Suita conjecture and the Ohsawa-Takegoshi extension theorem

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Abstract We prove a conjecture of N. Suita which says that for any bounded domain D in $\mathbb{C}$ one has $c_D^2 \le \pi K_D$, where $c_D(z)$ is the logarithmic capacity of $\mathbb{C} \setminus D$ with respect to $z \in D$ and K_D the Bergman kernel on the diagonal. We also obtain optimal constant in the Ohsawa-Takegoshi extension theorem.

1 Introduction

Suita [16] (see also [15, p. 179]) conjectured that for a bounded domain D in $\mathbb{C}$ one has

$$c_D^2 \le \pi K_D. \tag{1}$$

Here

$$c_D(z) = \exp\Big(\lim_{\zeta \to z}\big(G_D(\zeta, z) - \log|\zeta - z|\big)\Big)$$

is the logarithmic capacity of the complement of D with respect to $z \in D$, where $G_D(\cdot, z)$ the (negative) Green function for D with pole at z, and

$$K_D(z) = \sup\left\{|f(z)|^2 : f \text{ holomorphic in } D, \int_D |f|^2 d\lambda \le 1\right\}$$

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denotes the Bergman kernel on the diagonal. One can easily show that we have equality in (1) if D is simply connected. Suita in [16], using the theory of elliptic functions, proved strict inequality in (1) when D is an annulus, and hence also any regular doubly connected domain.

This conjecture has geometric interpretation and consequences. Suita also observed that

$$K_D = \frac{1}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} (\log c_D)$$

and thus that (1) is equivalent to the fact that the curvature of the invariant metric $c_D |dz|$ is bounded above by -4. Therefore (1) means precisely that maximal curvature is attained on the boundary (if D is smooth then one can show that this curvature is equal to -4 on ∂D, or equivalently that the ratio $c_D^2 / \pi K_D$ is equal to 1 there). This is not the case for other invariant metrics related to the Bergman kernel, e.g. $K_D |dz|^2$ and the Bergman metric $(\log K_D)_{z\bar{z}} |dz|^2$ (see [9] and [19]). On the other hand, this property is satisfied for the Carathéodory metric (see [17] and [18]) but this is much simpler to prove.

The relation of the Suita conjecture to the extension theorem was first observed and explored by Ohsawa [12]: to prove (1) for a given $z \in D$ we have to find a holomorphic f in D with $f(z) = 1$ and

$$\int_D |f|^2 d\lambda \leq \frac{\pi}{(c_D(z))^2}.$$

Using methods developed in [14] he proved the estimate

$$c_D^2 \leq C \pi K_D$$

with $C = 750$. This was later improved to $C = 2$ in [5], where the main tool was an estimate from [1]. A slightly better constant $C = 1.954$ was recently obtained in [10], see also [6] for a much simpler proof.

Our goal is to show the following version of the Ohsawa-Takegoshi theorem with optimal constant which in particular implies the Suita conjecture:

Theorem 1 *Assume that $\Omega \subset \mathbb{C}^{n-1} \times D$ is pseudoconvex, where D is a bounded domain in $\mathbb{C}$ containing the origin. Then for any holomorphic f in $\Omega' := \Omega \cap \{z_n = 0\}$ and φ plurisubharmonic in Ω one can find a holomorphic extension F of f to Ω with*

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

This kind of result, with a worse constant though, was first formulated in [8] (it used the proof of the Ohsawa-Takegoshi theorem from [2]), see also [13].

