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The Bergman kernel and pluripotential theory

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Abstract.

We survey recent developments relating the notions of the Bergman kernel and pluripotential theory and indicate some open problems.

§1. Introduction

We will discuss recent results relating the Bergman kernel and pluripotential theory. For $n = 1$ that there is such a relation is perhaps not surprising, since then the Bergman kernel can be expressed in terms of the Green function

$$K_\Omega = -\frac{2}{\pi}\frac{\partial^2 g_\Omega}{\partial z \partial \overline{w}}.$$

No counterpart of this is known for $n \geq 2$. Nevertheless, the pluricomplex Green function in several variables turned out to be a very useful tool in the theory of the Bergman kernel and Bergman metric. We will concentrate on the results that directly relate these two notions.

First we collect basic definitions, notations and assumptions. Good general references are for example [19], [25], [20] (for the Bergman kernel) and [23] (for pluripotential theoretic notions). Throughout Ω will always denote a bounded pseudoconvex domain in $\mathbb{C}^n$ (if $n = 1$ then every domain is pseudoconvex). The Bergman kernel $K_\Omega(z, w)$, $z, w \in \Omega$, is determined by

$$f(w) = \int_\Omega f(z)\overline{K_\Omega(z,w)}d\lambda(z), \quad w \in \Omega,\ f \in H^2(\Omega),$$

where $H^2(\Omega)$ is the (Hilbert) space of all holomorphic functions in Ω that belong to $L^2(\Omega)$. By k_Ω we will denote the Bergman kernel on the

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diagonal

$$(1.1)\quad k_\Omega(z) = K_\Omega(z,z) = \sup\left\{\frac{|f(z)|^2}{||f||^2} : f \in H^2(\Omega)\setminus\{0\}\right\}, \quad z \in \Omega,$$

($||f||$ is the L^2-norm). Then $\log k_\Omega$ is a smooth strongly plurisubharmonic function in Ω and the Bergman metric B_Ω is the Kähler metric given by the potential $\log k_\Omega$, that is

$$B_\Omega^2(z;X) = \sum_{j,k=1}^{n} \frac{\partial^2 \log k_\Omega}{\partial z_j \partial \overline{z}_k}(z) X_j \overline{X}_k, \quad z \in \Omega,\ X \in \mathbb{C}^n.$$

The Bergman metric defines the Bergman distance in Ω which will be denoted by $dist_\Omega$. We will call Ω *Bergman complete* if it is complete w.r.t. $dist_\Omega$, and *Bergman exhaustive* if $\lim_{z\to\partial\Omega} k_\Omega(z) = \infty$.

For a fixed $w \in \Omega$ the *pluricomplex Green function* with pole at w is defined by $g_w := g_\Omega(z,w) = \sup \mathcal{B}_w$, where

$$\mathcal{B}_w = \{u \in PSH(\Omega) : u < 0, \limsup_{z\to w}\big(u(z) - \log|z-w|\big) < \infty\}.$$

Then $g_w \in \mathcal{B}_w$ and

$$c_\Omega(w) = \exp \limsup_{z\to w}\big(g_w(z) - \log|z-w|\big)$$

is the logarithmic capacity of Ω w.r.t. w. One of the main differences between one and several complex variables is the symmetry of g_Ω: of course it is always symmetric if $n = 1$ and usually not true for $n \geq 2$ (the first counterexample was found by Bedford-Demailly [1]).

The domain Ω is called *hyperconvex* if it admits a bounded plurisubharmonic exhaustion function, that is there exists $u \in PSH(\Omega)$ such that $u < 0$ in Ω and $\lim_{z\to\partial\Omega} u(z) = 0$ (of course, if $n = 1$ then hyperconvexity is equivalent to the regularity of Ω). It was shown by Demailly [12] that if Ω is hyperconvex then g_Ω is continuous on $\overline{\Omega} \times \Omega$ (off the diagonal, vanishing on the boundary) but it is still an open problem if it is continuous on $\Omega \times \partial\Omega$ (for partial results see [8], [7], [17] and [6]).

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§2. Bergman completeness, Bergman exhaustivity and hyperconvexity

In this section we will concentrate on the relations between these three notions. We start with the following two results.

