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Bergman kernel and pluripotential theory

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Dedicated to Duong Phong on the occasion of his 60th birthday

Abstract. We survey recent applications of pluripotential theory for the Bergman kernel and metric.

Introduction

The fact that the theory of the Bergman kernel and metric for domains on the complex plane is closely related to the classical potential theory is a well known fact. However, for example the fact that regular domains are complete with respect to the Bergman metric was proved only in late 90's by B.-Y. Chen [**16**]. This result was a by-product of methods of several complex variables. In the 90's it turned out that also pluripotential theory is very useful in multi-dimensional theory of the Bergman kernel. Recent years brought some new results in this direction, by the way also solving some one-dimensional problems.

The main new input from several complex variables to one-dimensional problems was the technique of weighted L^2-estimates for the $\bar\partial$-operator going back to Hörmander [**27**]. Usually optimal weights in this particular context are constructed using the pluricomplex Green function, one of the basic notions of pluripotential theory. This method seems to have been completely missing in older one-dimensional techniques, although Hörmander's theorem was published already in 1965.

This paper surveys these recent developments. In Section 1 we discuss the notion of Bergman completeness and the Kobayashi criterion for it. Section 2 presents some notions and results from pluripotential theory. They are applied in Section 3 to obtain some results on the Bergman kernel and metric. Finally, in Section 4 we discuss the recently settled Suita conjecture which is another example of a one-dimensional result eventually proved using methods of several complex variables. We also present several open problems related to this subject.

1. Bergman completeness, Kobayashi criterion

Let Ω be a bounded domain in $\mathbb{C}^n$. By $H^2(\Omega)$ we will denote the space of holomorphic funtions in $L^2(\Omega)$ and by $||\cdot||$ the L^2-norm in Ω. The Bergman kernel

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$K_\Omega(\cdot,\cdot)$, defined on $\Omega\times\Omega$, is determined by

$$f(w)=\int_\Omega f\,\overline{K_\Omega(\cdot,w)}d\lambda,\quad f\in H^2(\Omega),\ w\in\Omega.$$

On the diagonal of $\Omega\times\Omega$, by some abuse of notation, we write

$$K_\Omega(w)=K_\Omega(w,w)=\sup\{|f(w)|^2 : f\in\mathcal{O}(\Omega),\ ||f||\le 1\}.$$

The Bergman metric on Ω is the Kähler metric with the potential $\log K_\Omega$, we write $B_\Omega = i\partial\bar\partial\log K_\Omega$. The riemannian distance given by this metric will be denoted by $dist_\Omega^B$. We say that Ω is *Bergman complete* if it complete w.r.t. $dist_\Omega^B$.

The basic tool used to prove Bergman completeness is due to Kobayashi:

THEOREM 1 (Kobayashi Criterion [**30**]). *Assume that Ω is a bounded domain in $\mathbb{C}^n$. If*

$$\lim_{w\to\partial\Omega}\frac{|f(w)|^2}{K_\Omega(w)}=0,\quad f\in H^2(\Omega), \tag{1}$$

then Ω is Bergman complete.

The converse is not true even for $n=1$, as shown by Zwonek [**39**].

The proof of the Kobayashi Criterion is based on the following idea: the mapping

$$\iota:\Omega\ni w\longmapsto [K_\Omega(\cdot,w)]\in\mathbb{P}(H^2(\Omega))$$

embeds Ω into infinitely dimensional projective space $\mathbb{P}(H^2(\Omega))$ equipped with the Fubini-Study metric ω_{FS}. One can show that

$$B_\Omega=\iota^*\omega_{FS} \tag{2}$$

and this is sometimes called the Kobayashi alternative definition of the Bergman metric. Suppose that $w_k\in\Omega$ is a Cauchy sequence w.r.t. $dist_\Omega^B$ which is not convergent. Without loss of generality we may assume that $w_k\to\partial\Omega$. Since by (2) the embedding ι is distance decreasing and since $\mathbb{P}(H^2(\Omega))$ is complete, it follows that $\iota(w_k)$ converges to some $[f]$, where $f\neq 0$. But this means that for some $\lambda_k\in\mathbb{C}$ with $|\lambda_k|=1$ we have

$$\lambda_k\frac{K_\Omega(\cdot,w_k)}{\sqrt{K_\Omega(w_k)}}\longrightarrow\frac{f}{||f||}$$

in $H^2(\Omega)$. This implies that $f(w_k)/\sqrt{K_\Omega(w_k)}\to ||f||$ which contradicts (1).

