Geodesics in the Spaces of Kähler Metrics and Volume Forms

Zbigniew Błocki
Uniwersytet Jagielloński, Kraków, Poland
http://gamma.im.uj.edu.pl/~blocki

Complex Geometry and Cauchy-Riemann Equation
August 23, 2016
Center for Advanced Study, Oslo

(M,ω) compact Kähler manifold
We can write $\{\omega\} \simeq \mathcal{H}/\sim$, where

$$\mathcal{H} = \{\varphi \in C^\infty(M) \colon \omega_\varphi := \omega + dd^c\varphi > 0\}$$

is the space of Kähler potentials, and

$$\varphi_1 \sim \varphi_2 \;\Leftrightarrow\; \varphi_1 - \varphi_2 = const.$$

Riemannian structure on $\mathcal{H}$ (Mabuchi, 1987 / Donaldson, 1999)

$$\langle\langle \psi, \eta \rangle\rangle := \frac{1}{V}\int_M \psi\,\eta\,\omega_\varphi^n, \quad \psi, \eta \in T_\varphi\mathcal{H} \simeq C^\infty(M),$$

where $V = \int_M \omega^n$.
Levi-Civita connection: if $\varphi \in C^\infty([0,1],\mathcal{H}) \subset C^\infty(M \times [0,1])$ and ψ is a vector field along φ (i.e. $\psi \in C^\infty(M \times [0,1])$, then

$$\nabla_{\dot\varphi}\psi = \dot\psi - \langle \nabla\psi, \nabla\dot\varphi \rangle,$$

so that $\frac{d}{dt}\langle\langle \psi, \eta \rangle\rangle = \langle\langle \nabla_{\dot\varphi}\psi, \eta \rangle\rangle + \langle\langle \psi, \nabla_{\dot\varphi}\eta \rangle\rangle$.

Normalization Aubin-Yau functional $I : \mathcal{H} \to \mathbb{R}$ is uniquely defined by

$$I(0) = 0, \quad \left.\frac{d}{dt}\right|_{t=0} I(\varphi + t\psi) = \frac{1}{V} \int_M \psi \omega_\varphi^n, \ \varphi \in \mathcal{H}, \ \psi \in C^\infty(M).$$

One can show that

$$I(\varphi) = \frac{1}{n+1} \sum_{p=0}^{n} \frac{1}{V} \int_M \varphi \, \omega_\varphi^p \wedge \omega^{n-p}.$$

Then $\mathcal{H}_0 = I^{-1}(0) \simeq \{\omega\}$ defines a natural Riemannian structure on $\{\omega\}$ which is independent of the choice of ω.

Geodesics A curve $\varphi : [0,1] \to \mathcal{H}$ is a *geodesic* if $\nabla_{\dot\varphi}\dot\varphi = 0$, that is

$$\ddot\varphi - |\nabla\dot\varphi|^2 = 0.$$

Locally write $u = g + \varphi$, where $\omega = dd^c g$. Then it is equivalent to

$$u_{tt} - u^{i\bar{j}} u_{ti} u_{t\bar{j}} = 0,$$

which is equivalent to

$$\det \begin{pmatrix} & & & u_{1t} \\ & (u_{j\bar{k}}) & & \vdots \\ & & & u_{nt} \\ u_{t\bar{1}} & \dots & u_{t\bar{n}} & u_{tt} \end{pmatrix} = 0.$$

This means that

$$(\omega + dd^c\varphi)^{n+1} = 0,$$

where $t = \log|z_{n+1}|$ (Semmes, 1992 / Donaldson, 1999).

To find a geodesic connecting $\varphi_0, \varphi_1 \in \mathcal{H}$ one has to solve HCMA

$$(\omega + dd^c\varphi)^{n+1} = 0$$

in $M \times \{0 \leq \log |z_{n+1}| \leq 1\}$ with boundary condition.

Donaldson Conjecture, 1999: Every $\varphi_0, \varphi_1 \in \mathcal{H}$ can be joined by a smooth geodesic.