Theorem 2.1. (Ohsawa [26], [27]) *If Ω is hyperconvex then it is Bergman exhaustive.*

Theorem 2.2. (Herbort [16], Błocki-Pflug [7]) *If Ω is hyperconvex then it is Bergman complete.*

Theorem (2.2) was proved independently in [16] and [7] ([7] heavily relied on [9], where Theorem 2.2 was proved in particular for $n = 1$, whereas [16] was written independently of both [7] and [9]).

We are now going to sketch the main ideas behind the proof of Theorem 2.2. As a byproduct, the method also gives Theorem 2.1 (the original Ohsawa proofs from [26] and [27] were different, we will discuss the one from [27] later). First, we use the theory of the complex Monge-Ampère operator to estimate the volume of the sublevel sets $\{g_w < -1\}$ for w near the boundary. In [5] it was shown that for hyperconvex Ω there exists a unique $u_\Omega \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u_\Omega = 0$ on $\partial\Omega$ and $(dd^c u_\Omega)^n = d\lambda$. Then integrating by parts (see [4])

$$\begin{aligned} vol(\{g_w < -1\}) &\leq \int_\Omega |g_w|^n (dd^c u_\Omega)^n \\ &\leq n!\, ||u_\Omega||^{n-1}_{L^\infty(\Omega)} \int_\Omega |u_\Omega| (dd^c g_w)^n \\ &\leq C(n, diam\,\Omega)\, |u_\Omega(w)|. \end{aligned} \tag{2.1}$$

In particular,

$$\Omega \text{ is hyperconvex } \Rightarrow \lim_{w \to \partial\Omega} vol(\{g_w < -1\}) = 0. \tag{2.2}$$

The above proof of (2.2) is taken from [7]. It was also independently shown in [16] (the argument there was due to Coman), where a result from [8] was used.

Before proceeding further, let us comment on the implication (2.2). As noticed in [32] (see p. 53), the reverse implication is true if $n = 1$. The following example from [16]

$$\{(z, w) \in \mathbb{C}^2 : |w| < e^{-1/|z|} < e^{-1}\}$$

shows that it is no longer true for $n \geq 2$ (see the review of [16] in Mathematical Reviews). (2.1) also shows that $g_w \to 0$ in $L^n(\Omega)$ as

$w \to \partial\Omega$ from which one can easily get that $g_w \to 0$ in $L^p(\Omega)$ for every $p < \infty$. The open problem of continuity of g_Ω on $\Omega \times \partial\Omega$ (for hyperconvex Ω) is equivalent to locally uniform convergence $g_w \to 0$ in Ω as $w \to \partial\Omega$.

To finish the proof of Theorem 2.2 we use the following estimate from [16] (it is proved using Hörmander's L^2-estimate for the $\overline{\partial}$ operator [18]; see also [9] and [6])

$$\frac{|f(w)|^2}{k_\Omega(w)} \le c_n \int_{\{g_w < -1\}} |f|^2 d\lambda, \quad f \in H^2(\Omega),\ w \in \Omega. \tag{2.3}$$

Combining (2.2) with (2.3) we get, if Ω is hyperconvex,

$$\lim_{w \to \partial\Omega} \frac{|f(w)|^2}{k_\Omega(w)} = 0, \quad f \in H^2(\Omega). \tag{2.4}$$

This is precisely the criterion of Kobayashi [24] and we conclude that Ω is Bergman complete. In addition, if we use (2.3) with $f \equiv 1$ and (2.1) we obtain the following quantitative version of Theorem 2.1, which also gives a comparison between the Bergman kernel and the solution to the complex Monge-Ampère equation

$$k_\Omega \ge \frac{1}{C(n, diam\,\Omega)\,|u_\Omega|}. \tag{2.5}$$

The reverse implications in Theorems 2.1 and 2.2 are false even for $n = 1$. Ohsawa [26] considered Zalcman-type domains

$$\Delta(0,1) \setminus \overline{\bigcup_{k=1}^{\infty} \Delta(2^{-k}, r_k)}, \tag{2.6}$$

where $\Delta(z, r)$ denotes the disk centered ar z with radius r and r_k is a sequence decreasing to 0 such that $r_k < 2^{-k}$ and $\overline{\Delta}(2^{-k}, r_k) \cap \overline{\Delta}(2^{-j}, r_j) = \emptyset$ for $k \ne j$. From Wiener's criterion it then follows that (2.6) is hyperconvex if and only if

$$\sum_{k=1}^{\infty} \frac{k}{-\log r_k} = \infty.$$

On the other hand, Ohsawa [26] showed that if for example $r_k = 2^{-k^3}$ (for $k \ge 2$) then (2.6) is Bergman exhaustive. Chen [9] proved that then (2.6) is also Bergman complete, we thus get a counterexample to reverse implications in Theorems 2.1 and 2.2.

