PARTIAL DIFFERENTIAL EQUATIONS II

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1. Weak Differentiation

Regularization. Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be such that $\rho \geq 0$, $\rho(x)$ depends only on $|x|$, $\operatorname{supp}\rho \subset \bar{B}(0,1)$ and $\int \rho\, d\lambda = 1$. For $\varepsilon > 0$ set $\rho_\varepsilon(y) := \varepsilon^{-n}\rho(y/\varepsilon)$. Then $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\rho_\varepsilon \subset \bar{B}(0,\varepsilon)$ and $\int \rho_\varepsilon\, d\lambda = 1$. For any $u \in L^1_{loc}(\Omega)$ we set $u_\varepsilon := u * \rho_\varepsilon$, that is

$$\begin{aligned} u_\varepsilon(x) &= \int_\Omega u(y)\rho_\varepsilon(x-y)d\lambda(y) \\ &= \int_{B(0,\varepsilon)} u(x-y)\rho_\varepsilon(y)d\lambda(y) \\ &= \int_{B(0,1)} u(x-\varepsilon y)\rho(y)d\lambda(y) \end{aligned}$$

(note that the first integral is in fact over $B(x,\varepsilon)$). The function u_ε is defined in the set

$$\Omega_\varepsilon := \{x \in \Omega : B(x,\varepsilon) \subset \Omega\}.$$

Theorem 1.1. *i) $u_\varepsilon \to u$ pointwise almost everywhere as $\varepsilon \to 0$.*

ii) If $u \in C(\Omega)$ then $u_\varepsilon \to u$ locally uniformly as $\varepsilon \to 0$.

iii) For $p \geq 1$ if $u \in L^p_{loc}(\Omega)$ then $u_\varepsilon \to u$ in $L^p_{loc}(\Omega)$ (that is in $L^p_{loc}(\Omega')$ for $\Omega' \Subset \Omega$) as $\varepsilon \to 0$.

Proof. i) By the Lebesgue differentiation theorem for almost all x we have

$$\lim_{\varepsilon\to 0} \frac{1}{\lambda(B(x,\varepsilon))}\int_{B(x,r)} |u(y)-u(x)|\, d\lambda(y) = 0.$$

For such an x

$$\begin{aligned} |u_\varepsilon(x)-u(x)| &\leq \int_{B(x,\varepsilon)} \rho_\varepsilon(x-y)|u(y)-u(x)|d\lambda(y) \\ &\leq \frac{C}{\lambda(B(x,\varepsilon))}\int_{B(x,\varepsilon)} |u(y)-u(x)|\, d\lambda(y). \end{aligned}$$

ii) We have

$$|u_\varepsilon(x) - u(x)| \le \int_{B(0,\varepsilon)} |u(x-y) - u(x)|\rho_\varepsilon(y)d\lambda)y) \le \sup_{B(0,\varepsilon)} |u - u(x)|$$

and the convergence follows because continuous functions are locally uniformly continuous.

iii) We first estimate by Hölder's inequality

$$|u_\varepsilon(x)|^p \le \int_{B(0,\varepsilon)} |u(x-y)|^p \rho_\varepsilon(y)d\lambda(y).$$

Integrating over x we will get

$$\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \le \|u\|_{L^p(\Omega)}.$$

For every $\delta > 0$ there exists $v \in C_0(\Omega)$ with $\|v - u\|_p \le \delta$ (this is a consequence of Lusin's theorem). Then for sufficiently small ε

$$\|u_\varepsilon - u\| \le \|u_\varepsilon - v_\varepsilon\| + \|v_\varepsilon - v\| + \|v - u\|$$

(with norms in $L^p(\Omega')$ for a fixed $\Omega' \Subset \Omega$). We have

$$\|u_\varepsilon - v_\varepsilon\| \le \|u - v\|_p \le \delta,$$

thus

$$\|u_\varepsilon - u\| \le 2\delta + \|v_\varepsilon - v\|$$

and it is enough to use ii). $\square$

Weak differentiation. We will use the notation

$$D_j = \frac{\partial}{\partial x_j}, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$

where $j = 1, \dots, n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$. Ω will denote a domain in $\mathbb{R}^n$. By Stokes' theorem we have

$$\int_\Omega \varphi\, D^\alpha u\, d\lambda = (-1)^{|\alpha|} \int_\Omega u\, D^\alpha \varphi\, d\lambda$$

for $u \in C^{|\alpha|}(\Omega)$, $\varphi \in C_0^{|\alpha|}(\Omega)$. Now for $u, v \in L^1_{loc}(\Omega)$ we say that $v = D^\alpha u$ *in the weak sense* if

$$\int_\Omega \varphi\, v\, d\lambda = (-1)^{|\alpha|} \int_\Omega u\, D^\alpha \varphi\, d\lambda, \quad \varphi \in C_0^{|\alpha|}(\Omega).$$

The function v, if exists, is determined almost everywhere.

Exercise 1. *Set* $u(x) := |x| \in L^1_{loc}(\mathbb{R})$*. Show, directly from the definition, that* u' *does exist but* u'' *does not.*

One can easily show that for the weak differentiation we also have $D^\alpha D^\beta = D^{\alpha+\beta}$.

Differentiating under the sign of integration, we see that

$$D^\alpha u_\varepsilon = u * D^\alpha \rho_\varepsilon$$

(in the strong sense) and $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$.

Proposition 1.2. *If* $D^\alpha u$ *exists in the weak sense then*

$$D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon.$$

Proof. We have

$$\begin{aligned} D^\alpha u_\varepsilon(x) &= \int_\Omega u(y) D^\alpha \rho_\varepsilon(x-y) d\lambda(y) \\ &= (-1)^{|\alpha|} \int_\Omega u\, D^\alpha(\rho_\varepsilon(\cdot - y)) d\lambda \\ &= (D^\alpha u)_\varepsilon(x). \qquad \square \end{aligned}$$

Sobolev Spaces. For $k = 1, 2, \dots$ and $p \geq 1$ define

$$W^{k,p}(\Omega) := \{u \in L^p_{loc}(\Omega) : D^\alpha u \in L^p(\Omega) \text{ if } |\alpha| \leq k\}.$$

This is a Banach space with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^p \, d\lambda \right)^{1/p}.$$

One can easily check that

$$\sum_{|\alpha| \leq k} \|D^\alpha u\|_p$$

(where we use the notation $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$) is an equivalent norm. Of course $W^{k,p}_{loc}(\Omega)$ will denote the class of those functions that belong to $W^{k,p}(\Omega')$ for $\Omega' \Subset \Omega$.

The case $p = 2$ is special because $W^{k,2}(\Omega)$ is a Hilbert space. It is often denoted by $H^k(\Omega)$.

Proposition 1.3. *For* $u \in W^{k,p}_{loc}(\Omega)$ *we have* $u_\varepsilon \to u$ *in* $W^{k,p}_{loc}(\Omega)$ *as* $\varepsilon \to 0$.

