Bergman Completeness

$\Omega$ bounded domain in $\mathbb{C}^n$

$H^2(\Omega) = \mathcal{O}(\Omega) \cap L^2(\Omega)$

$K_\Omega(\cdot, \cdot)$ Bergman kernel

$$f(w) = \int_{\Omega} f \overline{K_\Omega(\cdot, w)} \, d\lambda, \quad w \in \Omega, \ f \in H^2(\Omega)$$

$$K_\Omega(w) = K_\Omega(w, w) = \sup\{|f(w)|^2 : f \in H^2(\Omega), \|f\| \leq 1\}$$

$\Omega$ is called Bergman complete if it is complete w.r.t. the Bergman metric $B_\Omega = i\partial\bar{\partial}\log K_\Omega$
Kobayashi Criterion (1959) If

$$\lim_{w \to \partial \Omega} \frac{|f(w)|^2}{K_{\Omega}(w)} = 0, \quad f \in H^2(\Omega),$$

then $\Omega$ is Bergman complete.

The opposite is not true even for $n = 1$ (Zwonek, 2001).

Kobayashi Criterion easily follows using the embedding

$$\iota : \Omega \ni w \mapsto [K_{\Omega}(\cdot, w)] \in P(H^2(\Omega))$$

and the fact that $\iota^* \omega_{FS} = B_{\Omega}$.

Since $\iota$ is distance decreasing,

$$\text{dist}^B_{\Omega}(z, w) \geq \arccos \left( \frac{|K_{\Omega}(z, w)|}{\sqrt{K_{\Omega}(z)K_{\Omega}(w)}} \right).$$
Some Pluripotential Theory

Ω is called hyperconvex if it admits a negative plurisubharmonic (psh) exhaustion function \( u \in PSH^-(\Omega) \) s.th. \( u = 0 \) on \( \partial \Omega \).

Demailly (1985) If \( \Omega \) is pseudoconvex with Lipschitz boundary then it is hyperconvex.

**Pluricomplex Green function** For a pole \( w \in \Omega \) we set

\[
G_\Omega(\cdot, w) = G_w = \sup \{ v \in PSH^-(\Omega) : v \leq \log |\cdot - w| + C \}
\]

Lempert (1981) \( \Omega \) convex \( \Rightarrow \) \( G_\Omega \) symmetric

Demailly (1985) \( \Omega \) hyperconvex \( \Rightarrow \) \( e^{G_\Omega} \in C(\bar{\Omega} \times \Omega) \)

Open Problem \( e^{G_\Omega} \in C(\bar{\Omega} \times \bar{\Omega} \setminus \Delta_{\partial \Omega}) \)

Equivalently: \( G(\cdot, w_k) \to 0 \) loc. uniformly as \( w_k \to \partial \Omega \)?

True if \( \partial \Omega \in C^2 \) (Herbort, 2000)
Demailly (1985) If Ω is hyperconvex then $G_w = G_\Omega(\cdot, w)$ is the unique solution to

$$
\begin{cases}
  u \in PSH(\Omega) \cap C(\bar{\Omega}\setminus\{w\}) \\
  (dd^c u)^n = (2\pi)^n \delta_w \\
  u = 0 \text{ on } \partial \Omega \\
  u \leq \log |\cdot - w| + C
\end{cases}
$$

B. (2000) If Ω is smooth and strongly pseudoconvex then $G_w \in C^{1,1}(\bar{\Omega}\setminus\{w\})$.

B. (1995) If Ω is hyperconvex then $\exists! \; u = u_\Omega$ s.th.

$$
\begin{cases}
  u \in PSH(\Omega) \cap C(\bar{\Omega}) \\
  (dd^c u)^n = 1 \, d\lambda \\
  u = 0 \text{ on } \partial \Omega.
\end{cases}
$$

Hyperconvex domains are Bergman complete

Herbert If \( \Omega \) is pseudoconvex then

\[
\frac{|f(w)|^2}{K_\Omega(w)} \leq c_n \int |f|^2 d\lambda, \quad w \in \Omega, \ f \in H^2(\Omega).
\]

\( \{G_w < -1\} \)

