kuperberg, namely $c=(\pi/4)^3$, his proof was very interesting from our point of view because he used several complex variables and Hörmander's estimate. In [28] it was shown that Theorem 8.1 can be used in Nazarov's approach instead but of course Hörmander's estimate is hidden there.

Before we present Nazarov's proof of Theorem 9.1, let us look at his equivalent formulation of the Mahler conjecture as a problem in several complex variables. For $u\in L^2(K')$ and its Fourier transform $\widehat{u}\in\mathcal{O}(\mathbb{C}^n)$ by the Schwarz inequality and the Plancherel formula we have

$$|\widehat{u}(0)|^2=\left|\int_{K'}u\,d\lambda\right|^2\le\lambda(K')||u||^2_{L^2(K')}=(2\pi)^{-n}\lambda(K')||\widehat{u}||^2_{L^2(\mathbb{R}^n)}$$

and the equality holds if u is the characteristic function of K'. Therefore

$$\lambda(K')=(2\pi)^n\sup_{f\in\mathcal{P}}\frac{|f(0)|^2}{||f||^2_{L^2(\mathbb{R}^n)}}, \tag{9.2}$$

where

$$\mathcal{P}=\{\widehat{u}\colon u\in L^2(K')\}$$

is a family of entire holomorphic functions. In fact, using the Paley–Wiener theorem one can completely characterize the class $\mathcal{P}$ (see e.g. [103] for details): it consists of those $f \in \mathcal{O}(\mathbb{C}^n)$ that are of exponential growth (that is $|f(z)| \le Ce^{C|z|}$ for some constant C) and such that

$$|f(iy)| \le Ce^{q_K(y)}, \quad y \in \mathbb{R}^n,$$

where q_K is the Minkowski function for K (that is the norm in $\mathbb{R}^n$ with unit ball K). The usefulness of the formula for the volume of the dual (9.2) is that K' itself does not appear on the right-hand side. Therefore the Mahler conjecture is equivalent to finding $f \in \mathcal{P}$ such that $f(0) = 1$ and

$$\int_{\mathbb{R}^n} |f(x)|^2 d\lambda(x) \le \frac{n!\pi^n}{2^n}\lambda(K).$$

Nazarov, instead of constructing a holomorphic function on the entire $\mathbb{C}^n$, considered the convex tube in $\mathbb{C}^n$ defined by K:

$$T_K := \operatorname{int} K + i\mathbb{R}^n.$$

He proved the following bounds for the Bergman kernel in T_K:

$$K_{T_K}(0, 0) \le \frac{n!}{\pi^n}\frac{\lambda(K')}{\lambda(K)} \tag{9.3}$$

and

$$K_{T_K}(0, 0) \ge \left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda(K))^2}. \tag{9.4}$$

Combining them we get (9.1) with $c = (\pi/4)^3$. Note that (9.4) follows immediately from Theorem 8.1 and the following:

Proposition 9.2 *For a convex symmetric body K in $\mathbb{R}^n$ we have*

$$I^K_{T_K}(0) \subset \frac{4}{\pi}(K + iK).$$

Proof Let Φ be a conformal mapping from the strip $(-1, 1) + i\mathbb{R}$ to Δ such that $\Phi(0) = 0$, then $|\Phi'(0)| = \pi/4$. Fix $y \in K'$, then

$$F(z) := \Phi(z \cdot y) \in \mathcal{O}(T_K, \Delta)$$

satisfies $F(0) = 0$. For $X = \varphi'(0) \in I^K_{T_K}(0)$ by the Schwarz lemma we have $|(F \circ \varphi)'(0)| \le 1$ and therefore $|X \cdot y| \le 4/\pi$. This means that

$$I^K_{T_K}(0) \subset \frac{4}{\pi}\{z \in \mathbb{C}^n : |z \cdot y| \le 1 \text{ for all } y \in K'\} \subset \frac{4}{\pi}(K'' + iK'')$$

and the proposition follows since $K'' = K$. □

For smooth strongly convex K Lempert's theory [85] can be used to obtain more precise description of $I^K_{T_K}(0)$ in terms of the Gauss mapping of ∂K, see [28].