Proof. It follows immediately from Proposition 1.2 and Theorem 1.1.iii. $\square$

Proposition 1.4. *$C^\infty \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let $\psi_j \in C_0^\infty(\Omega)$ be a partition of unity in Ω (that is $\sum_j \psi_j = 1$ and the sum is locally finite). Fix $u \in W^{k,p}(\Omega)$ and $\delta > 0$. For every j we can find ε_j sufficiently small so that

$$\|(\psi_j u)_{\varepsilon_j} - \psi_j u\|_{W^{k,p}(\Omega)} \le \frac{\delta}{2^j}$$

and so that the sum

$$v := \sum_j (\psi_j u)_{\varepsilon_j}$$

is locally finite. It follows that $v \in C^\infty(\Omega)$ and $\|u - v\|_{W^{k,p}(\Omega)} \le \delta$. □

By $W_0^{k,p}(\Omega)$ we will denote the closure of $C_0^k(\Omega)$ in $W^{k,p}(\Omega)$. From Proposition 1.3 it follows that if $u \in W^{k,p}(\Omega)$ has compact support then $u \in W_0^{k,p}(\Omega)$.

Theorem 1.5 (Sobolev). *For $p < n$ we have $W_0^{1,p}(\Omega) \subset L^{np/(n-p)}(\Omega)$ and*

$$\|u\|_{np/(n-p)} \le C(n,p)\|Du\|_p, \quad u \in W_0^{1,p}(\Omega). \tag{1.1}$$

Proof. It is enough to show the *Sobolev inequality* (1.1) for $u \in C_0^1(\mathbb{R}^n)$. First assume that $p = 1$. We clearly have

$$|u(x)| \le \int_{\mathbb{R}} |D_j u|\, dx_j$$

and the right-hand side is a function in $\mathbb{R}^n$ independent of x_j. We thus have

$$\begin{aligned}
\int_{\mathbb{R}} |u|^{n/(n-1)} dx_1 &\le \int_{\mathbb{R}} \prod_{j=1}^n \left(\int_{\mathbb{R}} |D_j u| dx_j \right)^{1/(n-1)} dx_1 \\
&= \left(\int_{\mathbb{R}} |D_1 u| dx_1 \right)^{1/(n-1)} \int_{\mathbb{R}} \prod_{j=2}^n \left(\int_{\mathbb{R}} |D_j u| dx_j \right)^{1/(n-1)} dx_1 \\
&\le \left(\int_{\mathbb{R}} |D_1 u| dx_1 \right)^{1/(n-1)} \prod_{j=2}^n \left(\int_{\mathbb{R}^2} |D_j u| dx_1 dx_j \right)^{1/(n-1)}
\end{aligned}$$

by Hölder's inequality. Proceeding further we obtain similarly

$$\begin{aligned}
\int_{\mathbb{R}^2} |u|^{n/(n-1)} dx_1 dx_2 \le &\left(\int_{\mathbb{R}^2} |D_1 u| dx_1 dx_2 \right)^{1/(n-1)} \left(\int_{\mathbb{R}^2} |D_2 u| dx_1 dx_2 \right)^{1/(n-1)} \\
&\prod_{j=3}^n \left(\int_{\mathbb{R}^3} |D_j u| dx_1 dx_2 dx_j \right)^{1/(n-1)}
\end{aligned}$$

and eventually

$$\|u\|_{n/(n-1)} \le \left(\prod_{j=1}^n \int_{\mathbb{R}^n} |D_j u|\, d\lambda \right)^{1/n}.$$

From the inequality between geometric and arithmetic means we get

$$\|u\|_{n/(n-1)} \le \frac{1}{n}\int_{\mathbb{R}^n}\sum_{j=1}^{n}|D_j u|\,d\lambda \le \frac{1}{\sqrt{n}}\|Du\|_1.$$

For arbitrary p set $\widetilde{u} := |u|^q$ for some $q > 1$. Then $D_j\widetilde{u} = q|u|^{q-1}D_j u$ and $|D\widetilde{u}| = q|u|^{q-1}|Du|$, therefore

$$\begin{aligned}\|u\|^q_{qn/(n-1)} &= \|\widetilde{u}\|_{n/(n-1)} \le \frac{1}{\sqrt{n}}\|D\widetilde{u}\|_1\\ &= \frac{q}{\sqrt{n}}\int_{\mathbb{R}^n}|u|^{q-1}|Du|\,d\lambda \le \frac{q}{\sqrt{n}}\left(\int_{\mathbb{R}^n}|u|^{p'(q-1)}\,d\lambda\right)^{1/p'}\|Du\|_p\end{aligned}$$

by Hölder's inequality, where $1/p + 1/p' = 1$. We now solve $qn/(n-1) = p'(q-1)$ in q and get $q = (n-1)p/(n-p)$ (since $p < n$, we have $q > 1$). We thus obtain

$$\|u\|_{np/(n-p)} \le \frac{(n-1)p}{\sqrt{n}(n-p)}\|Du\|_p. \quad \square$$

Corollary 1.6. *For $p < n$ one has $W^{1,p}_{loc} \subset L^{np/(n-p)}_{loc}$.*

Proof. For $\Omega' \Subset \Omega'' \Subset \Omega$ choose $\psi \in C_0^\infty(\Omega'')$ with $\psi = 1$ in Ω'. Then for $u \in W^{1,p}(\Omega'')$ we have $\psi u \in W_0^{1,p}(\Omega'')$ (this is because directly from the definition of weak differentiation we have

$$D_j(\psi u) = D_j\psi u + \psi D_j u)$$

and the result follows. $\square$

Exercise 2. *Show that*

$$|x|^\alpha \in L^q_{loc}(\mathbb{R}^n) \iff \alpha > -n/q \quad \text{and} \quad |x|^\alpha \in W^{1,p}_{loc}(\mathbb{R}^n) \iff \alpha > 1 - n/p.$$

Conclude that the exponent $np/(n-p)$ in the Sobolev theorem is optimal for every $1 \le p < n$.

Theorem 1.7 (Morrey). *For $p > n$ we have $W_0^{1,p}(\Omega) \subset C^{0,1-n/p}(\bar{\Omega})$. Moreover, for $u \in W_0^{1,p}(\Omega)$*

$$\frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \le C(n,p)\|Du\|_p, \quad x, y \in \Omega,\ x \ne y. \tag{1.2}$$

Proof. We claim that it is enough to show *Morrey's inequality* (1.2) for $u \in C_0^1(\mathbb{R}^n)$. For if $u \in W_0^{1,p}(\Omega)$ and $u_j \in C_0^1(\Omega) \subset C_0^1(\mathbb{R}^n)$ are such that $u_j \to u$ in $W^{1,p}(\Omega)$ and pointwise almost everywhere (because from every sequence converging in L^1_{loc} one can choose a subsequence converging pointwise almost everywhere) then it follows that (1.2) holds almost everywhere, and thus everywhere.