Consequence: uniqueness of constant scalar curvature (csc) metrics up to holomorphic automorphisms

X.X. Chen, 2000: There exists unique, weak $(\omega + dd^c\varphi \geq 0)$, almost $C^{1,1}$ $(\Delta\varphi \in L^\infty)$ geodesic.

Lempert-Vivas, 2013: A geodesic need not be C^3.

Darvas-Lempert, 2012: A geodesic need not be C^2.

Remaining question: Are geodesics fully $C^{1,1}$?

B., 2012: If *bisec*$(M) \geq 0$ then geodesics are $C^{1,1}$.

Theorem Assume that (M, ω) is a compact Kähler manifold with boundary (possibly empty). Let $\varphi \in C^4(M)$ be such that $\omega_\varphi > 0$ and $\omega_\varphi^n = f\omega^n$. Then

$$|\nabla^2\varphi| \le C,$$

where C depends only on upper bounds for n, $|R|$, $|\nabla R|$, $|\varphi|$, $|\nabla\varphi|$, $\Delta\varphi$, $\sup_{\partial M}|\nabla^2\varphi|$, $||f^{1/n}||_{C^{1,1}(M)}$, $|\nabla(f^{1/2n})|$ and a lower positive bound for f. If M has nonnegative bisectional curvature then the estimate is independent of the latter.

Sketch of proof $\alpha := |\nabla^2\varphi| + |\nabla\varphi|^2 - A\varphi$, where

$$|\nabla^2\varphi| = \max_{X\neq 0}\frac{\langle\nabla_X\nabla\varphi, X\rangle}{|X|^2}$$

and $A \gg 0$. α attains max for some $x_0 \in M$ and $X \in T_{x_0}M$.

$$\widetilde{\alpha} = \frac{\langle\nabla_X\nabla\varphi, X\rangle}{|X|^2} + |\nabla\varphi|^2 - A\varphi$$

also attains max at x_0 but is smooth! Then

$$\frac{\partial^2}{\partial z^p\partial\bar{z}^p}\left(\frac{\langle\nabla_X\nabla\varphi, X\rangle}{g_{j\bar{k}}X^j\bar{X}^k}\right) = \cdots + X^j\bar{X}^k R_{j\bar{k}p\bar{p}}D_X^2\varphi.$$

Weak Solutions to CMA

Kołodziej, 1998 Let (M, ω) be the compact Kähler manifold. If $f \in L^p(M)$ for some $p > 1$ is such that $f \geq 0$ and $\int_M f\omega^n = \int_M \omega^n$ then there exists unique (up to an additive constant) $\varphi \in C(M)$ such that $\omega_\varphi \geq 0$ and

$$\omega_\varphi^n = f\omega^n.$$

Yau, 1978: $f > 0,\ f \in C^\infty \ \Rightarrow\ \varphi \in C^\infty$

B., 2002: $f \geq 0,\ f^{1/(n-1)} \in C^{1,1} \ \Rightarrow\ \Delta\varphi \in L^\infty$

B., 2009: $f \geq 0,\ f^{1/n} \in C^{0,1} \ \Rightarrow\ \varphi \in C^{0,1}$

$bisec(M) \geq 0,\ f \geq 0,\ f^{1/n} \in C^{1,1} \ \Rightarrow\ \varphi \in C^{1,1}$

Space of volume forms (Donaldson, 2010)

(M, g) compact Riemannian manifold
$dV_0 = \sqrt{\det(g_{ij})}$ Riemannian volume form on M , $V_0 = \int_M dV_0$

$$\mathcal{V} := \{dV \text{ volume form on } M \text{ with } \int_M dV = V_0\}$$

Then every element of $\mathcal{V}$ can be written in the form
$dV = (\Delta\varphi + 1)dV_0$, and $\mathcal{V} = \mathcal{H}/\sim$, where

$$\mathcal{H} = \{\varphi \in C^\infty(M) \colon \Delta\varphi + 1 > 0.\}$$

and $\varphi_1 \sim \varphi_2 \;\Leftrightarrow\; \varphi_1 - \varphi_2 = const.$