As shown in [93] the upper bound (9.3) follows easily from the formula for the Bergman kernel in convex tube domains due to Rothaus [102] (see also [68]):

$$K_{T_K}(z,w) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i(z-\bar{w})\cdot y}}{J_K(y)} d\lambda(y),$$

where

$$J_K(y) = \int_K e^{-2x\cdot y} d\lambda(x).$$

Fix $y \in \mathbb{R}^n$ and $\widetilde{x} \in K$. Then, since K is symmetric,

$$J_K(y) \geq 2^{-n}\int_K e^{-(x+\widetilde{x})\cdot y} d\lambda(x) \geq 2^{-n}\lambda(K)e^{-\widetilde{x}\cdot y}.$$

Minimizing the right-hand side over $\widetilde{x}$ we get

$$J_K(y) \geq 2^{-n} e^{q_{K'}(y)}.$$

Since for any convex body K one has

$$\int_{\mathbb{R}^n} e^{-q_K} d\lambda = \int_{\mathbb{R}^n}\int_{q_K(y)}^{\infty} e^{-t} dt\, d\lambda(y) = \int_0^{\infty} e^{-t}\lambda(\{q_K < t\})dt = n!\lambda(K),$$

the upper bound (9.3) follows.

10 Isoperimetric inequalities and symmetrization

One of the interesting open problems is whether in the lower bound for the Bergman kernel (7.4) the right-hand side is monotone in t. This would mean in particular that the best bound is obtained when $t \to -\infty$, that is that Theorem 7.5 is the optimal version of this estimate. We start with the following result from [31] showing that this is indeed the case for $n = 1$:

Theorem 10.1 *Assume that $w \in \Omega \subset \mathbb{C}$. Then the function*

$$(-\infty, 0] \ni t \longmapsto e^{-2t}\lambda(\{G_\Omega(\cdot, w) < t\})$$

is non-decreasing.

Proof With the notation $G := G_\Omega(\cdot, w)$ set

$$f(t) := \log\lambda(\{G < t\}) - 2t.$$

It is enough to show that if t is a regular value of G then $f'(t) \geq 0$. By the co-area formula

$$\lambda(\{G < t\}) = \int_{-\infty}^{t} \int_{\{G=s\}} \frac{d\sigma}{|\nabla G|}\, ds$$

and therefore by the Schwarz inequality

$$\frac{d}{dt}\lambda(\{G < t\}) = \int_{\{G=t\}} \frac{d\sigma}{|\nabla G|} \geq \frac{\sigma(\{G = t\})^2}{\displaystyle\int_{\{G=t\}} |\nabla G| d\sigma}.$$

We have

$$\int_{\{G=t\}} |\nabla G| d\sigma = \int_{\{G=t\}} \frac{\partial G}{\partial n} d\sigma = \int_{\{G<t\}} \Delta G = 2\pi$$

and by the isoperimetric inequality

$$\sigma(\{G = t\})^2 \geq 4\pi\lambda(\{G < t\}).$$

It follows that

$$f'(t) = \frac{\displaystyle\int_{\{G=t\}} \frac{d\sigma}{|\nabla G|}}{\lambda(\{G < t\})} - 2 \geq 0.$$

□

Note that the proof also shows that the problem whether for pseudoconvex $\Omega \subset \mathbb{C}^n$ the function

$$(-\infty, 0] \ni t \longmapsto e^{-2nt}\lambda(\{G_\Omega(\cdot, w) < t\}) \tag{10.1}$$

is non-decreasing is equivalent to the following "pluripolar isoperimetric inequality": if Ω is bounded, smooth and strongly pseudoconvex in $\mathbb{C}^n$ then for $w \in \Omega$ one has

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G_\Omega(\cdot, w)|} \geq 2\lambda(\Omega).$$

Similarly as in Lempert's proof of Theorem 7.3, the monotonicity of (10.1) would follow if we knew that the function

$$(-\infty, 0] \ni t \longmapsto \log \lambda(\{G_\Omega(\cdot, w) < t\})$$

was convex. This was conjectured in [31] but Fornæss [55] found a counterexample to that already for $n = 1$.

The method of proof of Theorem 10.1 was in fact inspired by the proof of a symmetrization result for the Laplacian due to Talenti [111]. For a measurable subset A in $\mathbb{R}^n$ its *Schwarz symmetrization* (or *rearrangement*) A^* is the ball centered at the origin such that $\lambda(A^*) = \lambda(A)$. For a nonnegative measurable function f defined on a measurable subset A of $\mathbb{R}^n$ its Schwarz symmetrization f^* is the radially symmetric (that is $f^*(x)$ depends only on $|x|$) function defined on A^* which is non-increasing in radius and such that $\lambda(\{f^* > t\}) = \lambda(\{f > t\})$ for every real t. If f is nonpositive than we set $f^* := -(-f)^*$ or equivalently require that f^* is non-decreasing in radius and the volumes of sublevel (instead of superlevel) sets are the same. One of the useful properties of rearrangements is that they preserve the L^p-norms, or more generally

$$\int_{\Omega^*} \gamma(|f^*|)\, d\lambda = \int_{\Omega} \gamma(|f|)\, d\lambda$$

for any increasing γ and f either nonpositive or nonnegative. For an introduction to rearrangements we refer to [36].