Corollary \( \lim_{w \to \partial \Omega} \lambda(\{G_w < -1\}) = 0 \Rightarrow \Omega \) is Bergman complete

Proposition If \( \Omega \) is hyperconvex then

\[
\lim_{w \to \partial \Omega} \|G_w\|_{L^n(\Omega)} = 0.
\]

Sketch of proof \( \|G_w\|^n_n = \int_\Omega |G_w|^n (dd^c u_\Omega)^n \leq n! \|u_\Omega\|_\infty^{n-1} \int_\Omega |u_\Omega|(dd^c G_w)^n \leq C(n, \lambda(\Omega)) |u_\Omega(w)| \)
Lower Bound for the Bergman Distance

Diederich-Ohsawa (1994), B. (2005) If $\Omega$ is pseudoconvex with $C^2$ boundary then

$$\dist_B^\Omega(\cdot, w) \geq \frac{\log \delta^{-1}_\Omega}{C \log \log \delta^{-1}_\Omega},$$

where $\delta_\Omega(z) = \dist_\Omega(z, \partial \Omega)$.

Pluripotential theory is the main tool in proving this estimate, in particular we have the following:

B. (2005) If $\Omega$ is pseudoconvex and $z, w \in \Omega$ are such that

$$\{ G_z < -1 \} \cap \{ G_w < -1 \} = \emptyset$$

then

$$\dist_B^\Omega(z, w) \geq c_n > 0.$$  

Open Problem $\dist_B^\Omega(\cdot, w) \geq \frac{1}{C} \log \delta^{-1}_\Omega$
From Herbort’s estimate

\[
\frac{|f(w)|^2}{K_\Omega(w)} \leq c_n \int |f|^2 d\lambda, \quad w \in \Omega, \ f \in H^2(\Omega),
\]

\{
G_w < -1\}

for \(f \equiv 1\) we get

\[
K_\Omega(w) \geq \frac{1}{c_n \lambda(\{G_w < -1\})}.
\]

To find the optimal constant \(c_n\) here turns out to have very interesting consequences!

Herbert (1999) \(c_n = 1 + 4e^{4n+3+R^2}\), where \(\Omega \subset B(z_0, R)\)

(Main tool: Hörmander’s estimate for \(\bar{\partial}\))

B. (2005) \(c_n = (1 + 4/Ei(n))^2\), where \(Ei(t) = \int_t^\infty \frac{ds}{se^s}\)

(Main tool: Donnelly-Fefferman’s estimate for \(\bar{\partial}\))
Suita Conjecture

$D$ bounded domain in $\mathbb{C}$

$$c_D(z) := \exp \lim_{\zeta \to z} (G_D(\zeta, z) - \log |\zeta - z|)$$

$$(\text{logarithmic capacity of } \mathbb{C} \setminus D \text{ w.r.t. } z)$$

$c_D|dz|$ is an invariant metric (Suita metric)

$$\text{Curv}_{c_D|dz|} = -\frac{(\log c_D)_{z\bar{z}}}{c_D^2}$$

Suita Conjecture (1972): $\text{Curv}_{c_D|dz|} \leq -1$

- “=” if $D$ is simply connected
- “<” if $D$ is an annulus (Suita)
- Enough to prove for $D$ with smooth boundary
- “=” on $\partial D$ if $D$ has smooth boundary
$Curv_{c_D|dz}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log|z|$
$\text{Curv}_{(\log K_D) \bar{z} z} \lvert dz \rvert^2$ for $D = \{ e^{-5} < |z| < 1 \}$ as a function of $\log |z|$
\[
\frac{\partial^2}{\partial z \partial \bar{z}} (\log c_D) = \pi K_D \quad \text{(Suita)}
\]

Therefore the Suita conjecture is equivalent to

\[ c_D^2 \leq \pi K_D. \]

Ohsawa (1995) observed that it is really an extension problem: for \( z \in D \) find holomorphic \( f \) in \( D \) such that \( f(z) = 1 \) and

\[
\int_D |f|^2 d\lambda \leq \frac{\pi}{(c_D(z))^2}.
\]

