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Regularity of the fundamental solution for the Monge-Ampère operator

Introduction

Let Ω be a bounded convex domain in $\mathbb{R}^n$. As in [2] we define a function $g_\Omega : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ as follows: for $y \in \Omega$ let $g_\Omega(\cdot, y)$ be a unique solution to the following Dirichlet problem

$$\begin{cases} u \in \mathrm{CVX}(\Omega) \cap \mathrm{C}(\overline{\Omega}) \\ Mu = \delta_y \\ u|_{\partial\Omega} = 0. \end{cases}$$

Here M is the Monge-Ampère operator which for smooth functions takes the form

$$Mu = \det D^2 u$$

and can be well defined for arbitrary convex functions as a nonnegative Borel measure (see [3]). The function $g_\Omega(\cdot, y)$ vanishes at $\partial\Omega$ and is affine along the intervals joining y with $\partial\Omega$. Therefore, it is determined by its value at y and that is why we are concerned with the function $h_\Omega(y) := g_\Omega(y, y)$.

In [2] it was proved in particular that g_Ω is continuous on $\overline{\Omega} \times \overline{\Omega}$ (with $g_\Omega := 0$ on $\partial(\Omega \times \Omega)$) and it is never symmetric unless $n = 1$. In this paper, in fact not relying on the results from [2], we investigate the regularity of the function h_Ω. We show in particular that h_Ω is always smooth (C^∞) and convex as conjectured in [2].

Throughout the paper Ω is always meant to be a bounded convex domain in $\mathbb{R}^n$.

1. Preliminaries

We will need several simple facts:

Proposition 1.1. *If $\Omega_j \uparrow \Omega$ then $g_{\Omega_j} \downarrow g_\Omega$; in particular $h_{\Omega_j} \downarrow h_\Omega$.*

Proof. Fix $y \in \Omega$ and set $u_j := g_{\Omega_j}(\cdot, y)$, $u := g_\Omega(\cdot, y)$. Then $u_{j+1} \leq 0 = u_j$ on $\partial\Omega_{j+1}$ and $Mu_{j+1} = Mu_j = \delta_y$. Therefore, by the comparison principle (see [3]) $u \leq u_{j+1} \leq u_j$. We have

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$u_j \downarrow v \in \mathrm{CVX}(\Omega)$, $u \le v \le 0$ and by the continuity of the Monge-Ampère operator ([3], Theorem 3.7), $Mv = \lim Mu_j = \delta_y$. This means that $u = v$. ■

Lemma 1.2. *Let Ω be smooth and fix $y \in \Omega$. Let u be such that $u(y) = -1$, $u = 0$ on $\partial\Omega$ and u is affine along half-lines beginning at y. (In fact $u = |h_\Omega(y)|^{-1} g_\Omega(\cdot, y)$ on Ω.) Then for $x \in \partial\Omega$ we have $\nabla u(x) = n_x / \langle x - y, n_x \rangle$.*

Proof. It is easy to see that $\nabla(x) = n_x/\mathrm{dist}(y, T_x)$, where T_x is the affine tangent hyperplane to $\partial\Omega$ at x. Let y^* denote a point from T_x, where $\mathrm{dist}(y, T_x) = |y^* - y|$. We have $y^* - y = \alpha n_x$ for some $\alpha > 0$ and $\langle y^* - x, n_x \rangle = 0$. Combining these gives $\alpha = \langle x - y, n_x \rangle$ and the lemma follows. ■

Lemma 1.3. *Let D be a convex domain in $\mathbb{R}^n$ containing the origin. For $w \in \partial B$, the unit sphere, by $f(w)$ denote a positive number such that $f(w)w \in \partial\Omega$. Then*

$$\lambda(D) = \frac{1}{n} \int_{\partial B} f(w)^n d\sigma(w).$$

Proof. It follows immediately if we use the polar change of coordinates:

$$J : (0, \infty) \times \partial B \ni (r, x) \longrightarrow rx \in \mathbb{R}^n \setminus \{0\},$$

and observe that $\mathrm{Jac}\, J = r^{n-1}$. ■

2. The integral formula

Let Ω be a smooth. Then we can define a mapping

$$S : \partial\Omega \ni x \longrightarrow n_x \in \partial B.$$

One can show that if Ω is strictly convex then S is a smooth diffeomorphism.

Our basic tool in studying the regularity of h_Ω will be the following integral formula:

Theorem 2.1. *Let Ω be smooth and strictly convex. Then*

$$h_\Omega(y) = -\left(\frac{1}{n} \int_{\partial B} \langle S^{-1}(w) - y, w \rangle^{-n} d\sigma(w) \right)^{-1/n}, \quad y \in \Omega.$$

Proof. Let u be as in Lemma 1.2 and by E denote the gradient image of u at y (see [3] for the definition of a gradient image). Then $\lambda(E) = \int_\Omega Mu$ and $h_\Omega(y) = -\lambda(E)^{-1/n}$. Moreover, since Ω is smooth, we have $\partial E = \nabla u(\partial\Omega)$. By Lemma 1.2 at $x \in \partial\Omega$ one has $\nabla u(x) = n_x/\langle x-y, n_x\rangle$. Now Theorem 2.1 follows immediately from Lemma 1.3.