Riemannian structure on $\mathcal{H}$

$$\langle\langle \psi, \eta \rangle\rangle = \frac{1}{V_0} \int_M \psi\, \eta\, (1+\Delta\varphi) dV_0, \quad \varphi \in \mathcal{H},\ \psi, \eta \in T_\varphi \mathcal{H} \simeq C^\infty(M).$$

Levi-Civita connection: if $\varphi \in C^\infty([0,1], \mathcal{H}) \subset C^\infty(M \times [0,1])$ and ψ is a vector field along φ (i.e. $\psi \in C^\infty(M \times [0,1])$, then

$$\nabla_{\dot\varphi}\psi = \dot\psi - \frac{\langle \nabla\psi, \nabla\dot\varphi \rangle}{\Delta\varphi + 1}.$$

Geodesics $\varphi : [0,1] \to \mathcal{H}$ is a *geodesic* if

$$(\Delta\varphi + 1)\varphi_{tt} - |\nabla\varphi_t|^2 = 0.$$

Chen-He, 2011: Given $\varphi_0, \varphi_1 \in \mathcal{H}$, there exists unique, weak, almost $C^{1,1}$ geodesic connecting them.

By Darvas-Lempert we cannot expect better regularity than C^2.

B.-Gu If M has nonnegative sectional curvature then geodesics are $C^{1,1}$.

Sketch of proof Define

$$\alpha = |\nabla^2\varphi| + |\nabla\varphi|^2 + A(-\varphi + t^2/2).$$

Then α attains max for some $(x_0, t_0) \in M \times (0,1)$ and $X \in T_{x_0}M$. We may assume $X = e_1$, then

$$\widetilde{\alpha} = \nabla_{11}\varphi + |\nabla\varphi|^2 + A(-\varphi + t^2/2).$$

One can show that

$$\nabla_{11}\nabla_{ii}\varphi - \nabla_{ii}\nabla_{11}\varphi = -2R_{11ii}(\nabla_{11}\varphi - \nabla_{ii}\varphi) - \nabla_i R^m_{1i1}\varphi_m - \nabla_1 R^m_{1ii}\varphi_m \le C.$$

Relation to Nahm's equations $T_1, T_2, T_3 : (0,2) \to U(n)$

$$\frac{dT_1}{dt} = [T_2, T_3], \quad \frac{dT_2}{dt} = [T_3, T_1], \quad \frac{dT_3}{dt} = [T_1, T_2].$$

Fixing $B \in GL(n, \mathbb{C})$, Donaldson (1984) showed that they are equivalent to a 2nd order ODE for $h(t)$ valued in the space of positive Hermitian matrices $\mathcal{H} \simeq GL(n, \mathbb{C})/U(n)$ which is Euler-Lagrange for the Lagrangian

$$E(h) = \int \left(\left| \frac{dh}{dt} \right|^2_{\mathcal{H}} + V_B(h) \right) dt,$$

where $V_B(h) = Tr(hBh^{-1}B^*)$. He proved that given $h_0, h_1 \in \mathcal{H}$ one can find unique $h(t)$ joining them. ($h(t)$ is a path of a particle moving under the influence of a potential $-V_B$.)

If M is a Riemann surface then the space of Kähler potentials $\mathcal{H}$ behaves similarly as $\mathcal{G}^c/\mathcal{G}$, where $\mathcal{G}$ is the group of area preserving diffeomorphisms (although $\mathcal{G}^c$ does not really exist!).

Recent developments (Székelyhidi, Tossatti-Weinkove, Chu-Tossatti-Weinkove, Székelyhidi-Tossatti-Weinkove, ...)
Various C^2-estimates

Lemma Let φ be a C^4 function defined near $x_0 \in \mathbb{R}^n$. Assume that $D^2\varphi$ is diagonal at x_0 and $\varphi_{11} > \varphi_{ii}$, $i > 1$, there. Near x_0 define $\lambda := \lambda_{max}(D^2\varphi)$. Then at x_0 we have $\lambda = \varphi_{11}$, $\lambda_p = \varphi_{11p}$ and

$$\lambda_{pp} = \varphi_{11pp} + 2\sum_{i>1} \frac{\varphi_{1ip}^2}{\varphi_{11} - \varphi_{ii}}.$$

Thank you!