Our proof is in big part motivated by the recent work of Chen [7] who showed that the Ohsawa-Takegoshi theorem can deduced directly from the classical Hörmander estimate for the $\bar{\partial}$-equation [11]. We use Chen's idea in the proof of Theorem 2 below. It improves a related $\bar{\partial}$-estimate from [6] which gave Theorem 1 but only with the same constant as in [10]. This $\bar{\partial}$-estimate was one of the crucial improvements compared to [6]. Another one was the solution of the following ODE problem: find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ such that h is convex and decreasing, $g + \log t$, $h + \log t$ vanish at ∞, and

$$\left(1 - \frac{(g')^2}{h''}\right)e^{2g-h+t} \geq 1.$$

2 The proof

Similarly as in [7] our main tool will be the Hörmander estimate: if Ω is pseudoconvex in $\mathbb{C}^n$ and

$$\alpha = \sum_j \alpha_j d\bar{z}_j \in L^2_{loc,(0,1)}(\Omega)$$

is such that $\bar{\partial}\alpha = 0$ then for any smooth, strongly plurisubharmonic φ in Ω one can find $u \in L^2_{loc}(\Omega)$ such that

$$\bar{\partial}u = \alpha$$

and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq \int_\Omega |\alpha|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} d\lambda. \qquad (2)$$

Here

$$|\alpha|^2_{i\partial\bar{\partial}\varphi} = \sum_{j,k} \varphi^{j\bar{k}} \bar{\alpha}_j \alpha_k$$

(where $(\varphi^{j\bar{k}})$ is the inverse transposed of the complex Hessian $(\partial^2\varphi/\partial z_j\partial\bar{z}_k)$) denotes the length of α with respect to the Kähler metric $i\partial\bar{\partial}\varphi$.

It was observed in [4] that (2) makes also sense (and indeed remains true) for arbitrary plurisubharmonic φ: instead of $|\alpha|^2_{i\partial\bar\partial\varphi}$ one should take any $H \in L^\infty_{loc}(\Omega)$ with

$$i\bar\alpha \wedge \alpha \leq H\, i\partial\bar\partial\varphi.$$

In this convention we formulate our new estimate for the $\bar\partial$-equation:

Theorem 2 *Let $\alpha \in L^2_{loc,(0,1)}(\Omega)$ be a $\bar\partial$-closed form in a pseudoconvex domain Ω in $\mathbb{C}^n$. Assume that φ is plurisubharmonic in Ω, $\psi \in W^{1,2}_{loc}(\Omega)$ is locally bounded from above and they satisfy $|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \leq H < 1$ in Ω for some $H \in L^\infty_{loc}(\Omega)$. Then there exists $u \in L^2_{loc}(\Omega)$ solving $\bar\partial u = \alpha$ and such that for every $b > 0$*

$$\int_\Omega |u|^2(1-H)e^{2\psi-\varphi}d\lambda \leq \left(1 + \frac{1}{b}\right)\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi}\frac{1+bH}{1-H}e^{2\psi-\varphi}d\lambda.$$

If in addition $H \leq \delta < 1$ on $\operatorname{supp}\alpha$ then

$$\int_\Omega |u|^2(1-H)e^{2\psi-\varphi}d\lambda \leq \frac{1+\sqrt\delta}{1-\sqrt\delta}\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi}e^{2\psi-\varphi}d\lambda. \qquad (3)$$

Proof By standard approximation we may assume that φ is smooth up to the boundary, strongly plurisubharmonic, and that ψ is bounded in Ω. Note that then $L^2(\Omega, e^{\psi-\varphi})$, $L^2(\Omega, e^{-\varphi})$ and $L^2(\Omega)$ are the same sets. We use a trick from [3]: define u to be the minimal solution to $\bar\partial u = \alpha$ in $L^2(\Omega, e^{\psi-\varphi})$ which means that u is perpendicular to $\ker\bar\partial$ there. Then $v := ue^\psi$ is perpendicular to $\ker\bar\partial$ in $L^2(\Omega, e^{-\varphi})$ and thus it is the minimal solution to $\bar\partial v = \beta$, where

$$\beta := e^\psi(\alpha + u\,\bar\partial\psi) \in L^2_{loc,(0,1)}(\Omega).$$

By Hörmander's estimate (2)

$$\int_\Omega |v|^2 e^{-\varphi}d\lambda \leq \int_\Omega |\beta|^2_{i\partial\bar\partial\varphi}e^{-\varphi}d\lambda$$

which means that

$$\int_\Omega |u|^2 e^{2\psi-\varphi}d\lambda \leq \int_\Omega |\alpha + u\,\bar\partial\psi|^2_{i\partial\bar\partial\varphi}e^{2\psi-\varphi}d\lambda$$

$$\leq \int_\Omega \left(|\alpha|^2_{i\partial\bar\partial\varphi} + 2|u|\sqrt{H}|\alpha|_{i\partial\bar\partial\varphi} + |u|^2 H\right)e^{2\psi-\varphi}d\lambda.$$