We see from this proof that a slightly weaker condition than (1)

$$\limsup_{w\to\partial\Omega}\frac{|f(w)|^2}{K_\Omega(w)}<||f||^2,\quad f\in H^2(\Omega)\setminus\{0\} \tag{3}$$

is also sufficient for Bergman completeness.

PROBLEM 1. *Is* (3) *necessary for a bounded Ω to be Bergman complete?*

The fact that the Kobayashi embedding ι is distance decreasing translates to

$$dist_\Omega^B(z,w)\ge\arccos\frac{|K_\Omega(z,w)|}{\sqrt{K_\Omega(z)K_\Omega(w)}}. \tag{4}$$

An interesting consequence is

$$K_\Omega(z,w)=0\ \Rightarrow\ dist_B(z,w)\ge\frac{\pi}{2}.$$

It was shown in [**21**] that the constant $\pi/2$ is optimal here.

2. Some pluripotential theory

Again we assume that Ω is a bounded domain in $\mathbb{C}^n$. We say that it is *hyperconvex* if it admits a negative plurisubharmonic (psh) exhaustion function. In other words, there exists $u \in PSH^-(\Omega)$ such that $u = 0$ on $\partial\Omega$ (uniformly). Demailly [**17**] proved that any pseudoconvex domain with Lipschitz boundary is hyperconvex. For $n = 1$ hyperconvexity is equivalent to regularity of the boundary.

PROBLEM 2. *Assume that Ω is a bounded pseudoconvex domain with continuous boundary (that is $\partial\Omega$ is locally a graph of a continuous function). Does Ω have to be hyperconvex?*

There are some of those exhaustion functions in hyperconvex domains that are of particular interest. Probably the most important one is the *pluricomplex Green function*: for a pole $w \in \Omega$ we set

$$G_\Omega(\cdot, w) = G_w := \sup\{v \in PSH^-(\Omega) : v \leq \log|\cdot - w| + C\}.$$

The fundamental result is due to Demailly [**17**] who showed that it satisfies the complex Monge-Ampère equation

$$(dd^c G_w)^n = (2\pi)^n \delta_w \tag{5}$$

and that if Ω is hyperconvex then G_Ω is continuous on $\bar{\Omega} \times \Omega$ away from the diagonal (vanishing on $\partial\Omega \times \Omega$), see also [**8**] for a slightly different proof of the latter result.

It is in fact an open problem whether G_Ω is continuous on $\bar{\Omega} \times \bar{\Omega}$ away from the diagonal if Ω is hyperconvex. Equivalently, one can formulate it as follows:

PROBLEM 3. *If Ω is bounded and hyperconvex, is it true that $G_w \to 0$ locally uniformly in Ω as $w \to \partial\Omega$?*

One can show a weaker convergence:

PROPOSITION 2 ([**13**]). *Assume that Ω is bounded and hyperconvex. Then for every $p < \infty$ we have $G_w \to 0$ in $L^p(\Omega)$ as $w \to \partial\Omega$.*

On the other hand, Herbort [**26**] showed that the locally uniform convergence holds for pseudoconvex domains with C^2 boundary (see also [**18**] and [**9**]). Problem 3 has of course a positive answer if G_Ω is symmetric. This is however usually not the case if $n > 1$, as proved by Bedford and Demailly [**1**] (for a simpler counterexample see [**29**]). It follows from the results of Lempert [**32**] that G_Ω is symmetric if Ω is convex.

Another important feature of hyperconvex domains is that the Dirichlet problem for the inhomogeneous complex Monge-Ampère operator can be solved on them. Generalizing the fundamental result of Bedford and Taylor [**2**] it was proved in [**6**] that if Ω is hyperconvex, $f \in C(\bar{\Omega})$, $f \geq 0$, and $\varphi \in C(\partial\Omega)$ is a restriction of some psh u in Ω, continuous on $\bar{\Omega}$, then the following Dirichlet problem has a unique solution

$$\begin{cases} u \in PSH(\Omega) \cap C(\bar{\Omega}) \\ (dd^c u)^n = f\, d\lambda \\ u = \varphi \text{ on } \partial\Omega. \end{cases}$$

Especially interesting case is for $f \equiv 1$ and $\varphi \equiv 0$, we denote the resulting solution by u_Ω.

