Estimates for the Bergman Kernel and Logarithmic Capacity

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Notation

$\Omega \subset \mathbb{C}^n$ domain, $w \in \Omega$

$K_\Omega(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 d\lambda \leq 1\}$

(Bergman kernel on the diagonal)

$$\begin{aligned} G_w(z) &= G_\Omega(z, w) \\ &= \sup\{u(z) : u \in PSH^-(\Omega) : \overline{\lim_{z\to w}} \left(u(z) - \log|z-w|\right) < \infty\} \end{aligned}$$

(pluricomplex Green function)

For $n = 1$ one can define

$c_\Omega(w) = \exp\left(\lim_{z\to w}(G_\Omega(z, w) - log|z-w|)\right)$ (logarithmic capacity)

B. 2013 For $\Omega \subset \mathbb{C}$ we have $c_\Omega^2 \leq \pi K_\Omega$ (Suita conjecture)

Original proof: $\bar\partial$-equation, special ODE, optimal version of the Ohsawa-Takegoshi extension theorem

Guan-Zhou 2015 "$=$" $\Leftrightarrow$ $\Omega \simeq \Delta \setminus F$, where Δ is the unit disk and F is closed and polar

B. 2014 If Ω is pseudoconvex in $\mathbb{C}^n$, $w \in \Omega$ and $t \leq 0$ then

$$K_\Omega(w) \geq \frac{1}{e^{-2nt}\lambda(\{G_w < t\})}.$$

For $n = 1$ we have

$$\lim_{t\to-\infty} \frac{1}{e^{-2nt}\lambda(\{G_w < t\})} = \frac{c_\Omega(w)^2}{\pi}.$$

Proof 1 (sketch) Using Donnelly-Fefferman estimate for $\bar\partial$ one can show

$$K_\Omega(w) \geq \frac{1}{c_n \lambda(\{G_w < -1\})},$$

where $c_n = (1 + 4/Ei(n))^2$ and $Ei(t) = \int_t^\infty \frac{ds}{se^s}$

(Herbort 1999, B. 2005)

Now use the tensor power trick: $\widetilde{\Omega} = \Omega \times \cdots \times \Omega \subset \mathbb{C}^{nm}$,
$\widetilde{w} = (w, \dots, w)$ for $m \gg 0$. Then

$$K_{\widetilde{\Omega}}(\widetilde{w}) = (K_\Omega(w))^m, \quad \lambda(\{G_{\widetilde{w}} < -1\}) = (\lambda(\{G_w < -1\}))^m,$$

and for $\widetilde{\Omega}$ we get

$$K_\Omega(w) \geq \frac{1}{c_{nm}^{1/m} \lambda(\{G_w < -1\})}.$$

But $\lim_{m\to\infty} c_{nm}^{1/m} = e^{2n}$. Similarly we can get the estimate for every t. $\square$

Proof 2 (Lempert) By Berndtsson's result on log-(pluri)subharmonicity of the Bergman kernel for sections of a pseudoconvex domain (Maitani-Yamaguchi in dimension two) it follows that $\log K_{\{G_w<t\}}(w)$ is convex for $t \in (-\infty, 0]$. Therefore

$$t \longmapsto 2nt + \log K_{\{G_w<t\}}(w)$$

is convex and bounded, hence non-decreasing. It follows that

$$K_\Omega(w) \geq e^{2nt} K_{\{G_w<t\}}(w) \geq \frac{e^{2nt}}{\lambda(\{G_w < t\})}. \quad \square$$

Berndtsson-Lempert: This method can be improved to show the Ohsawa-Takegoshi extension theorem with optimal constant.

Proof 1 uses infinitely many dimensions, whereas Proof 2 works in dimension $n+1$. No known proof in dimension n.

Convex Domains

B. 2014 If $\Omega \subset \mathbb{C}^n$ is convex then

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega^K(w))}, \quad w \in \Omega,$$

where $I_\Omega^K(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$ (Kobayashi indicatrix).