Talenti [111] proved the following:

Theorem 10.2 *Let Ω be a bounded regular domain in $\mathbb{R}^n$ and let u be a (possibly weak) solution to the following Dirichlet problem*

$$\begin{cases} \Delta u = f \geq 0 & \text{in } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega \end{cases}.$$

If v solves

$$\begin{cases} \Delta v = f^* & \text{in } \ \Omega^* \\ v = 0 & \text{on } \ \partial\Omega^* \end{cases},$$

then $v \leq u^*$ in Ω^*.

Proof By approximation we may assume that u is smooth and strongly subharmonic. By the Hardy–Littlewood inequality for $t \leq 0$ we have

$$\int_{\{u<t\}} f\, d\lambda \leq \int_{\{u^*<t\}} f^* d\lambda = \int_{B(0,r)} \Delta v\, d\lambda = n\omega_n r^{n-1}\gamma'(r),$$

where r is such that $\{u^* < t\} = B(0,r)$, $v(x) = \gamma(|x|)$, ω_n is the volume of the unit ball in $\mathbb{R}^n$, and the last equality follows from the fact that

$$\Delta v = \gamma'' + (n-1)\frac{\gamma'}{r} = r^{1-n}\frac{d}{dr}(r^{n-1}\gamma').$$

On the other hand, if t is a regular value of u then by the Schwarz inequality

$$\int_{\{u<t\}} f\,d\lambda = \int_{\{u=t\}} |\nabla u| d\sigma \geq \frac{\sigma(\{u=t\})^2}{\displaystyle\int_{\{u=t\}} \frac{d\sigma}{|\nabla u|}}.$$

By the isoperimetric inequality

$$\sigma(\{u=t\}) \geq n\omega_n^{1/n} \lambda(\{u<t\})^{1-1/n}$$

and by the co-area formula

$$\int_{\{u=t\}} \frac{d\sigma}{|\nabla u|} = \frac{d}{dt}\lambda(\{u<t\}).$$

Therefore

$$\int_{\{u<t\}} f\,d\lambda \geq n^2\omega_n^{2/n} \frac{\lambda(\{u<t\})^{2-2/n}}{\dfrac{d}{dt}\lambda(\{u<t\})}.$$

Write $u^*(x) = \eta(|x|)$. Since $\{u^* < t\} = B(0,r)$, we have $t = \eta(r)$ and $\lambda(\{u<t\}) = \omega_n r^n$. Therefore

$$\int_{\{u<t\}} f\,d\lambda \geq n\omega_n r^{n-1}\eta'(r)$$

and it follows that $\eta' \leq \gamma'$. Since $\eta(R) = \gamma(R) = 0$, where $\Omega^* = B(0,R)$, we obtain that $\eta \geq \gamma$. □

For a corresponding symmetrization result for the real Monge–Ampère equation one has to symmetrize convex u with respect to a different measure. For a bounded convex domain Ω in $\mathbb{R}^n$ its quermassintegrals $V_m(\Omega)$, $m = 0, 1, \dots, n$, are defined by the formula

$$\lambda(\Omega + t\mathbb{B}) = \sum_{m=0}^{n} \binom{n}{m} V_{n-m}(\Omega)t^m,$$

where $\mathbb{B}$ is the unit ball in $\mathbb{R}^n$ and $t \geq 0$. Then $V_n(\Omega) = \lambda(\Omega)$, $V_{n-1}(\Omega) = \sigma(\partial\Omega)/n$ (if Ω is smooth) and $V_0(\Omega) = \omega_n$. We also have $V_m(B(0,r)) = \omega_n r^m$. Alexandrov–Fenchel inequalities state that the expression

$$(V_m(\Omega)/\omega_n)^{1/m}$$

is non-increasing in m and we have equality at any stage only for balls.

If Ω is in addition smooth then $V_m(\Omega)$ can be expressed in terms of an integral over $\partial\Omega$ of a proper curvature of $\partial\Omega$. If $\kappa_1, \dots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$ then the mth mean curvature of $\partial\Omega$, $m = 1, \dots, n-1$, is defined by

$$H_m := \sum_{1 \le i_1 < \dots < i_m \le n-1} \kappa_{i_1} \dots \kappa_{i_m}.$$

For $m = 0$ we set $H_0 \equiv 1$. Then H_1 is the mean curvature and H_{n-1} the Gauss curvature of $\partial\Omega$. Then for $m = 0, 1, \dots, n-1$ we have

$$V_m(\Omega) = \frac{1}{n\binom{n-1}{m}} \int_{\partial\Omega} H_{n-m-1} d\sigma.$$

We refer to [35] and [106] for more details.