Using the methods of the Ohsawa-Takegoshi extension theorem he showed the estimate

\[ c_D^2 \leq C \pi K_D \]

with \( C = 750 \).

\[
C = 2 \quad \text{(B., 2007)}
\]

\[
C = 1.95388 \ldots \quad \text{(Guan-Zhou-Zhu, 2011)}
\]
Ohsawa-Takegoshi extension theorem (1987) with optimal constant (B., 2013)

$0 \in D \subset \mathbb{C}, \quad \Omega \subset \mathbb{C}^{n-1} \times D, \quad \Omega$ pseudoconvex,

$\varphi \in PSH(\Omega)$

$f$ holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$

Then there exists a holomorphic extension $F$ of $f$ to $\Omega$ such that

$$
\int_{\Omega} |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{c_D(0)^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.
$$

For $n = 1$ and $\varphi \equiv 0$ we get the Suita conjecture.

Main tool: Hörmander’s estimate for $\bar{\partial}$

B.-Y. Chen (2011) proved that the Ohsawa-Takegoshi theorem (without optimal constant) follows from Hörmander’s estimate.
Tensor Power Trick

We have
\[ \frac{1}{c_n \lambda(\{ G_w < -1 \})} \]
where \( c_n = (1 + 4/Ei(n))^2 \).

Take \( m \gg 0 \) and set \( \tilde{\Omega} := \Omega^m \subset \mathbb{C}^{nm}, \tilde{w} := (w, \ldots, w) \). Then
\[ K_{\tilde{\Omega}}(\tilde{w}) = (K_{\Omega}(w))^m, \quad \lambda_{2nm}(\{ G_{\tilde{w}} < -1 \}) = (\lambda_{2n}(\{ G_w < -1 \}))^m. \]

Therefore
\[ \frac{1}{c_{nm}^{1/m} \lambda(\{ G_w < -1 \})} \]
but
\[ \lim_{m \to \infty} c_{nm}^{1/m} = e^{2n}. \]
Repeating this argument for any sublevel set we get

**Theorem 1** Assume $\Omega$ is pseudoconvex in $\mathbb{C}^n$. Then for $t \leq 0$ and $w \in \Omega$

$$K_\Omega(w) \geq \frac{1}{e^{-2nt} \lambda(\{G_w < t\})}.$$ 

Lempert recently noticed that this estimate can also be proved using Berndtsson’s result on positivity of direct image bundles.

What happens when $t \to -\infty$?

For $n = 1$ we get $K_\Omega \geq c_\Omega^2 / \pi$ (another proof of Suita Conjecture).

**Theorem 2** If $\Omega$ is a convex domain in $\mathbb{C}^n$ then for $w \in \Omega$

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega(w))},$$

$I_\Omega(w) = \{ \varphi'(0) : \varphi \in \mathcal{O}(-\Delta, \Omega), \varphi(0) = w \}$ (Kobayashi indicatrix).
Mahler Conjecture

$K$ - convex symmetric body in $\mathbb{R}^n$

$$K' := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } x \in K\}$$

Mahler volume := $\lambda(K)\lambda(K')$

Mahler volume is an invariant of the Banach space defined by $K$: it is independent of linear transformations and of the choice of inner product.

Santaló Inequality (1949) Mahler volume is maximized by balls

Mahler Conjecture (1938) Mahler volume is minimized by cubes

Hansen-Lima bodies: starting from an interval they are produced by taking products of lower dimensional HL bodies and their duals.

$n = 2$: square
$n = 3$: cube & octahedron
$n = 4$: ...
Bourgain-Milman (1987) There exists $c > 0$ such that

$$\lambda(K)\lambda(K') \geq c^n \frac{4^n}{n!}.$$

Mahler Conjecture: $c = 1$

G. Kuperberg (2006) $c = \pi/4$

Nazarov (2012)