Assume therefore that $u \in C_0^1(\mathbb{R}^n)$ and denote $r = |x-y|$. Let B any closed ball of radius R containing x and y. Then $r \le 2R$ and $B \subset B(x, r+R) \subset B(x, 3R)$. We have, assuming for simplicity that $x = 0$,

$$u(y) - u(0) = \int_0^r \frac{d}{d\rho} u(\rho \frac{y}{|y|}) d\rho = \int_0^r \langle Du(\rho \frac{y}{|y|}), \frac{y}{|y|} \rangle d\rho. \tag{1.3}$$

Set

$$u_B := \frac{1}{\lambda(B)} \int_B u \, d\lambda$$

and

$$V(x) := \begin{cases} |Du(x)|, & x \in B \\ 0, & x \notin B. \end{cases}$$

Integrating (1.3) over B w.r.t. y we can estimate

$$\begin{aligned}
\lambda(B)|u_B - u(0)| &\le \int_B \int_0^r V\big(\rho \frac{y}{|y|}\big) d\rho \, d\lambda(y) \\
&\le \int_0^{2R} \int_{B(0,3R)} V\big(\rho \frac{y}{|y|}\big) d\lambda(y) \, d\rho \\
&= \int_0^{3R} \int_0^{3R} t^{n-1} dt \int_{|\omega|=1} V(\rho\omega) d\sigma(\omega) \, d\rho \\
&= \frac{(3R)^n}{n} \int_B |y|^{1-n} |Du(y)| d\lambda(y) \\
&\le \frac{(3R)^n}{n} \|Du\|_p \left(\int_B |y|^{(1-n)p'} d\lambda(y) \right)^{1/p'}
\end{aligned}$$

where $1/p + 1/p' = 1$. Since

$$\begin{aligned}
\int_B |y|^{(1-n)p'} d\lambda(y) &\le \int_{B(0,3R)} |y|^{(1-n)p'} d\lambda(y) \\
&= c_n \int_0^{3R} \rho^{(n-1)(1-p')} d\rho \\
&= c_n' R^{n+p'(1-n)}
\end{aligned}$$

and $n/p' + 1 - n = 1 - n/p$, we now get

$$|u_B - u(x)| \le C(n,p) R^{1-n/p} \|Du\|_p$$

and

$$|u(x) - u(y)| \le |u_B - u(x)| + |u_B - u(y)| \le 2C(n,p) R^{1-n/p} \|Du\|_p. \qquad \square$$

From the proof we can deduce the following estimate:

Theorem 1.8. *Assume that B is an open ball with radius R and $u \in W^{1,p}(B)$ for some $p > n$. Then for $x, y \in B$*

$$|u(x) - u(y)| \le C(n,p) R^{1-n/p} \|Du\|_{L^p(B)}.$$

Proof. By the proof of Theorem 1.7 the inequality holds for $u \in C^1 \cap W^{1,p}(B)$. For general u we can now use Proposition 1.4 to get it for almost all x, y. But since, by Morrey's theorem, u is in particular continuous, the theorem follows. □

We also have the following counterpart of Corollary 1.6 (with the same proof):

Corollary 1.9. *For $p > n$ we have $W^{1,p}_{loc}(\Omega) \subset C^{0,1-n/p}(\Omega)$.* □

Exercise 3. *Considering again the function $|x|^\alpha$ show that the Hölder exponent $1 - n/p$ in Morrey's theorem is optimal.*

Morrey's theorem for $p = \infty$ asserts that functions from $W^{1,\infty}_{loc}$ are locally Lipschitz continuous. In fact in this case the opposite also holds:

Theorem 1.10. *We have $W^{1,\infty}_{loc} = C^{0,1}$.*

Proof. $\subset$ follows from Morrey's theorem but we can in fact show it independently. For $u \in W^{1,\infty}(\Omega)$ we have

$$|Du_\varepsilon(x)| = |(Du)_\varepsilon| \leq \|Du\|_\infty$$

and

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq \|Du\|_\infty |x - y|$$

(if Ω is convex). Therefore for almost all $x, y \in \Omega_\varepsilon$

$$|u(x) - u(y)| \leq \|Du\|_\infty |x - y|,$$

and thus for all $x, y \in \Omega$.

On the other hand, take Lipschitz continuous u with compact support. For $h \neq 0$ consider the difference quotient

$$D^h_j u(x) = \frac{u(x + he_j) - u(x)}{h}.$$

Then $|D^h_j u(x)| \leq C$ and by the Banach-Alaoglu theorem there exists a sequence $h_m \to 0$ and $v_j \in L^\infty(\mathbb{R}^n)$ such that $D^{h_m}_j u(x) \to v_j$ weakly in $L^2(\mathbb{R}^n)$. Then for $\varphi \in C^\infty_0(\mathbb{R}^n)$

$$\begin{aligned} \int_{\mathbb{R}^n} u\, D_j \varphi \, d\lambda &= \lim_{m\to\infty} \int_{\mathbb{R}^n} u\, D^{-h_m}_j \varphi \, d\lambda \\ &= - \lim_{m\to\infty} \int_{\mathbb{R}^n} D^{h_m}_j u\, \varphi \, d\lambda \\ &= - \int_{\mathbb{R}^n} v_j\, \varphi \, d\lambda \end{aligned}$$

and $v_j = D_j u$ weakly. □

Iterating the Sobolev theorem we will get

$$W^{k,p}_{loc} \subset W^{k-1,np/(n-p)}_{loc} \subset W^{k-2,np/(n-2p)}_{loc} \subset \cdots \subset L^{np/(n-kp)}_{loc}$$

provided that $p < n/k$. If p is such that $n/(j+1) < p < n/j$ then

$$W^{k,p}_{loc} \subset W^{k-j,np/(n-jp)}_{loc} \subset C^{k-j-1,j+1-n/p}$$

(we may denote the latter as $C^{k-n/p}$) by Morrey's theorem. We thus get:

Theorem 1.11. *Let $p \geq 1$ and $k = 1, 2, \dots$ If $p < n/k$ then $W^{k,p}_{loc} \subset L^{np/(n-kp)}_{loc}$. For $p > n/k$ such that $p \neq n/j$ for $j = 1, \dots, k-1$ we have $W^{k,p}_{loc} \subset C^{k-n/p}$.* □

For $p = 1$, without invoking neither Sobolev nor Morrey's theorems, one can show in a simple way that $W^{k,1}_{loc} \subset C^{k-n}$, where $k \geq n$, proceeding as follows:

Exercise 4. *Prove that:*
i) $\|u\|_\infty \leq \|D_1 \dots D_n u\|_1$ *if* $u \in C_0^\infty(\mathbb{R}^n)$;
ii) $u_\varepsilon \to u$ *uniformly as* $\varepsilon \to 0$ *if* $u \in W^{n,1}(\mathbb{R}^n)$ *has compact support.*
Conclude that $W^{n,1}_{loc} \subset C$ and then that $W^{k,1}_{loc} \subset C^{k-n}$.

In particular we have $W^{1,n}_{loc} \subset C$ if $n = 1$. This is however no longer true for $n \geq 2$:

Exercise 5. *Show the function* $\log(-\log|x|)$ *is in $W^{1,n}_{loc}$ near the origin for $n \geq 2$ but not for $n = 1$.*

It shows that the second part of Theorem 1.11 is not true for $p = n/j$.