PROBLEM 4. *Is $u_\Omega \in C^\infty(\Omega)$ for an arbitrary bounded hyperconvex Ω?*

Of course it is true for smooth strongly pseudoconvex Ω by the classical result of Krylov [**31**] and Caffarelli-Kohn-Nirenberg-Spruck [**15**]. Then it is even smooth up to the boundary, something that cannot be expected in general. The only other case when the problem is known to have an affirmative answer is a polydisk, see [**7**]. The reason why one could expect it to hold in general is that in the analogous case of the real Monge-Ampère equation on arbitrary bounded convex domain without any regularity assumption the solution is indeed smooth, as proved by Pogorelov [**37**].

But even the continuity of u_Ω on $\bar\Omega$ is useful. We can for example prove Proposition 2: by [**5**] we have

$$\int_\Omega |G_w|^n d\lambda = \int_\Omega |G_w|^n (dd^c u_\Omega)^n \leq n! ||u_\Omega||_\infty^{n-1} \int_\Omega |u_\Omega(w)| (dd^c G_w)^n.$$

By (5) we will get

$$||G_w||_n^n \leq C|u_\Omega(w)|, \tag{6}$$

where C depends only on n and the volume of Ω. This gives Proposition 2 for $p = n$ and the general case easily follows from it.

3. Applications for the Bergman kernel and metric

We start with the result proved independently in [**13**] and [**25**]:

THEOREM 3. *Hyperconvex domains are Bergman complete.*

We can use the following inequality of Herbort [**25**]:

$$\frac{|f(w)|^2}{K_\Omega(w)} \leq C \int_{\{G_w < -1\}} |f|^2 d\lambda, \quad w \in \Omega,\ f \in H^2(\Omega), \tag{7}$$

where C depends only on n and the diameter of Ω. It holds in arbitrary pseudoconvex domain. This estimate was proved using Hörmander's estimate for $\bar\partial$ [**27**], the main use of the pluricomplex Green function in this context is from the fact that e^{-2nG_w} is not locally integrable near w.

Kobayashi Criterion and (7) imply that for a bounded pseudoconvex domain the condition

$$\lim_{w \to \partial\Omega} \lambda(\{G_w < -1\}) = 0$$

is sufficient for Bergman completeness. But for hyperconvex domains it follows immediately from Proposition 2 and we get Theorem 3.

Herbort's inequality (7) with $f \equiv 1$ gives

$$K_\Omega(w) \geq \frac{1}{C\lambda(\{G_w < -1\})} \tag{8}$$

Combining this with (6) we get the following lower bound for the Bergman kernel in terms of a solution of the complex Monge-Ampère equation:

$$K_\Omega \geq \frac{1}{C|u_\Omega|}.$$

Another very interesting application of pluripotemntial theory has been to prove a quantitative lower bound for the Bergman distance in pseudoconvex domains with smooth boundary. Of course, a bounded domain Ω in $\mathbb{C}^n$ is Bergman complete if and only if for a fixed $z_0 \in \Omega$ we have

$$\lim_{z \to \partial\Omega} dist_\Omega^B(z, z_0) = \infty.$$

Theorem 3 means that this is the case for hyperconvex domains but the proof does not give any quantitative behaviour of teh Bergman distance near the boundary. Diederich and Ohsawa [**19**] showed the following lower bound for domains with C^2-boundary:

$$dist_\Omega^B(\cdot, z_0) \geq \frac{1}{C} \log\log \delta_\Omega^{-1},$$

where $\delta_\Omega(z)$ is the euclidean distance of z to $\partial\Omega$ and a positive constant C depends on Ω and z_0. They used in particular a rather technical analogue of the Green function.

Their approach was simplified in [**9**], where an improved estimate was shown:

$$dist_\Omega^B(\cdot, z_0) \geq \frac{\log \delta_\Omega^{-1}}{C \log\log \delta_\Omega^{-1}}.$$

The key was the following result directly linking the pluricomplex Green function with the Bergman metric: if Ω is pseudoconvex and $z, w \in \Omega$ are such that $\{G_z < -1\} \cap \{G_w < -1\} = \emptyset$ then

$$dist_\Omega^B(z, w) \geq c_n > 0.$$

PROBLEM 5. *Does the following estimate hold for bounded pseudoconvex domains with C^2 boundary:*

$$dist_\Omega^B(\cdot, z_0) \geq \frac{1}{C} \log \delta_\Omega^{-1}?$$

It would be optimal. It is known for strongly pseudoconvex domains and also for convex ones (in the latter case without any additional assumptions on the boundary - see [**9**]).