For $t > 0$ we will get

$$\int_\Omega |u|^2 (1 - H) e^{2\psi - \varphi} d\lambda$$
$$\leq \int_\Omega \left[|\alpha|^2_{i\partial\bar\partial\varphi} \left(1 + t^{-1} \frac{H}{1 - H} \right) + t|u|^2 (1 - H) \right] e^{2\psi - \varphi} d\lambda.$$

We obtain the first estimate if we take $t := 1/(b + 1)$, whereas the second one follows if we let $b := \delta^{-1/2}$. $\qquad\square$

Proof of Theorem 1 Using appropriate approximation we may assume that Ω is bounded, smooth and strongly pseudoconvex, φ is smooth up to the boundary and that f is holomorphic in a neighborhood of $\overline{\Omega'}$. Fix $\varepsilon > 0$ and define

$$\alpha := \bar\partial \big(f(z') \chi(-2\log|z_n|) \big) = -\frac{f(z') \chi'(-2\log|z_n|)}{\bar z_n} d\bar z_n,$$

where we write $z = (z', z_n)$ and $\chi \in C^{0,1}(\mathbb{R})$ is non-decreasing and such that $\chi = 0$ on $\{t \leq -2\log\varepsilon\}$, $\chi(\infty) = 1$ (it will be precisely determined later). Note that α is defined in Ω for sufficiently small ε, $\mathrm{supp}\,\alpha \subset \{|z_n| \leq \varepsilon\}$, and for any solution of $\bar\partial u = \alpha$ the function

$$F := f(z') \chi(-2\log|z_n|) - u \tag{4}$$

is a holomorphic extension of f provided that $u = 0$ on Ω'.

In order to use Theorem 2 and define appropriate weights $\widetilde\varphi$, ψ the choice of the following two functions on $\mathbb{R}_+$ is absolutely crucial:

$$h(t) := -\log(t + e^{-t} - 1),$$
$$g(t) := -\log(t + e^{-t} - 1) + \log(1 - e^{-t}).$$

They solve the ODE problem formulated in the introduction: one can check that they are decreasing, convex,

$$\left(1 - \frac{(g')^2}{h''} \right) e^{2g - h + t} = 1 \tag{5}$$

and

$$\lim_{t \to \infty} \big(h(t) - 2g(t) + \log(-h'(t)) \big) = 0. \tag{6}$$

(In fact, as noticed by the referee, we have $g = \log(-h')$. This substitution reduces (5) to

$$\frac{d^2}{dt^2}(e^{-h}) = e^{-t}$$

and enables to solve the desired ODE problem immediately.)

For $G := G_D(\cdot, 0)$ we can write $G = \log |\zeta| + v$, where v is a bounded harmonic function in D. There are positive constants C_0, C_1 such that

$$\left|2G - 2\log|\zeta|\right| \leq C_0 \quad \text{in } D$$

and

$$\left|2G_\zeta - \frac{1}{\zeta}\right| \leq C_1 \text{ near the origin.}$$

For $M := -2\log\varepsilon - C_0$ we define

$$\eta(t) := \begin{cases} h(t), & t < M, \\ -\delta\log(t - M + a) + b, & t \geq M \end{cases}$$

and

$$\gamma(t) := \begin{cases} g(t), & t < M, \\ -\delta\log(t - M + a) + \widetilde{b}, & t \geq M, \end{cases}$$

where $\delta = M^{-1/2}$ and $a, b, \widetilde{b}$ are chosen in such a way that $\eta \in C^{1,1}(\mathbb{R}_+)$ and $\gamma \in C^{0,1}(\mathbb{R}_+)$. Then

$$a = a(\varepsilon) = -\frac{\delta}{h'(M)},$$

$$b = b(\varepsilon) = h(M) + \delta\log a,$$

$$\widetilde{b} = \widetilde{b}(\varepsilon) = g(M) + \delta\log a.$$

Note that δ is selected so that

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0, \qquad \lim_{\varepsilon \to 0} a(\varepsilon) = \infty. \tag{7}$$