Extremely accurate estimate:

B.-Zwonek 2015 For $\Omega = \{|z_1| + |z_2|^2 < 1\}$ and $w = (b, 0),\ b \in [0, 1)$ one has

$$\lambda(I_\Omega^K(w)) K_\Omega(w) = 1 + \frac{(1-b)^3 b^2}{3(1+b)^3} \leq 1.0047.$$

B.-Zwonek 2015 If $\Omega \subset \mathbb{C}^n$ is convex then

$$K_\Omega(w) \leq \frac{4^n}{\lambda(I_\Omega^K(w))}, \quad w \in \Omega,$$

General Case

B. 2014 If $\Omega \subset \mathbb{C}^n$ is pseudoconvex then

$$K_\Omega(w) \geq \frac{1}{e^{-2nt}\lambda(\{G_w < t\})}, \quad w \in \Omega, \ t \leq 0.$$

B.-Zwonek 2015 If $\Omega \subset \mathbb{C}^n$ is pseudoconvex then

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega^A(w))}, \quad w \in \Omega,$$

where $I_\Omega^A(w) = \{X \in \mathbb{C}^n : \overline{\lim}_{\zeta \to 0} \left(G_w(w + \zeta X) - \log|\zeta|\right) < 0\}$
(Azukawa indicatrix).

Conjecture For Ω pseudoconvex and $w \in \Omega$ the function

$$t \longmapsto e^{-2nt}\lambda(\{G_w < t\})$$

is non-decreasing in t.

It would easily follow if we knew that the function

$$t \longmapsto \log \lambda(\{G_w < t\}) - 2t \qquad (1)$$

is convex on $(-\infty, 0]$. Fornæss however constructed a counterexample to this (already for $n = 1$). Generalizing his example one can show

Theorem If t_0 is a critical value of G_w then

$$\frac{d}{dt}\lambda(\{G_w < t\})\Big|_{t=t_0} = \infty.$$

In particular, the function (1) is not convex.

B.-Zwonek 2015 For $n = 1$ the function (1) is non-decreasing.

The conjecture for arbitrary n is equivalent to the following *pluricomplex isoperimetric inequality* for smooth strongly pseudoconvex Ω

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G_w|} \geq 2n\lambda(\Omega).$$

Conjecture If $\Omega \Subset \mathbb{C}^n$ is smooth and strongly pseudoconvex and K is the Levi curvature of $\partial\Omega$ then

$$\int_{\partial\Omega} K \, d\sigma \geq c_n \sqrt{\lambda(\Omega)}.$$

Carleson (1963) showed that for $\Omega \subset \mathbb{C}$ the Bergman space

$$A^2(\Omega) := \mathcal{O} \cap L^2(\Omega)$$

is trivial iff $\mathbb{C} \setminus \Omega$ is polar. In other words,

$$K_\Omega(w) = 0 \iff c_\Omega(w) = 0.$$

The Suita inequality $c_\Omega^2 \leq \pi K_\Omega$ is a quantitative version of $\Rightarrow$.

Theorem For $\Omega \subset \mathbb{C}$, $w \in \Omega$ and $0 < r < \delta_\Omega(w) := \operatorname{dist}(w, \partial\Omega)$ we have

$$K_\Omega(w) \leq \frac{1}{-2\pi r^2 \max\limits_{z \in \overline{\Delta}(w,r)} G_\Omega(z,w)}.$$

Corollary $\exists$ uniform constant $C > 0$ s.th. for $w \in \Omega \subset \mathbb{C}$, we have

$$K_\Omega(w) \leq \frac{C}{\delta_\Omega(w)^2 \log\left(1/(\delta_\Omega(w) c_\Omega(w))\right)}.$$

Wiegerinck Conjecture

Wiegerinck, 1984

- $\forall\ k \in \mathbb{N}\ \exists\ \Omega \subset \mathbb{C}^2$ s.th. $\dim A^2(\Omega) = k$
- Conjecture If $\Omega \subset \mathbb{C}^n$ is pseudoconvex then either $A^2(\Omega) = \{0\}$ or $\dim A^2(\Omega) = \infty$
- True for $n = 1$

For $w \in \Omega \subset \mathbb{C}$ and $j = 0, 1, \dots$ define

$$K_\Omega^{(j)}(w) := \sup\{|f^{(j)}(w)|^2 \colon f \in A^2(\Omega),\ ||f|| \le 1,\\ f(w) = f'(w) = \dots = f^{(j-1)}(w) = 0\}.$$

Similarly as before one can show that

$$\frac{j!(j+1)!}{\pi}(c_\Omega(w))^{2j+2} \le K_\Omega^{(j)}(w) \le \frac{C_j}{\delta_\Omega(w)^{2+j}\log\left(1/(\delta_\Omega(w)c_\Omega(w))\right)}.$$