4. Suita Conjecture

Let D be a bounded domain in $\mathbb{C}$. For $z \in D$ the capacity of the complement of D with respect to z is given by

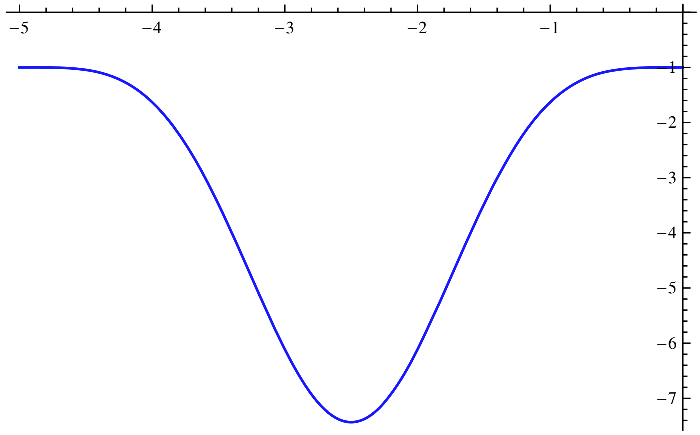
$$c_D(z) = \exp(\lim_{\zeta \to z} (G_D(\zeta, z) - \log|\zeta - z|)).$$

It is not invariant with respect to biholomorphic mappings but one can easily show that the metric $c_D|dz|$ is. It was considered by Suita [**38**] who conjectured the following upper bound for its curvature

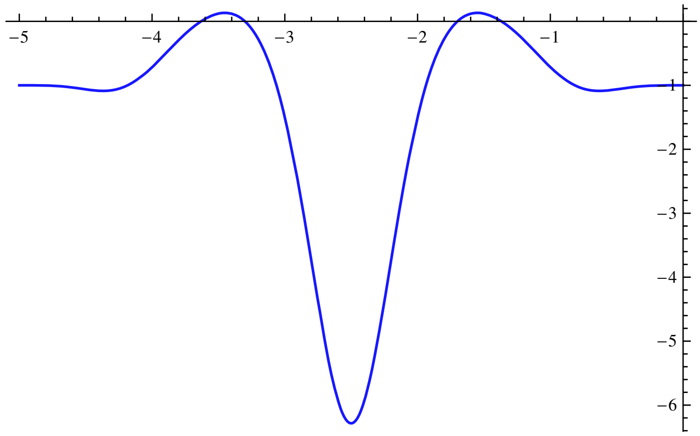
$$-\frac{(\log c_D)_{z\bar{z}}}{c_D^2} \leq -1. \tag{9}$$

One can easily show that we have equality in (9) if D is simply connected. For arbitrary D, by approximation it is enough to prove (9) for domains with smooth boundary. One can also show that then we have equality on the boundary. Therefore the Suita conjecture essentially asks whether the curvature of $c_D|dz|$ satisfies the maximum principle.

Suita [**38**], using elliptic functions, proved strict inequality in (9) for an annulus, and thus for any doubly connected regular domain. For example, if $D = \{e^{-5} < |z| < 1\}$ the curvature of $c_D|dz|$ as a function of $\log|z|$ looks as follows (pictures made with *Mathematica*):



On the other hand, it is not a common property of invariant metrics in complex analysis that their curvatures satisfy the maximum principle. For example, for the same annulus we have the following picture for the curvature of the Bergman metric $B_D = \frac{\partial^2}{\partial z \partial \bar{z}}(\log K_D)|dz|^2$



It was in fact shown in [**20**] (see also [**40**]) that the maximum of the curvature of the Bergman metric on the annulus $\{r < |z| < 1\}$ tends to 2, the optimal upper bound, as $r \to 0$.