Differentiability almost everywhere. As an application of Morrey's inequality we will get the following:

Theorem 1.12. *For $p > n$ functions from $W^{1,p}_{loc}$ are differentiable almost everywhere.*

Proof. By the Lebesgue differentiation theorem for almost all x

$$\lim_{r \to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |Du(y) - Du(x)|^p \, d\lambda(y) = 0,$$

where $Du = (D_1 u, \dots, D_n u)$ and $D_j u \in L^p_{loc}$. Fix such an x and set

$$v(y) := u(y) - u(x) - \langle Du(x), y - x \rangle.$$

Then by Theorem 1.8 with $B = B(x, R)$ and $R = r = 2|x - y|$

$$\begin{aligned} \frac{|v(y)|}{|x-y|} &\leq C_1 r^{-n/p} \|Dv\|_{L^p(B(x,r))} \\ &= C_2 \left(\frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |Du(z) - Du(x)|^p \, d\lambda(z) \right)^{1/p} \end{aligned}$$

and it converges to 0 as $r \to 0$. It follows that $Du(x)$ is the classical derivative of u at x. □

Corollary 1.13 (Rademacher). *Lipschitz continuous functions are differentiable almost everywhere.* □

Compactness. It will be important for the existence theorems later on to know when the imbedding in the Sobolev theorem is compact.

Theorem 1.14 (Rellich-Kondrachov). *Assume that Ω is bounded. Then for $p < n$ and $q < np/(n-p)$ the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is compact (that is continuous and images of bounded sets are relatively compact).*

Proof. Continuity is a consequence of the Sobolev inequality. We first show compactness for $q = 1$. Let $\mathcal{A}$ be a bounded set in $W_0^{1,p}(\Omega)$, without loss of generality we may assume that $\mathcal{A} \subset C_0^1(\mathbb{R}^n)$ with $\|u\|_{W^{1,p}(\Omega)} \le 1$ for $u \in \mathcal{A}$ and $\operatorname{supp} u \subset \Omega$. Fix $\widetilde{\Omega}$ with $\Omega \Subset \widetilde{\Omega} \Subset \mathbb{R}^n$ and for $\varepsilon > 0$ sufficiently small define $\mathcal{A}_\varepsilon := \{u_\varepsilon : u \in \mathcal{A}\} \subset C_0^1(\widetilde{\Omega})$. We have

$$|u_\varepsilon(x)| \le \int_{B(x,\varepsilon)} |u(y)|\rho_\varepsilon(x-y)d\lambda(y) \le \sup \rho_\varepsilon \, \|u\|_1 \le \sup \rho_\varepsilon$$

and similarly

$$|Du_\varepsilon(x)| \le \sup |D\rho_\varepsilon|.$$

It follows that $\mathcal{A}_\varepsilon$ is equicontinuous and from the Arzela-Ascoli theorem we deduce that $\mathcal{A}_\varepsilon$ is relatively compact in $L^1(\widetilde{\Omega})$ for every single ε.

We also have

$$\begin{aligned} |u_\varepsilon(x) - u(x)| &\le \int_{B(0,\varepsilon)} \rho_\varepsilon(y)|u(x-y) - u(x)| \, d\lambda(y) \\ &= \int_{B(0,\varepsilon)} \rho_\varepsilon(y)\Big| \int_0^1 \frac{d}{dt}u(x-ty)\, dt\Big| \, d\lambda(y) \\ &\le \varepsilon \int_{B(0,\varepsilon)} \rho_\varepsilon(y) \int_0^1 |Du(x-ty)| \, dt \, d\lambda(y) \end{aligned}$$

and thus, integrating w.r.t. x

$$\|u_\varepsilon - u\|_1 \le \varepsilon \|Du\|_1 \le \varepsilon \lambda(\Omega)^{1-1/p} \|Du\|_p.$$

It is now sufficient to use the following simple fact:

Lemma 1.15. *Let V be a Banach space with the following property: for every $u \in V$ and $\varepsilon > 0$ there exists $u_\varepsilon \in V$ with $\|u - u_\varepsilon\| \le C\varepsilon$ for some uniform constant C. Assume moreover that $\mathcal{A}$ is a bounded subset of V such that for every $\varepsilon > 0$ the set $\mathcal{A}_\varepsilon := \{u_\varepsilon : u \in \mathcal{A}\}$ is relatively compact. Then $\mathcal{A}$ is relatively compact.*

Proof. We have to show that every sequence u_m in $\mathcal{A}$ has a convergent subsequence. For $\delta > 0$ set $\varepsilon := C/\delta$. We can find a subsequence $u_{m_j,\varepsilon}$ such that $\|u_{m_j,\varepsilon} - u_{m_k,\varepsilon}\| \le \delta$ for all j, k, and by the assumption $\|u_{m_j} - u_{m_k}\| \le 3\delta$. Using the diagonal method we will now easily get a Cauchy subsequence of u_m. □

Proof of Theorem 1.14, continued. For $q > 1$ from Hölder's inequality we infer, if $0 \le \lambda < 1$,

$$\|u_\varepsilon - u\|_q^q \le \|u_\varepsilon - u\|_1^\lambda \|u_\varepsilon - u\|_{(q-\lambda)/(1-\lambda)}^{q-\lambda}.$$

We choose λ with $(q-\lambda)/(1-\lambda) = np/(n-p) =: \mu$, that is $\lambda = (\mu - q)/(\mu - 1)$ (note that $\mu > q > 1$). By the Sobolev inequality

$$\|u_\varepsilon - u\|_q \le C\|u_\varepsilon - u\|_1^{\lambda/q}$$

and we can use the previous part. □

2. Elliptic Equations of Second Order

We will consider second order operators in divergence form

$$Lu := D_i(a^{ij}D_ju) + b^iD_iu + cu, \tag{2.1}$$

where a^{ij}, b^i, c are functions defined in Ω, $a^{ij} = a^{ji}$. Note that operators in non-divergence form

$$a^{ij}D_iD_ju + b^iD_iu + cu$$

can be written in divergence form

$$D_i(a^{ij}D_ju) + (b^i - D_ia^{ij})D_iu + cu$$

provided that a^{ij} are sufficiently regular.

A function u is a weak solution of the equation

$$Lu = f \tag{2.2}$$

if

$$-\mathcal{L}(u,\varphi) = \int_\Omega f\,\varphi\,d\lambda, \quad \varphi \in C_0^\infty(\Omega),$$

where

$$\mathcal{L}(u,v) = \int_\Omega a^{ij}D_iu\,D_jv\,d\lambda - \int_\Omega \big(b^iD_iu + cu\big)v d\lambda.$$

The equation (2.2) makes sense for $u \in W^{1,2}_{loc}(\Omega)$ and $a^{ij}, b^i, c, f \in L^2_{loc}(\Omega)$. We can also write $Lu \geq 0$ if $-\mathcal{L}(u,\varphi) \geq 0$ for $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. On the other hand, the definition of $\mathcal{L}(u,v)$ makes sense for $u, v \in W^{1,2}(\Omega)$ if

$$a^{ij}, b^i, c \in L^\infty(\Omega). \tag{2.3}$$

We can also impose weak boundary condition: for $u, \varphi \in W^{1,2}(\Omega)$ we say that $u = \varphi$ on $\partial\Omega$ if $u - \varphi \in W_0^{1,2}(\Omega)$. We will say that $u \leq \varphi$ on $\partial\Omega$ if $(u-\varphi)^+ \in W_0^{1,2}(\Omega)$ (where $u^+ := \max\{u, o\}$). We will need a simple fact:

Lemma 2.1. *If $u \in W^{1,p}(\Omega)$ then $u^+ \in W^{1,p}(\Omega)$ and $D(u^+) = \chi_{\{u>0\}}Du$.*

Proof. Let $\rho \in C^\infty(\mathbb{R})$ be such that $\rho(t) = 0$ for $t \leq -1$, $\rho(t) = t$ for $t \geq 1$ and $\rho' \geq 0$. For $\varepsilon > 0$ define $\rho_\varepsilon(t) := \varepsilon\rho(t/\varepsilon)$. Then $\rho_\varepsilon \in C^\infty(\mathbb{R})$, $\rho_\varepsilon(t) = 0$ for $t \leq -\varepsilon$, $\rho(t) = t$ for $t \geq \varepsilon$ and ρ_ε decreases to t^+ as ε decreases to 0.

The sequence $\rho_\varepsilon \circ u$ decreases to u^+. Using Proposition 1.4 one can show that for $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \rho_\varepsilon \circ u\,D_j\varphi\,d\lambda = -\int_\Omega \varphi\,\rho'_\varepsilon \circ u\,D_ju\,d\lambda.$$

Therefore

$$\int_\Omega u^+D_j\varphi\,d\lambda = -\lim_{\varepsilon\to 0}\int_\Omega \varphi\,\rho'_\varepsilon \circ u\,D_ju\,d\lambda = -\int_\Omega \varphi\,\chi_{\{u>0\}}\,D_ju\,d\lambda. \quad \square$$

The operator (2.1) is called *uniformly elliptic* if there exists a constant $\lambda > 0$ such that

(2.4) $$a^{ij}\zeta_i\zeta_j \geq \lambda|\zeta|^2, \quad \zeta \in \mathbb{R}^n,$$

that is the lowest eigenvalue of the matrix $(a^{ij}(x))$ is $\geq \lambda$ for every $x \in \Omega$.

Dirichlet problem. From now on we will always assume that L satisfies (2.3), (2.4), and that Ω is a bounded domain. We will analyze existence and uniqueness of solutions of the Dirichlet problem

(2.5) $$\begin{cases} Lu = f \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

for $f \in L^2(\Omega)$ and $\varphi \in W^{1,2}(\Omega)$. We will concentrate on the zero-value boundary problem

(2.6) $$\begin{cases} Lu = f \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

It will be essentially no loss of generality:

Remark (reduction to $\varphi = 0$). Clearly uniqueness for (2.5) and (2.6) is equivalent. If $\widetilde{u}$ solves

$$\begin{cases} L\widetilde{u} = f - L\varphi \\ \widetilde{u} = 0 \text{ on } \partial\Omega \end{cases}$$

then $u = \widetilde{u} + \varphi$ solves (2.5), but we have to assume in addition that $L\varphi \in L^2(\Omega)$, whereas in general we are only guaranteed that $L\varphi \in L^1(\Omega)$. To get around this problem one can consider a more general equation than (2.2)

(2.2') $$Lu = f + D_i f^i,$$

where $f^i \in L^2(\Omega)$. A function u is a weak solution of this if

$$-\mathcal{L}(u, \varphi) = \int_\Omega f\,\varphi\,d\lambda - \int_\Omega f^i D_i\varphi\,d\lambda, \quad \varphi \in C_0^\infty(\Omega),$$

or more generally $\varphi \in W_0^{1,2}(\Omega)$. It turns out that the results below also hold for (2.2') replaced with (2.2). Then however

$$f + D_i f^i - L\varphi = f - b^i D_i\varphi - c\varphi + D_i\big(f^i - a^{ij}D_j\varphi\big)$$

and now the problem reduces to $\varphi = 0$ without any problem.

Exercise 6. *Find all $\sigma \in \mathbb{R}$ for which the problem*

$$\begin{cases} u'' - \sigma u = 0 \\ u(0) = u(1) = 0 \end{cases}$$

has a nonzero smooth solution.

The main tool will be Hilbert space methods, namely the following result:

Theorem 2.3 (Lax-Milgram). *Let B be a bilinear form on a Hilbert space H such that*

$$|B(x,y)| \le C\|x\|\,\|y\|$$

and

$$|B(x,x)| \ge c\|x\|^2$$

for some positive constants C, c and all $x, y \in H$. Then for any $f \in H'$ there exists unique $x \in H$ with

$$f(y) = B(x,y), \quad y \in H.$$

In other words, the mapping

$$H \ni x \longmapsto B(x,\cdot) \in H'$$

is bijective.

Proof. By the Riesz theorem, which says that the mapping

$$H \ni x \longmapsto \langle x,\cdot\rangle \in H'$$

is bijective, we get

$$T : H \longrightarrow H$$

given by

$$B(x,\cdot) = \langle Tx,\cdot\rangle, \quad x \in H.$$

By the Riesz theorem again it suffices to show that T is bijective. It is clear that T is linear, by the assumptions we have moreover

$$c\|x\| \le \|Tx\| \le C\|x\|, \quad x \in H.$$

It follows that T is one-to-one and has closed range (the latter by the Banach-Alaoglu theorem). If x is perpendicular to the range then in particular $0 = \langle Tx, x\rangle = B(x,x)$, and thus $x = 0$. Therefore T is onto. □

Of course, if B is in addition symmetric then it is another scalar product in H and in this case the Lax-Milgram theorem is a direct consequence of the Riesz theorem.

We first check the assumptions of the Lax-Milgram theorem for $\mathcal{L}$ and the Hilbert space $H = W_0^{1,2}(\Omega)$.

Proposition 2.4. *For $u, v \in W^{1,2}(\Omega)$ we have*

$$|\mathcal{L}(u,v)| \le C\|u\|_{W^{1,2}(\Omega)}\|v\|_{W^{1,2}(\Omega)}$$

and

$$\mathcal{L}(u,u) \ge \frac{\lambda}{2}\int_\Omega |Du|^2 d\lambda - C\int_\Omega u^2 d\lambda,$$

where C depends only on λ, n and an upper bound for the coefficients of L.

Proof. The first part is a consequence of the Schwarz inequality. On the other hand,

$$\mathcal{L}(u,u) \geq \lambda \int_\Omega |Du|^2 d\lambda - C_1 \int_\Omega |Du|\,|u|\,d\lambda - C_2 \int_\Omega u^2 d\lambda.$$

The desired inequality now easily follows, since for every $\varepsilon > 0$

$$2|Du|\,|u| \leq \varepsilon |Du|^2 + \frac{1}{\varepsilon} u^2. \quad \square$$

The following result is an easy consequence of the Lax-Milgram theorem and Proposition 2.4:

Theorem 2.5. *There exists $\mu_0 \geq 0$ depending only on L such that for every $\mu \geq \mu_0$ and every $f \in L^2(\Omega)$ the problem*

$$\begin{cases} Lu - \mu u = f \\ u = 0 \ \ on\ \partial\Omega \end{cases}$$

has a unique solution in $W^{1,2}(\Omega)$.