Using Theorem 2.1 and the fact that $K := \operatorname{Jac} S$ is the Gauss curvature of $\partial\Omega$ one can show the following:

Theorem 2.2 *Let Ω be smooth. Then*

$$h_\Omega(y) = -\left(\frac{1}{n}\int_{\partial\Omega} \langle x-y, n_x\rangle^{-n} K(x)\, d\sigma(w)\right)^{-1/n}, \quad y \in \Omega. \ \blacksquare$$

3. The main results

Theorem 3.1. *Let Ω be an arbitrary bounded convex domain in $\mathbb{R}^n$. Then h_Ω is smooth and for $y \in \Omega$ the following estimate holds:*

$$\left|\frac{\partial^\alpha(|h_\Omega|^{-n})}{\partial y^\alpha}(y)\right| \le \frac{(n+|\alpha|-1)!}{n!}\frac{\sigma(\partial B)}{\operatorname{dist}(y,\partial\Omega)^{n+|\alpha|}}. \tag{3.1}$$

Proof. By Proposition 1.1 and Sobolev theorem it will be sufficient if we prove (3.1) in smooth and strictly convex domains. Set $f := |h_\Omega|^{-n}$. Then by Theorem 2.1

$$f(y) = \frac{1}{n}\int_{\partial B} F(y,w)^{-n} d\sigma(w),$$

where $F(y,w) = \langle S^{-1}(w) - y, w\rangle = \operatorname{dist}(y, T_{S^{-1}(w)})$. F is smooth and positive on $\Omega \times \partial B$ and we can differentiate under the sign of integration. Then for a multi-indice $\alpha = (\alpha_1, \ldots, \alpha_n)$ we have

$$\frac{\partial^\alpha f}{\partial y_\alpha}(y) = (n+1)\ldots(n+|\alpha|-1)\int_{\partial B} F(y,w)^{-n-|\alpha|} w_1^{\alpha_1}\ldots w_n^{\alpha_n}\, d\sigma(w)$$

and, since $F(y,w) \ge \operatorname{dist}(y,\partial\Omega)$, the estimate (3.1) follows. $\blacksquare$

Theorem 3.2. *Take $y \in \Omega$ and $\zeta \in \partial B$. Then*

$$\frac{\partial^2 h_\Omega}{\partial\zeta\partial\zeta}(y) \ge c_n(\operatorname{diam}\Omega)^{-2n-2}|h_\Omega(y)|^{2n+1},$$

where $c_n > 0$ depends only on n. In particular h_Ω is strictly convex.

Proof. We may assume that $\zeta = (1, 0, \ldots, 0)$ and, by Proposition 1.1, that Ω is smooth and strictly convex. By Theorem 2.1

$$f(y) := (-h_\Omega(y))^{-n} = \frac{1}{n}\int_{\partial B} F(y,w)^{-n} d\sigma(w), \tag{3.2}$$

where

$$F(y,w) := \langle S^{-1}(w) - y, w\rangle = \mathrm{dist}(y, T_{S^{-1}(w)}) \le \mathrm{diam}\Omega.$$

We can compute that

$$(h_\Omega)_{11} \left(= \frac{\partial^2 h_\Omega}{\partial y_1^2}\right) = \frac{1}{n}(-h_\Omega)^{2n+1}\left(f\, f_{11} - \frac{n+1}{n} f_1^2\right).$$

Differentiating (3.2) under the sign of integration we obtain

$$f_1 = \int_{\partial B} F(y,w)^{-n-1} w_1 d\sigma(w)$$

and

$$f_{11} = (n+1)\int_{\partial B} F(y,w)^{-n-2} w_1^2 d\sigma(w).$$

Let C^+ and C^- denote the half-spheres $\{w \in \partial B : w_1 \ge 0\}$ and $\{w \in \partial B : w_1 \le 0\}$, respectively. Then

$$\begin{aligned} f_1^2 = &\left(\int_{\partial B} F(y,w)^{-n-1}|w_1| d\sigma(w)\right)^2 \\ &- 4\int_{C^+} F(y,w)^{-n-1}|w_1| d\sigma(w) \int_{C^-} F(y,w)^{-n-1}|w_1| d\sigma(w). \end{aligned} \tag{3.3}$$

From the Schwarz inequality we infer

$$\begin{aligned} \left(\int_{\partial B} F(y,w)^{-n-1}|w_1| d\sigma(w)\right)^2 &\le \int_{\partial B} F(y,w)^{-n} d\sigma(w) \int_{\partial B} F(y,w)^{-n-2} w_1^2 d\sigma(w) \\ &= \frac{n}{n+1} f\, f_{11}. \end{aligned}$$

Combining this with (3.3) and the fact that $F(y,w) \le \mathrm{diam}\Omega$ we obtain

$$\begin{aligned} f\, f_{11} - \frac{n+1}{n} f_1^2 & \\ &\ge 4\frac{n+1}{n}\int_{C^+} F(y,w)^{-n-1}|w_1| d\sigma(w) \int_{C^-} F(y,w)^{-n-1}|w_1| d\sigma(w) \\ &\ge 4\frac{n+1}{n}\left(\int_{C^+} |w_1| d\sigma(w)\right)^2 (\mathrm{diam}\Omega)^{-2n-2} \end{aligned}$$

and the theorem follows. ■

Theorem 3.2 gives a lower bound for the eigenvalues of the matrix $D^2 h_\Omega(y)$. We conjecture that Mh_Ω, which is in fact the product of all eigenvalues, tends to ∞ as y tends to $\partial\Omega$. This would in particular imply Theorem A in [1].

References

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