Suita [**38**] proved that

$$K_D = \frac{1}{\pi}(\log c_D)_{z\bar{z}},$$

it can in fact be easily deduced from the Schiffer formula

$$K_D(z, w) = \frac{2}{\pi} \frac{\partial^2 G_D}{\partial z \partial \bar{w}}(z, w), \quad z \neq w.$$

This means that (9) is equivalent to

$$c_D^2 \leq \pi K_D. \tag{10}$$

A breakthrough came with a paper of Ohsawa [**35**] who realized that this is really an extension problem: it is equivalent, given $z \in D$, to construct a holomorphic f

in D with $f(z) = 1$ and

$$\int_D |f|^2 d\lambda \leq \frac{\pi}{(c_D(z))^2}.$$

Using the methods of the original proof of the Ohsawa-Takegoshi extension theorem [**36**] he obtained the estimate

$$c_D^2 \leq C\pi K_D$$

with $C = 750$. This was later improved to $C = 2$ in [**10**] and to $C = 1.95388\dots$ in [**24**].

The inequality (10) was eventually showed in [**11**] where also a version of the Ohsawa-Takegoshi extension theorem with optimal constant was obtained. Of course the same proof of (10) works also for Riemann surfaces admitting a Green function (that is the ones admitting a bounded nonconstant subharmonic function). Guan and Zhou [**23**] have recently answered in the affirmative a more detailed question of Suita: strict inequality in (10) holds for all Riemann surfaces except for those that are biholomorphic to the unit disk with possibly a closed polar subset removed.

It turns out that finding the best constant in Herbort's estimate (7) and in (8) leads to a simpler proof of the Suita conjecture than in [**11**]. It was recently explored in [**12**]. Herbort [**25**] originally showed (7) with

$$C = 1 + 4e^{4n+3+R^2}.$$

where R is such that $\Omega \subset B(z_0, R)$ for some z_0. It was improved in [**9**] to

$$C = C_n = (1 + 4/Ei(n))^2, \tag{11}$$

where

$$Ei(t) = \int_t^\infty \frac{ds}{se^s}.$$

The main tool was an estimate for $\bar\partial$ due to Donnelly and Fefferman [**22**] (Berndtsson [**3**] proved that in fact it can be quite easily deduced from Hörmander's estimate).

A way to improve this constant is to use the tensor power trick: for a positive integer m consider the domain $\widetilde\Omega = \Omega \times \cdots \times \Omega \subset \mathbb{C}^{nm}$, $\widetilde w = (w, \dots, w)$ and $\widetilde f(z^1, \dots, z^m) = f(z^1)\dots f(z^m)$. Then

$$K_{\widetilde\Omega}(z^1, \dots, z^m) = K_\Omega(z^1)\dots K_\Omega(z^m)$$

and by [**28**]

$$\{G_{\widetilde w} < -1\} = \{G_w < -1\} \times \cdots \times \{G_w < -1\}.$$

Now by (7) with the constant (11) we get

$$\frac{|f(w)|^2}{K_\Omega(w)} \leq C_{nm}^{1/m} \int_{\{G_w < -1\}} |f|^2 d\lambda.$$

Since

$$\lim_{m\to\infty} C_{nm}^{1/m} = e^{2n},$$

we obtain (7) with $C = e^{2n}$ and this constant is optimal (take for example $f \equiv 1$ in a ball and w its center).

Doing the same for an arbitrary sublevel set we obtain the following result:

THEOREM 4. *Assume that* Ω *is pseudoconvex,* $f \in H^2(\Omega)$*,* $w \in \Omega$ *and* $a \geq 0$*. Then*

$$\frac{|f(w)|^2}{K_\Omega(w)} \leq e^{2na} \int_{\{G_w < -a\}} |f|^2 d\lambda.$$

For $f \equiv 1$ we get the main estimate from [**12**]:

$$K_\Omega(w) \geq \frac{1}{e^{2na}\lambda(\{G_w < -a\})}. \tag{12}$$

Now, letting $a \to \infty$, we easily get (10). This way we have obtained a nontrivial one-dimensional result making use in an essential way of many complex variables. As noticed by Lempert [**33**], the estimate (12) can be also deduced from a result of Berndtsson [**4**] on log-subharmonicity of sections of the Bergman kernel.

This estimate can also be used to simplify the complex analytic proof of the Bourgain-Milman inequality [**14**] due to Nazarov [**34**], for details see [**12**].

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