Proof. For the operator $\widetilde{L}u = Lu - \mu u$ the associated form is

$$\widetilde{\mathcal{L}}(u,v) = \mathcal{L}(u,v) + \mu\langle u, v\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\Omega)$. Then for $\mu \geq \lambda/2 + C$ (where C is the constant from Proposition 2.4) we have

$$\widetilde{\mathcal{L}}(u,u) \geq \frac{\lambda}{2} \|u\|^2_{W^{1,2}(\Omega)}.$$

By the Lax-Milgram theorem for $f \in L^2(\Omega)$ there exists unique $u \in W_0^{1,2}(\Omega)$ with

$$\widetilde{\mathcal{L}}(u,v) = -\int_\Omega f v\, d\lambda, \quad v \in W_0^{1,2}(\Omega). \quad \square$$

Theorem 2.6 (Fredholm alternative). *For a given operator L precisely one of the following statements holds:*

i) either for every $f \in L^2(\Omega)$ the equation $Lu = f$ has a unique solution in $W_0^{1,2}(\Omega)$;

ii) or there exists a nonzero $u \in W_0^{1,2}(\Omega)$ such that $Lu = 0$.

Proof. Let μ, given by Theorem 2.5, be such that the equation

$$Lu - \mu u = g$$

is uniquely solvable in $W_0^{1,2}(\Omega)$ for $g \in L^2(\Omega)$. In other words, we have a well defined operator

$$\widetilde{L}^{-1} : L^2(\Omega) \to W_0^{1,2}(\Omega),$$

where $\widetilde{L}u = Lu - \mu u$. Now the equation $Lu = f$ is equivalent to $\widetilde{L}u = f - \mu u$, which means that $u = \widetilde{L}^{-1}(f - \mu u)$. We can write it as

$$u - Tu = h,$$

where $T = -\mu\widetilde{L}^{-1}$ and $h = \widetilde{L}^{-1}f$.

If $\widetilde{L}u = g$ then by the proof of Theorem 2.5

$$\frac{\lambda}{2}\|u\|_2^2 \le \widetilde{L}(u,u) = -\langle g, u\rangle \le \|g\|_2\|u\|_2.$$

It follows that

$$\|Tg\|_2 \le \frac{2\mu}{\lambda}\|g\|_2, \quad g \in L^2(\Omega).$$

Therefore the linear operator

$$T : L^2(\Omega) \to L^2(\Omega)$$

is bounded. Since the range of T is contained in $W_0^{1,2}(\Omega)$, by the Rellich-Kondrachov theorem we infer that T is also compact.

To finish the proof it now suffices to use the following fact from functional analysis:

Theorem 2.7. *Let H be a Hilbert space and $T : H \to H$ a compact linear operator such that* $\ker(I - T) = \{0\}$. *Then $I - T$ is onto.*

Proof. Suppose $H_1 := (I - T)(H) \subsetneq H$. Then $H_2 := (I - T)(H_1) = (I - T)^2(H) \subsetneq H_1$ (because $I-T$ is one-to-one) and we can define subspaces $H_k := (I-T)^k(H)$ such that $H_{k+1} \subsetneq H_k$. We claim that H_k are closed. For this it will be enough to show that if $\widetilde{H}$ is a closed subspace of H then $(I-T)(\widetilde{H})$ is also closed. Take a convergent sequence $y_j = x_j - Tx_j$, where $x_j \in \widetilde{H}$. We may assume that $x_j \in \widetilde{H} \cap (\ker(I-T))^\perp$. If we show that for some constant C

$$\|x\| \le C\|x - Tx\|, \quad x \in (\ker(I - T))^\perp, \tag{2.7}$$

then $\|x_j - x_k\| \le C\|y_j - y_k\|$ and x_j will also be convergent. To show that $(I-T)(\widetilde{H})$ is closed it therefore remains to prove (2.7).

Suppose (2.7) does not hold. Then we can find $\widetilde{x}_j \in (\ker(I-T))^\perp$ with $\|\widetilde{x}_j\| = 1$ and such that

$$\widetilde{x}_j - T\widetilde{x}_j \to 0. \tag{2.8}$$

Since T is compact, choosing a subsequence if necessary, we may assume that $T\widetilde{x}_j$ is convergent and thus by (2.8) $\widetilde{x}_j$ is also convergent to some $\widetilde{x}$. But then $\widetilde{x} \in \ker(I - T) \cap (\ker(I - T))^\perp$ and $\|\widetilde{x}\| = 1$ which is a contradiction. We thus showed that $(I - T)(\widetilde{H})$ is closed and therefore so are the subspaces H_k.

We can now choose $\widehat{x}_k \in H_k \cap H_{k+1}^\perp$ with $\|\widehat{x}_k\| = 1$. For $k > l$ write

$$T\widehat{x}_k - T\widehat{x}_l = -(\widehat{x}_k - T\widehat{x}_k) + (\widehat{x}_l - T\widehat{x}_l) + \widehat{x}_k - \widehat{x}_l.$$

Since $H_{k+1} \subsetneq H_k \subset H_{l+1}$, we have $\widehat{x}_k - T\widehat{x}_k,\ \widehat{x}_l - T\widehat{x}_l,\ \widehat{x}_k \in H_{l+1}$. But $\widehat{x}_l \in H_{l+1}^{\perp}$ and thus $\|T\widehat{x}_k - T\widehat{x}_l\| \geq \|\widehat{x}_l\| = 1$ which contradicts the fact that T is compact. □

As a consequence of the Fredholm alternative we will get in particular the following improvement of Theorem 2.5:

Theorem 2.8. *Assume that $c \leq 0$. Then for every $f \in L^2(\Omega)$ the equation $Lu = f$ has a unique solution in $W_0^{1,2}(\Omega)$.*

This result follows immediately from the following weak maximum principle which excludes the case ii) in Theorem 2.6:

Theorem 2.9. *Assume that $c \leq 0$. Let $u \in W^{1,2}(\Omega)$ be such that $u \leq 0$ on $\partial\Omega$ and $Lu \geq 0$. Then $u \leq 0$ in Ω.*

Proof. By approximation we have $\mathcal{L}(u,v) \leq$ for $v \in W_0^{1,2}(\Omega)$ with $v \geq 0$. Therefore, since $c \leq 0$, for $v \in W_0^{1,2}(\Omega)$ with $v \geq 0$ and $uv \geq 0$ we obtain

$$\int_\Omega a^{ij} D_i u\, D_j v\, d\lambda \leq \int_\Omega b^i D_i u\, v\, d\lambda \leq C \int_\Omega |Du|\, v\, d\lambda.$$