The relation between Bergman exhaustivity and Bergman completeness is also of interest. The problem is related to the Kobayashi criterion

(2.4). For if (2.4) was equivalent to Bergman completeness (this problem was posed by Kobayashi) then Bergman completeness would imply Bergman exhaustiveness (putting $f \equiv 1$ in (2.4)). Let us first look at (2.4). By (1.1) we have

$$\frac{|f(z)|}{\sqrt{k_\Omega(z)}} \leq \frac{|h(z)|}{\sqrt{k_\Omega(z)}} + ||f-h||, \quad f, h \in H^2(\Omega),\ z \in \Omega,$$

and we easily see that to verify (2.4) it is enough to check it, for a given sequence $\Omega \ni w_j \to w_0 \in \partial\Omega$, for f belonging to a dense subspace of $H^2(\Omega)$. Therefore, if Ω is Bergman exhaustive and $H^\infty(\Omega, w_0)$, the space of holomorphic functions in Ω that are bounded near w_0, is dense in $H^2(\Omega)$ for every $w_0 \in \partial\Omega$ then Ω satisfies (2.4) and is thus also Bergman complete. We use the following.

Theorem 2.3. (Hedberg [15], Chen [10]) *If $n = 1$ then $H^\infty(\Omega, w_0)$ is dense in $H^2(\Omega)$ for every $w_0 \in \partial\Omega$.*

Corollary 2.1. (Chen [10]) *If $n = 1$ then Bergman exhaustiveness implies Bergman completeness.*

The above results are false for $n \geq 2$ and the counterexample is the Hartogs triangle $\{(z, w) \in \mathbb{C}^2 : |w| < |z| < 1\}$. They hold however if one in addition assumes that for every $w_0 \in \partial\Omega$ there exists a neighborhood basis U_j of w_0 such that $\Omega \cup U_j$ is pseudoconvex for every j (in the case of Hartogs triangle this is not true at the origin) - see [6].

The remaining problem is therefore whether Bergman completeness implies Bergman exhaustiveness. It was settled in the negative by Zwonek [33] who showed that the following domain

$$\Delta(0,1) \setminus \overline{\bigcup_{k=2}^{\infty} \bigcup_{j=0}^{k^5-1} \Delta\big(k^{-5}e^{2\pi i j/k^5}, e^{-k^{19}}\big)}, \tag{2.7}$$

is Bergman complete but not Bergman exhaustive (see also [22]). Note that any such an example, by Theorem 2.3, does not satisfy (2.4) which shows that the Kobayashi criterion is not necessary for Bergman completeness.

It is possible to characterize Bergman exhaustive domains in terms of potential theory in dimension 1.

Theorem 2.4. (Zwonek [34]) *Assume $n = 1$. Then Ω is Bergman exhaustive if and only if*

$$\int_0^{1/2} \frac{dt}{-t^3 \log cap\,(\overline{\Delta}(z,t) \setminus \Omega)} = \infty, \quad z \in \partial\Omega.$$

From Theorem 2.4 it follows in particular that (2.6) is Bergman exhaustive if and only if

$$\sum_{k=1}^{\infty} \frac{4^k}{-\log r_k} = \infty.$$

No characterization of Bergman completeness in terms of potential theory is known. Jucha [21] however showed that (2.6) is Bergman complete if and only if

$$\sum_{k=1}^{\infty} \frac{2^k}{\sqrt{-\log r_k}} = \infty.$$

As a consequence, one can simplifiy the Zwonek example (2.7): it is sufficient to take (2.6) with $r_k = 2^{-k^2 4^k}$.

From the definition it easily follows that Bergman completeness is a biholomorphically invariant notion, whereas Bergman exhaustiveness is not: the Hartogs triangle is biholomorphic to $\Delta \times \Delta_*$, which is not Bergman exhaustive. To author's knowledge, no such example is known for $n = 1$ (it would of course also show that the Kobayashi criterion is not necessary for Bergman completeness).

In [6] it was shown that the Kobayashi criterion (2.4) for Bergman completeness can be replaced with the following

$$\limsup_{w \to \partial\Omega} \frac{|f(w)|^2}{k_\Omega(w)} < ||f||^2, \quad f \in H^2(\Omega) \setminus \{0\}.$$

It remains an open problem if this condition is necessary for Bergman completeness.