By $\widetilde{\ }$ we will denote the symmetrization with respect to V_1 instead of the Lebesgue measure λ. Note that by the Alexandrov–Fenchel inequalities we have $\Omega^* \subset \widetilde{\Omega}$ and $\widetilde{u} \le u^*$ for negative convex u. We have the following result for the real Monge–Ampère equation due to Talenti [112] in dimension 2 and Tso [113] in the general case.

Theorem 10.3 *Let Ω be a bounded convex domain. Assume that u is a (possibly weak) convex solution to the Dirichlet problem*

$$\begin{cases} \det D^2 u = f \ge 0 & in\ \ \Omega \\ u = 0 & on\ \ \partial\Omega \end{cases}.$$

Extend f^* by 0 from Ω^* to $\widetilde{\Omega}$. If convex v solves

$$\begin{cases} \det D^2 v = f^* & in\ \ \widetilde{\Omega} \\ v = 0 & on\ \ \partial\widetilde{\Omega} \end{cases},$$

then $v \le \widetilde{u}$ in $\widetilde{\Omega}$.

Proof By approximation we may assume that u is smooth and strongly convex. Similarly as in the proof of Theorem 10.2 we have

$$\int_{\{u<t\}} f\, d\lambda \le \int_{\{u^*<t\}} f^* d\lambda \le \int_{\{\widetilde{u}<t\}} f^* d\lambda = \int_{B(0,r)} \det D^2 v = \omega_n(\gamma'(r))^n,$$

where r is such that $\{\widetilde{u} < t\} = B(0, r)$ and $v(x) = \gamma(|x|)$, so that

$$\det D^2 v = r^{1-n}(\gamma')^{n-1}\gamma'' = \frac{1}{n} r^{1-n} \frac{d}{dr}((\gamma')^n).$$

On the other hand for the regular value t of u we have by the Hölder inequality

$$\int_{\{u<t\}} f\,d\lambda = \int_{\{u=t\}} |\nabla u|^n H_{n-1} d\sigma \geq \frac{\left(\int_{\{u=t\}} H_{n-1} d\sigma\right)^{n+1}}{\left(\int_{\{u=t\}} \frac{H_{n-1}}{|\nabla u|} d\sigma\right)^n}.$$

We have

$$\int_{\{u=t\}} H_{n-1} d\sigma = n\omega_n$$

and by Reilly [101]

$$\int_{\{u=t\}} \frac{H_{n-1}}{|\nabla u|} d\sigma = \frac{1}{n-1} \frac{d}{dt} \int_{\{u=t\}} H_{n-2} d\sigma = n \frac{d}{dt} V_1(\{u < t\}). \tag{10.2}$$

If $\widetilde{u}(x) = \eta(|x|)$ then, since $\{\widetilde{u} < t\} = B(0, r)$, we have $t = \eta(r)$. We will obtain

$$\int_{\{u<t\}} f\,d\lambda \geq \omega_n (\eta'(r))^n,$$

and thus $\eta' \leq \gamma'$. Since $\eta(R) = \gamma(R) = 0$, where $\widetilde{\Omega} = B(0, R)$, we get $\eta \geq \gamma$. □

It would be very desirable to prove a similar result for the complex Monge–Ampère equation. This would in particular immediately imply the following important estimate of Kołodziej [81] (see also [82]):

Theorem 10.4 *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let be a solution to the following Dirichlet problem*

$$\begin{cases} u \in PSH(\Omega) \cap C(\Omega) \\ (dd^c u)^n = f d\lambda & in\ \ \Omega \\ u = 0 & on\ \ \partial\Omega \end{cases}.$$

Then for every $p > 1$ one has

$$\sup_{\Omega} |u| \leq C ||f||^{1/n}_{L^p(\Omega)},$$

where C is a constant depending on n, p and the diameter of Ω.

Similarly as for convex domains, for a smooth pseudoconvex Ω one can consider the Levi principal curvatures of the boundary $\lambda_1, \ldots, \lambda_{n-1}$ and define the mth complex mean curvature K_m similarly as H_m, so that $K = K_{n-1}$ is the Levi curvature of the boundary. See [92] and [90] for basic results on complex mean curvatures. If one tries to repeat the method of the proof of Theorem 10.3 then two problems appear: first is the lack of complex counterparts of the Alexandrov–Fenchel inequalities and secondly it is not clear what the Reilly formula (10.2) should look like in the complex case. It is also not at all clear what the right symmetrization $\tilde{}$ should be now. One of interesting conjectures that arise (although not sufficient to prove a symmetrization result for the complex Monge–Ampère equation), is the following: for a bounded smooth strongly pseudoconvex Ω in $\mathbb{C}^n$ the following complex isoperimetric inequality holds:

$$\int_{\partial\Omega} K\, d\sigma \geq 2n\sqrt{\omega_{2n}\lambda(\Omega)}$$

with equality exactly for balls.

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