Suppose $\sup_\Omega u > 0$ and choose a with $0 < a < \sup_\Omega u$. Set $v := (u-a)^+$. Then $v \in W_0^{1,2}(\Omega)$ (by Lemma 2.1 and regularization), $v \geq 0$, $uv \geq 0$. Therefore by Lemma 2.1

$$\int_\Omega a^{ij} D_i v\, D_j v\, d\lambda \leq C_1 \int_\Omega |Dv|\, v\, d\lambda$$

and thus by (2.4)

$$\|Dv\|_2^2 \leq C_2 \int_\Omega |Dv|\, v\, d\lambda.$$

We will get

$$\|Dv\|_2 \leq C_3 \|v\|_{L^2(\{Dv \neq 0\})}$$

and by the Sobolev inequality for $n \geq 3$

$$\|v\|_{2n/(n-2)} \leq C_4 \|Dv\|_2 \leq C_5 \|v\|_{L^2(\{Dv\neq 0\})} \leq C_5 (\lambda(\{Dv \neq 0\}))^{1/n} \|v\|_{2n/(n-2)}$$

and thus

$$\lambda(\{Dv \neq 0\}) \geq c > 0, \tag{2.9}$$

where c does not depend on a. (For $n = 2$ we choose any p with $1 < p < 2$ and similarly obtain

$$\|v\|_{2p/(2-p)} \leq C\|Dv\|_p \leq C(\lambda(\Omega))^{1/p-1/2} \|Dv\|_2.)$$

By Lemma 2.1 (applied twice) we have $\{Dv \neq 0\} \subset \{a < u < \sup_\Omega u\}$ which easily contradicts (2.9). □

Eigenvalues. For a given operator L (which in turn depends also on Ω) by Σ we denote the set of eigenvalues of $-L$, that is those $\sigma \in \mathbb{R}$ such that the problem

$$\begin{cases} Lu + \sigma u = 0 \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has a nonzero solution in $W^{1,2}(\Omega)$. The set Σ is called a *spectrum* of $-L$.

Theorem 2.10. *For $\sigma \notin \Sigma$ the problem*

$$\begin{cases} Lu + \sigma u = f \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{2.10}$$

has a unique solution in $W^{1,2}(\Omega)$ for every $f \in L^2(\Omega)$. The set Σ is either finite or consists of a sequence converging to $+\infty$.

Proof. The first part follows directly from the Fredholm alternative applied to the operator $Lu + \sigma u$. Let $\mu > 0$, $\widetilde{L}u = Lu - \mu u$ and $T = -\mu\widetilde{L}^{-1}$ be as in the proof of Theorem 2.6. For $\sigma \in \Sigma$ we then have $\widetilde{L}u = -(\sigma + \mu)u$ for some nonzero $u \in W_0^{1,2}(\Omega)$ and thus

$$Tu = \frac{\mu}{\sigma + \mu} u.$$

Therefore, σ is an eigenvalue of $-L$ if and only if $\mu/(\sigma + \mu)$ is an eigenvalue of T. Since by Theorem 2.5 Σ is bounded from below, it is enough to use the following result:

Theorem 2.11. *Let $T : H \to H$ be a linear compact operator, where H is a Hilbert space. Then the set of nonzero eigenvalues of T is either finite or consists of a sequence converging to 0.*

Proof. If $Tw_k = \eta_k w_k$, where $\|w_k\| = 1$, then, choosing subsequence if necessary, by compactness we see that the sequence $\eta_k w_k$ is convergent, and thus η_k is bounded. We thus have to show that if $\eta_k \to \eta$, where all η_k are distinct, then $\eta = 0$. Suppose that $\eta \neq 0$ and $\eta_k \neq 0$. By H_k denote the space spanned by $w_1, \dots, w_k$. Then, since w_k are linearly independent, we have $H_k \subsetneq H_{k+1}$. For $k \geq 2$ we also have $(T - \eta_k I)(H_k) \subset H_{k-1}$. We can find $x_k \in H_k \cap H_{k-1}^{\perp}$ with $\|x_k\| = 1$. For $k > l$ we have $H_{l-1} \subsetneq H_l \subset H_{k-1} \subsetneq H_k$ and

$$\frac{Tx_k}{\eta_k} - \frac{Tx_l}{\eta_l} = \frac{Tx_k - \eta_k x_k}{\eta_k} - \frac{Tx_l - \eta_l x_l}{\eta_l} + x_k - x_l.$$

Now $Tx_k - \eta_k x_k,\ Tx_l - \eta_l x_l,\ x_l \in H_{k-1}$ and $x_k \in H_{k-1}^{\perp}$, therefore

$$\left\| \frac{Tx_k}{\eta_k} - \frac{Tx_l}{\eta_l} \right\| \geq \|x_k\| = 1.$$

We get a contradiction with the compactness of T. □

Theorem 2.12. *Assume that*

$$Lu = D_i(a^{ij} D_j u), \tag{2.11}$$

that is the coefficients b^i and c vanish. Then the eigenvalues of $-L$ are positive and there exists a complete orthonormal system in $L^2(\Omega)$ consisting of eigenfunctions of $-L$ from $W_0^{1,2}(\Omega)$. Eigenspaces of $-L$ are finite dimensional.

Proof. Positivity of the eigenvalues follows from Theorem 2.9. Together with the Fredholm alternative it also implies that the operator

$$L^{-1} : L^2(\Omega) \to W_0^{1,2}(\Omega)$$

is well defined. Thus

$$S : L^2(\Omega) \to L^2(\Omega)$$

given by $S := -L^{-1}$ is a compact operator by the Rellich-Kondrachov theorem. We claim that

$$\langle Sf, g\rangle = \langle f, Sg\rangle,$$

that is S is symmetric. This follows immediately from

$$\langle Lu, v\rangle = \langle u, Lv\rangle,$$

which we first prove for $u, v \in C_0^\infty(\Omega)$ (integrating by parts), and thus it holds for $u, v \in W_0^{1,2}(\Omega)$.

It is clear that $\ker S = \{0\}$. Therefore the eigenvalues of S are precisely $1/\sigma$, where σ is an eigenvalue of $-L$. By λ_k denote all eigenvalues of S and let $H_k = \ker(S - \lambda_k I)$ be the corresponding eigenspaces. Note that if $Sf = \lambda_k f$, $Sg = \lambda_l g$ then

$$\lambda_k\langle f, g\rangle = \langle Sf, g\rangle = \langle f, Sg\rangle = \lambda_l\langle f, g\rangle$$

and thus the spaces H_k and H_l are perpendicular for $k \neq l$.