§3. Other results

Diederich-Ohsawa [14] proved a quantitative estimate for the Bergman distance in smooth pseudoconvex domains. Pluripotential theory turned out to be one of the main tools in establishing this result. The estimate from [14] was improved in [6] with help of the following theorem.

Theorem 3.1. ([6]) *Assume that Ω is psudoconvex and $z, w \in \Omega$ are such that $\{g_z < -1\} \cap \{g_w < -1\} = \emptyset$. Then $dist_\Omega(z, w) \geq c_n > 0$.*

On the other hand, the following estimate was used in [13] (see also [11]) to show a quantitative bound for the Bergman metric in smooth pseudoconvex domains.

Theorem 3.2. (Diederich-Herbort [13]) *There exists a positive constant C, depending only on n and the diameter of Ω, such that for any psudoconvex Ω*

$$\frac{1}{C}B_{\{g_w<-1\}}(w;X) \le B_\Omega(w;X) \le CB_{\{g_w<-1\}}(w;X), \quad w\in\Omega,\ X\in\mathbb{C}^n.$$

No counterpart of Zwonek's Theorem 2.4, characterizing the domains where $\lim_{z\to\partial\Omega} k_\Omega(z)=\infty$ in terms of potential theory, is known for $n\ge 2$. However, the domains with $\limsup_{z\to\partial\Omega} k_\Omega(z)=\infty$ are characterized completely.

Theorem 3.3. (Pflug-Zwonek [29]) *The following are equivalent*

(1) *Ω is an L^2-domain of holomorphy (that is Ω is a domain of existence of a function from $H^2(\Omega)$);*
(2) *$\partial\Omega$ has no pluripolar part (that is if U is open then $U\cap\partial\Omega$ is either empty or non-pluripolar);*
(3) $\limsup_{z\to w} k_\Omega(z)=\infty,\ \ w\in\partial\Omega.$

The proof of Theorem 2.1 in [27] relied on the following quantitative estimate.

Theorem 3.4. (Ohsawa [27]) *Assume $n=1$. There exists a positive numerical constant C such that for any Ω*

$$C\sqrt{k_\Omega(w)}\ge c_\Omega(w), \quad w\in\Omega.$$

The above result of course gives Theorem 2.1 for $n=1$ and also provides another quantitative bound for the Bergman kernel from below in terms of potential theory, alternative to (2.5). Theorem 2.1 for arbitrary n then follows easily from the Ohsawa-Takegoshi extension theorem [28].

Ohsawa [27] obtained $C=\sqrt{750\pi}$ in Theorem 3.4. Berndtsson [3] proved this estimate with $C=\sqrt{6\pi}$. The Suita conjecture [30] asserts that the estimate holds with $C=\sqrt{\pi}$. This constant would be then optimal - it is attained for the disk.

In fact, one can easily generalize Theorem 3.4 to higher dimensions. Without loss of generality we may assume that Ω is hyperconvex (the general case can be obtained by approximation). For a fixed $w\in\Omega$ by [31] one can find $\zeta\in\mathbb{C}^n$, $|\zeta|=1$, such that

$$c_\Omega(w)=\exp\lim_{\lambda\to 0}(g_w(w+\lambda\zeta)-\log|\lambda|).$$

By D denote the one dimensional slice $\{\lambda\in\mathbb{C}: w+\lambda\zeta\in\Omega\}$ and by g the Green function for D with pole at 0. Then $g(\lambda)\ge g_w(w+\lambda\zeta)$

and thus $c_D(0) \geq c_\Omega(w)$. By Theorem 3.4 and the Ohsawa-Takegoshi extension theorem

$$c_\Omega(w) \leq c_D(0) \leq C_S\sqrt{k_D(0)} \leq C_S C_{OT}\sqrt{k_\Omega(w)},$$

where C_S is the constant from Theorem 3.4 and C_{OT} the constant from the Ohsawa-Takegoshi extension theorem (Berndtsson [2] showed that if $\Omega \subset \{|z_1| \leq 1\}$ then one can take $C_{OT} = 4\pi$).

We do not know if $\lim\limits_{w\to\partial\Omega} c_\Omega(w) = \infty$ for hyperconvex Ω (and $n \geq 2$). If this was the case then the above estimate would give another quantitative version of Theorem 2.1.

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