Now we are ready to define

$$\widetilde{\varphi} := \varphi + 2G + \eta(-2G), \qquad \psi := \gamma(-2G).$$

Then $\widetilde{\varphi}$ is plurisubharmonic in Ω and $\psi \in W_{loc}^{1,2}(D)$ is locally bounded from above. We have

$$|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\widetilde{\varphi}} \leq |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\eta(-2G))} = \frac{(\gamma'(-2G))^2}{\eta''(-2G)} \begin{cases} < 1 & \text{in } \Omega, \\ = \delta & \text{on } \operatorname{supp}\alpha \end{cases}$$

(note that $\operatorname{supp}\alpha \subset \{-2G \geq M\}$) and

$$|\alpha|^2_{i\partial\bar\partial\tilde\varphi} \leq |\alpha|^2_{i\partial\bar\partial(\eta(-2G))} = \frac{|f(z')|^2(\chi'(-2\log|z_n|))^2}{4|G_{z_n}|^2|z_n|^2\eta''(-2G)},$$

where we follow the convention discussed before Theorem 2. Since the function $-\delta\log(t-M+a)+t$ is increasing in t by (5) we have

$$\left(1-\frac{(\gamma')^2}{\eta''}\right)e^{2\gamma-\eta+t} \begin{cases} =1 & \text{on } \{t<M\}, \\ \geq (1-\delta)e^{2g(M)-h(M)+M} & \text{on } \{t\geq M\} \end{cases}$$

and thus for sufficiently small ε

$$\left(1-\frac{(\gamma')^2}{\eta''}\right)e^{2\gamma-\eta+t} \geq 1$$

on $\mathbb{R}_+$. Theorem 2 now gives a solution $u=u_\varepsilon$ of $\bar\partial u = \alpha$ such that

$$\int_\Omega |u|^2 e^{-\varphi}d\lambda \leq \int_\Omega |u|^2\left(1-|\bar\partial\psi|^2_{i\partial\bar\partial\tilde\varphi}\right)e^{2\psi-\tilde\varphi}d\lambda \leq \frac{1+\sqrt\delta}{1-\sqrt\delta}\mathcal{A}(\varepsilon), \qquad (8)$$

where

$$\mathcal{A}(\varepsilon) := \int_\Omega \frac{|f(z')|^2(\chi'(-2\log|z_n|))^2}{4|G_{z_n}|^2|z_n|^2\eta''(-2G)}e^{2\psi-\tilde\varphi}d\lambda.$$

We have

$$\varlimsup_{\varepsilon\to0}\mathcal{A}(\varepsilon) \leq \varlimsup_{\varepsilon\to0}\mathcal{B}(\varepsilon)\int_{\Omega'}|f|^2e^{-\varphi}d\lambda'$$

where

$$\mathcal{B}(\varepsilon) := \int_{\{|\zeta|\leq\varepsilon\}} \frac{(\chi'(-2\log|\zeta|))^2}{4|G_\zeta|^2|\zeta|^2\eta''(-2G)}e^{2\psi-\eta(-2G)-2G}d\lambda(\zeta).$$