- equivalent SCV formulation of the Mahler Conjecture via the Fourier transform and the Paley-Wiener theorem
- proof of the Bourgain-Milman Inequality ($c = (\pi/4)^3$) using Hörmander’s estimate for $\bar{\partial}$
$K$ - convex symmetric body in $\mathbb{R}^n$
Nazarov: consider $T_K := \text{int} K + i\mathbb{R}^n \subset \mathbb{C}^n$. Then

$$
(\frac{\pi}{4})^{2n} \frac{1}{(\lambda_n(K))^2} \leq K_{T_K}(0) \leq \frac{n! \lambda_n(K')}{\pi^n \lambda_n(K)}.
$$

Therefore

$$
\lambda_n(K)\lambda_n(K') \geq \left(\frac{\pi}{4}\right)^{3n} \frac{4^n}{n!}.
$$

To show the lower bound we can use Theorem 2:

$$
K_{T_K}(0) \geq \frac{1}{\lambda_{2n}(I_{T_K}(0))}.
$$

**Proposition** $I_{T_K}(0) \subset \frac{4}{\pi} (K + iK)$

In particular, $\lambda_{2n}(I_{T_K}(0)) \leq \left(\frac{4}{\pi}\right)^{2n} (\lambda_n(K))^2$

**Conjecture** $\lambda_{2n}(I_{T_K}(0)) \leq \left(\frac{4}{\pi}\right)^n (\lambda_n(K))^2$
Lempert Theory (1981)

Ω - bounded strongly convex domain in $\mathbb{C}^n$ with smooth boundary

$\varphi \in \mathcal{O}(\Delta, \Omega) \cap C(\bar{\Delta}, \bar{\Omega})$ is a geodesic if and only if $\varphi(\partial \Delta) \subset \partial \Omega$ and there exists $h \in \mathcal{O}(\Delta, \mathbb{C}^n) \cap C(\bar{\Delta}, \mathbb{C}^n)$ s.th. the vector $e^{it} \overline{h(e^{it})}$ is outer normal to $\partial \Omega$ at $\varphi(e^{it})$ for every $t \in \mathbb{R}$.

$\exists \ F \in \mathcal{O}(\Omega, \Delta)$ a left-inverse to $\varphi$ (i.e. $F \circ \varphi = id_\Delta$) s.th.

$$(z - \varphi(F(z))) \cdot h(F(z)) = 0, \quad z \in \Omega.$$  

Lempert’s Theory for Tube Domains (S. Zajàc, 2013)

$\Omega = T_K$, where $K$ is smooth and strongly convex in $\mathbb{R}^n$

Since $\text{Im}(e^{it} \overline{h(e^{it})}) = 0$, $h$ must be of the form

$$h(\zeta) = \tilde{w} + \zeta b + \zeta^2 w$$

for some $w \in \mathbb{C}^n$ and $b \in \mathbb{R}^n$. Therefore

$$\text{Re} \ \varphi(e^{it}) = \nu^{-1} \left( \frac{b + 2\text{Re}(\overline{e^{it}w})}{|b + 2\text{Re}(\overline{e^{it}w})|} \right),$$

where $\nu : \partial K \to S^{n-1}$ is the Gauss map.
By the Schwarz formula

\[ \varphi(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + \zeta}{e^{it} - \zeta} \nu^{-1} \left( \frac{b + 2\text{Re}(e^{it}w)}{|b + 2\text{Re}(e^{it}w)|} \right) dt + i\text{Im}\varphi(0). \]

If \( K \) is in addition symmetric then all geodesics in \( T_K \) with \( \varphi(0) = 0 \) are of the form

\[ \varphi(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + \zeta}{e^{it} - \zeta} \nu^{-1} \left( \frac{\text{Re}(e^{it}w)}{|\text{Re}(e^{it}w)|} \right) dt \]

for some \( w \in (\mathbb{C}^n)^* \). Then

\[ \varphi'(0) = \frac{1}{\pi} \int_0^{2\pi} e^{it} \nu^{-1} \left( \frac{\text{Re}(e^{it}\overline{w})}{|\text{Re}(e^{it}\overline{w})|} \right) dt \]

parametrizes \( \partial I_{T_K}(0) \) for \( w \in S^{2n-1} \).

Conjecture \( \lambda_{2n}(I_{T_K}(0)) \leq \left( \frac{4}{\pi} \right)^n (\lambda_n(K))^2 \)