Balanced Domains

A domain $\Omega \subset \mathbb{C}^n$ is called balanced if $z \in \Omega,\ \zeta \in \Delta \ \Rightarrow\ \zeta z \in \Omega$. Then

$$K_\Omega(0) = \frac{1}{\lambda(\Omega)}.$$

Since for any domain Ω and $w \in \Omega$ the Azukawa indicatrix

$$I^A_\Omega(w) = \{X \in \mathbb{C}^n : \overline{\lim_{\zeta \to 0}} \left(G_w(w + \zeta X) - \log|\zeta| \right) < 0\}$$

is a balanced domain, it follows that for pseudoconvex domains one has

$$K_\Omega(w) \geq K_{I^A_\Omega(w)}(0).$$

Similarly for $j = 0, 1, \dots$ and $X \in \mathbb{C}^n$

$$K^{(j)}_\Omega(w; X) \geq K^{(j)}_{I^A_\Omega(w)}(0; X),$$

where

$$K^{(j)}_\Omega(w; X) := \sup\{|f^{(j)}(w).X|^2 \colon f \in A^2(\Omega),\ ||f|| \leq 1,$$
$$f(w) = f'(w) = \cdots = f^{(j-1)}(w) = 0\}.$$

Corollary $\dim A^2(I_\Omega^A(w)) = \infty \;\Rightarrow\; \dim A^2(\Omega) = \infty$

Pflug-Zwonek 2016
Wiegerinck conjecture holds for balanced domains in $\mathbb{C}^2$

Problem $K_\Omega(w) > 0 \;\Leftrightarrow\; \lambda(I_\Omega^A(w)) < \infty$

A good upper bound for the Bergman kernel in terms of pluripotential theory would be needed.

Some Partial Results

Theorem If $\Omega \subset \mathbb{C}^n$ is pseudoconvex and such that $\dim A^2(\Omega) < \infty$ then for $w \in \Omega$ and $t \leq 0$

$$A^2(\{G_w < t\}) = \{f|_{\{G_w<t\}} \colon f \in A^2(\Omega)\}.$$

Sketch of proof We may assume that $G := G_w \not\equiv -\infty$. Clearly $\supset$ and it is enough to prove that

$$\dim A^2(\{G < t\}) \leq \dim A^2(\Omega).$$

Take linearly independent $f_1, \dots, f_k \in A^2(\{G < t\})$. One can find m such that the m-jets of $f_1, \dots, f_k$ at w are linearly independent. Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(s) = 1$ for $s \leq t - 3$, $\chi(s) = 0$ for $s \geq t - 1$ and $|\chi'| \leq 1$.

Set

$$\alpha := \bar{\partial}(f_j \chi \circ G) = f_j \chi' \circ G \, \bar{\partial} G,$$
$$\varphi := 2(n+m+1)G,$$
$$\psi := -\log(-G).$$

Since

$$i\bar{\alpha} \wedge \alpha \le |f_j|^2 G^2 i\partial\bar{\partial}\psi,$$

it follows from the Donnelly-Fefferman estimate that one can find a solution to $\bar{\partial} u = \alpha$ with

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le 4 \int_\Omega |f_j|^2 G^2 e^{-\varphi} d\lambda.$$

Therefore $F_j := f_j \chi \circ G - u \in A^2(\Omega)$ and F_j has the same m-jet at w as f_j and thus $F_1, \dots, F_k$ are also linearly independent. □

Theorem Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex and such that for some $w \in \Omega$ and $t \leq 0$ the set $\{G_w < t\}$ does not satisfy the Liouville property. Then $A^2(\Omega)$ is either trivial or infinitely dimensional.

Theorem Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex and $w_j \in \Omega$ be an infinite sequence, not contained in any analytic subset of Ω, and such that for some $t < 0$ and all $j \neq k$ one has $\{G_{w_j} < t\} \cap \{G_{w_k} < t\} = \emptyset$. Then $A^2(\Omega)$ is either trivial or infinitely dimensional.

Example (Siciak 1985) There exists a pseudoconvex balanced dense domain Ω in $\mathbb{C}^2$ such that $\dim A^2(\Omega) = \infty$.

Thank you!