We now want to replace G by $\log|\zeta|$ in the above integral. Note that

$$\lim_{\varepsilon\to0}\sup_{\{|\zeta|\leq\varepsilon\}}|\zeta|^2e^{-2G} = \frac{1}{(c_D(0))^2}$$

and near the origin

$$4|G_\zeta|^2|\zeta|^2 \geq \left(1-C_1|\zeta|\right)^2.$$

Moreover on $\{|\zeta|\leq\varepsilon\}$ we have

$$2\psi - \eta(-2G) \leq 2\gamma\left(-2\log|\zeta|-C_0\right) - \eta\left(-2\log|\zeta|-C_0\right)$$

$$\leq 2\gamma\left(-2\log|\zeta|\right) - \eta\left(-2\log|\zeta|\right) + \delta\log\left(1 + \frac{C_0}{a}\right)$$

and

$$\eta''(-2G) \geq \eta''\left(-2\log|\zeta| + C_0\right) \geq \left(\frac{a + C_0}{a + 2C_0}\right)^2 \eta''(-2\log|\zeta|).$$

Hence, from (7) it follows that

$$\overline{\lim_{\varepsilon \to 0}} \, \mathcal{B}(\varepsilon) \leq \frac{1}{(c_D(0))^2} \overline{\lim_{\varepsilon \to 0}} \, \widetilde{\mathcal{B}}(\varepsilon),$$

where

$$\widetilde{\mathcal{B}}(\varepsilon) = \int_{\{|\zeta| \leq \varepsilon\}} \frac{(\chi'(-2\log|\zeta|))^2}{|\zeta|^2 \eta''(-2\log|\zeta|)} e^{2\gamma(-2\log|\zeta|) - \eta(-2\log|\zeta|)} d\lambda(\zeta)$$

$$= \pi \int_{M+C_0}^{\infty} \frac{(\chi')^2 e^{2\gamma - \eta}}{\eta''} \, dt.$$

By the Schwarz inequality the optimal choice of increasing χ with

$$\chi(\infty) = \int_{M+C_0}^{\infty} \chi' \, dt = 1$$

is

$$\chi(t) := \begin{cases} 0, & t < M + C_0, \\ \frac{1}{c} \int_{M+C_0}^{t} w \, ds, & t \geq M + C_0, \end{cases}$$

where $w = \eta'' e^{\eta - 2\gamma}$ and $c = \int_{M+C_0}^{\infty} w \, dt$. Then

$$\widetilde{\mathcal{B}}(\varepsilon) = \frac{\pi}{\int_{M+C_0}^{\infty} \eta'' e^{\eta - 2\gamma} \, dt}$$

and

$$\int_{M+C_0}^{\infty} \eta'' e^{\eta - 2\gamma} \, dt = \delta e^{b - 2\widetilde{b}} \int_{C_0}^{\infty} (t + a)^{\delta - 2} dt$$

$$= \frac{1}{1 - \delta}\left(1 + \frac{C_0}{a}\right)^{\delta - 1} e^{h(M) - 2g(M) + \log(-h'(M))}.$$

By (6) and (7)

$$\lim_{\varepsilon \to 0} \widetilde{\mathcal{B}}(\varepsilon) = \pi$$

and we have thus obtained that

$$\overline{\lim_{\varepsilon \to 0}} \int_{\Omega} |u_\varepsilon|^2 e^{-\varphi} d\lambda \le \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Finally, we note that if $0 < \widetilde{\varepsilon} \le \varepsilon$ then

$$\int_{\{|\zeta| \le \widetilde{\varepsilon}\}} \left(1 - \frac{(\gamma'(-2G))^2}{\eta''(-2G)}\right) e^{2\psi - \eta(-2G) - 2G} d\lambda(\zeta)$$

$$\ge \frac{1}{C(\varepsilon)} \int_{\{|\zeta| \le \widetilde{\varepsilon}\}} \frac{e^{2\gamma(-2\log|\zeta|) - \eta(-2\log|\zeta|)}}{|\zeta|^2} d\lambda(\zeta)$$

$$= \frac{\pi e^{2\widetilde{b} - b}}{C(\varepsilon)} \int_{-2\log\widetilde{\varepsilon}}^{\infty} (t - M + a)^{-\delta} dt$$

$$= \infty$$

and (8) implies that $u = 0$ almost everywhere on Ω'. The weak limit of $F = F_\varepsilon$ given by (4) is the required extension. $\qquad\square$

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