Set $\widetilde{H} := \bigoplus H_k$ (that is $\widetilde{H}$ consists of finite linear combinations of elements from H_k). We have to show that $\widetilde{H}$ is dense in $L^2(\Omega)$. We clearly have $S(\widetilde{H}) \subset \widetilde{H}$. Set $\widehat{H} := \widetilde{H}^\perp$. If $f \in \widehat{H}$ and $g \in \widetilde{H}$ then $\langle Sf, g\rangle = \langle f, Sg\rangle = 0$, and thus $S(\widehat{H}) \subset \widehat{H}$. Since $\ker S = \{0\}$, for density of $\widetilde{H}$ it is enough to show that $S(\widehat{H}) = 0$. For that it suffices to prove that

$$\langle Sf, f\rangle = 0, \quad f \in \widehat{H} \tag{2.12}$$

(because the corresponding form $\langle Sf, g\rangle$ is symmetric). Suppose

$$M := \sup_{f\in\widehat{H},\ ||f||=1} \langle Sf, f\rangle > 0$$

(if the corresponding infimum is negative then we may consider $-S$ instead of S). We can find $f_j \in \widehat{H}$ with $||f_j|| = 1$ and such that $\langle Sf_j, f_j\rangle \to M$. By compactness we may assume in addition that $Sf_j \to \widetilde{f}$. We then have by the Schwarz inequality applied to the positive form $\langle Mf - Sf, g\rangle$

$$\begin{aligned} ||Mf_j - Sf_j|| &= \sup_{g\in\widehat{H},\ ||g||=1} |\langle Mf_j - Sf_j, g\rangle| \\ &\leq \sup_{g\in\widehat{H},\ ||g||=1} \langle Mg - Sg, g\rangle^{1/2}\ \langle Mf_j - Sf_j, f_j\rangle^{1/2}. \end{aligned}$$

It follows that $Mf_j - Sf_j \to 0$ and $S\widetilde{f} = M\widetilde{f}$. We thus get an eigevector in $\widehat{H} = \widetilde{H}^\perp$, which is a contradiction. Therefore (2.12) and the density of $\widetilde{H}$ follows.

The last statement of the theorem is a consequence of the following result.

Proposition 2.13. *Assume that $T : H \to H$ is a compact operator on a Hilbert space H. Then* $\dim\ker(T - I) < \infty$.

Proof. If the dimension were not finite then we would find an orthonormal sequence $x_k \in \ker(T - I)$. For $k \neq l$

$$||Tx_k - Tx_l||^2 = ||x_k - x_l||^2 = ||x_k||^2 - 2\langle x_k, x_l\rangle + ||x_l||^2 = 2.$$

Thus Tx_k has no convergent subsequence which contradicts compactness. □

The dimension of the corresponding eigenspace is called a *multiplicity* of an eigenvalue. Summing up, we see that eigenvalues of a symmetric elliptic operator (2.11) form a sequence of positive numbers converging to $+\infty$

$$0 < \sigma_1 \leq \sigma_2 \leq \dots$$

(we repeat an eigenvalue in this sequence k times, where k is the multiplicity). One can in fact show that the first eigenvalue is simple (multiplicity is 1), that is $\sigma_1 < \sigma_2$.

The famous problem *Can one hear the shape of a drum?* whether one can tell the shape of a domain knowing the eigenvalues of the Laplacian. It turned out that in general one cannot, but the problem is still open for example for smooth or convex domains.

Example. For $\Omega = (0, 2\pi)$ and $L = \Delta$ we have to solve

$$\begin{cases} u'' + \sigma u = 0 \\ u(0) = u(2\pi) = 0. \end{cases}$$

For a solution to exist we have to assume $\sigma > 0$, they are of the form $A\cos(\sqrt{\sigma}t) + B\sin(\sqrt{\sigma}t)$. The boundary condition implies that $A = 0$ and $\sin(2\pi\sqrt{\sigma}) = 0$, and thus

$$\sigma_k = \frac{k^2}{4}, \quad k = 1, 2, \dots,$$

whereas $u_k = \sin(kt/2)$ are the corresponding eigenfunctions.

Exercise 7. *Show that* $\sin(kt/2)$, $k = 1, 2, \dots$, *forms a complete orthogonal system in* $L^2((0, 2\pi))$.

The eigenvalue equation for the Laplacian

$$\Delta u + \sigma u = 0$$

is called the Helmholtz equation. For product domains it can be solved using the method of separation of variables (by Σ_Ω we denote the spectrum of $-\Delta$ for Ω).

Proposition 2.14. $\Sigma_{\Omega_1 \times \Omega_2} = \Sigma_{\Omega_1} + \Sigma_{\Omega_2}$.

Sketch of proof. Let $\sigma_j \in \Sigma_{\Omega_j}$, $j = 1, 2$, and let $u_j \in W_0^{1,2}(\Omega_j)$ be corresponding eigenfunctions. Set

$$w(x, y) := u_1(x)u_2(y), \quad x \in \Omega_1,\ y \in \Omega_2.$$

One can show that

$$\Delta w = v\Delta u + u\Delta v$$

(in the weak sense). Therefore

$$\Delta w + (\sigma_1 + \sigma_2)w = v(\Delta u + \sigma_1 u) + u(\Delta v + \sigma_2 v) = 0$$

and thus we have $\supset$. To show $\subset$ it is enough to prove (using Fubini theorem) that if $u_k(x)$ is a complete orthogonal system in $L^2(\Omega_1)$ and $v_l(y)$ a complete orthogonal system in $L^2(\Omega_2)$ then $u_k(x)v_l(y)$ is a complete orthogonal system in $L^2(\Omega_1 \times \Omega_2)$. $\square$

Example (rectangle). Similarly as in the previous example we can show that for $a > 0$

$$\Sigma_{(0,a)} = \{\frac{\pi^2 k^2}{a^2} : \ k = 1, 2 \dots\}.$$

Therefore, by Proposition 2.14

$$\Sigma_{(0,a)\times(0,b)} = \{\pi^2(\frac{k^2}{a^2} + \frac{l^2}{b^2}) : \ k, l = 1, 2, \dots\}.$$

Example (disc). It turns out that we can solve the Helmholtz equation in a disc also using separation of variables but applied to polar coordinates $x = r\cos\phi$, $y = r\sin\phi$. It is known that

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}.$$

Consider the function of the form

$$u(x, y) = R(r)\Phi(\phi).$$

Then

$$\Delta u + \sigma u = \big(R''(r) + \frac{1}{r}R'(r) + \sigma R(r)\big)\Phi(\phi) + \frac{1}{r^2}R(r)\Phi''(\phi).$$

To get a single variable equation we assume that

$$\Phi'' + c\Phi = 0.$$

We will get nontrivial periodic solutions of period 2π only if $c \geq 0$: $\Phi = A_0$ for $c = 0$ and

$$\Phi = A_k \cos(k\phi) + B_k \sin(k\phi)$$

for $c = k^2$, $k = 1, 2, \dots$. The equation for R now becomes

$$r^2 R_{rr} + r R_r + (\sigma r^2 - k^2)R = 0,$$

and, after the substitution $\rho = \sqrt{\sigma} r$,

$$\rho^2 R_{\rho\rho} + \rho R_\rho + (\rho^2 - k^2)R = 0.$$

The solutions are Bessel functions of order $k = 0, 1, \dots$:

$$J_k(\rho) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(k+j)!}\left(\frac{\rho}{2}\right)^{k+2j}.$$

We thus got the following solutions to the Helmholtz equation

$$J_k(\sqrt{\sigma} r)\big(A_k \cos(k\phi) + B_k \sin(k\phi)\big), \quad k = 0, 1, \dots,$$

where $B_0 = 0$ (one can check that these functions are smooth at the origin). The boundary condition in the unit disc gives

$$J_k(\sqrt{\sigma}) = 0,$$

therefore the eigenvalues are the squares of zeros of the Bessel functions (for J_0 the are of multiplicity 1 and for zeros of J_k, $k = 1, 2, \dots$, of multiplicity 2).