On the Bergman Kernel and the Kobayashi Pseudodistance in Convex Domains

Zbigniew Błocki, Włodzimierz Zwonek
Uniwersytet Jagielloński, Kraków, Poland
http://gamma.im.uj.edu.pl/~blocki

DMV-PTM Joint Meeting
Poznań, September 17–20, 2014
Thematic Session: Complex Analysis

Theorem 0 Ω convex domain in $\mathbb{C}^n$. Then for $w \in \Omega$

$$\frac{1}{\lambda(I_\Omega(w))} \le K_\Omega(w) \le \frac{4^n}{\lambda(I_\Omega(w))}, \qquad (1)$$

where

$$K_\Omega(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 d\lambda \le 1\}$$

is the Bergman kernel on the diagonal and

$$I_\Omega(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$$

is the Kobayashi indicatrix (Δ is the unit disk in $\mathbb{C}$).

The lower bound in (1) follows from the $\bar\partial$-equation and Lempert's theory. It is optimal: if Ω is balanced w.r.t. w then "$=$".

Sketch of proof of the upper bound Denote $I := int\, I_\Omega(w)$ and assume that $w = 0$. One can show that $I \subset 2\,\Omega$. Then

$$K_\Omega(0)\lambda(I) \le K_{I/2}(0)\lambda(I) = \frac{\lambda(I)}{\lambda(I/2)} = 4^n. \quad \square$$

If Ω is symmetric (w.r.t. $w = 0$) then $I \subset 4/\pi\,\Omega$ and $4 \rightsquigarrow 16/\pi^2$ in (1).

These estimates can be written as

$$1 \leq F_\Omega(w) \leq 4$$

where

$$F_\Omega(w) = \big(\lambda(I_\Omega(w))K_\Omega(w)\big)^{1/n}$$

is a biholomorphically invariant function.

- Find an example with $F_\Omega \not\equiv 1$.
- What are the properties of the function $w \longmapsto \lambda(I_\Omega(w))$?
- What is the optimal upper bound for F_Ω?

In this talk we will restrict ourselves to studying convex complex ellipsoids in $\mathbb{C}^2$:

$$\mathcal{E}(p,q) = \{z \in \mathbb{C}^2 \colon |z_1|^{2p} + |z_2|^{2q} < 1\}, \quad p,q \geq 1/2.$$

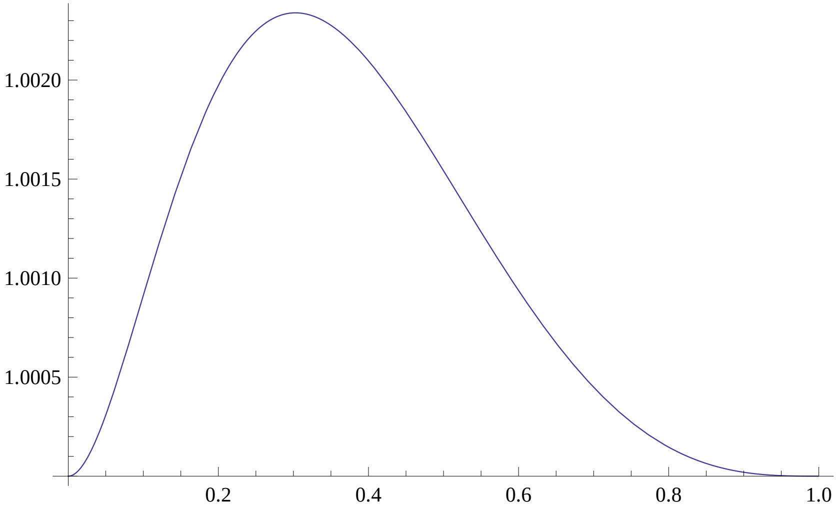
Blank-Fan-Klein-Krantz-Ma-Pang (1992) found implicit formulas for the Kobayashi function of $\mathcal{E}(m, 1)$. Thus they can be made explicit for $m = 1/2$. Using this one can prove

Theorem 1 For $\Omega = \{|z_1| + |z_2|^2 < 1\}$ and $b \in [0, 1)$ one has

$$\lambda(I_\Omega((b, 0))) = \frac{\pi^2}{3}(1 - b)^3(1 + 3b + 3b^2 - b^3)$$

and

$$\lambda(I_\Omega((b, 0)))K_\Omega((b, 0)) = 1 + \frac{(1 - b)^3 b^2}{3(1 + b)^3}.$$



$F_\Omega((b,0))$ for $\Omega = \{|z_1| + |z_2|^2 < 1\}$

Although the Kobayashi function of $\mathcal{E}(m,1)$ is given by implicit formulas, it turns out that the volume of the Kobayashi indicatrix can be computed explicitly:

Theorem 2 For $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$, $m \geq 1/2$, and $b \in [0,1)$ one has

$$\lambda(I_\Omega((b,0))) \\ = \pi^2 \left[-\frac{m-1}{2m(3m-2)(3m-1)} b^{6m+2} - \frac{3(m-1)}{2m(m-2)(m+1)} b^{2m+2} \right. \\ \left. + \frac{m}{2(m-2)(3m-2)} b^6 + \frac{3m}{3m-1} b^4 - \frac{4m-1}{2m} b^2 + \frac{m}{m+1} \right].$$

For $m = 2/3$

$$\lambda(I_\Omega((b,0))) = \frac{\pi^2}{80} \left(-65b^6 + 40b^6 \log b + 160b^4 - 27b^{10/3} - 100b^2 + 32 \right),$$

and $m = 2$

$$\lambda(I_\Omega((b,0))) = \frac{\pi^2}{240} \left(-3b^{14} - 25b^6 - 120b^6 \log b + 288b^4 - 420b^2 + 160 \right).$$

About the proof Main tool: Jarnicki-Pflug-Zeinstra (1993) formula for geodesics in convex complex ellipsoids. If

$$\mathbb{C} \supset U \ni z \longmapsto (f(z), g(z)) \in \partial I$$

is a parametrization of an S^1-invariant portion of ∂I then the volume of the corresponding part of I is given by

$$\frac{\pi}{2} \int_U |H(z)| d\lambda(z), \tag{2}$$

where

$$H = |f|^2(|g_{\bar{z}}|^2 - |g_z|^2) + |g|^2(|f_{\bar{z}}|^2 - |f_z|^2) + 2\mathrm{Re}\left(f\bar{g}(\overline{f_z}g_z - \overline{f_{\bar{z}}}g_{\bar{z}})\right).$$

Both H and the integral (2) are computed with the help of *Mathematica*. The same method is used for computations in other ellipsoids.

For $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ the formula for the Bergman kernel is well known:

$$K_\Omega(w) = \frac{1}{\pi^2}(1-|w_2|^2)^{1/m-2}\frac{(1/m+1)(1-|w_2|^2)^{1/m} + (1/m-1)|w_1|^2}{\left((1-|w_2|^2)^{1/m} - |w_1|^2\right)^3},$$

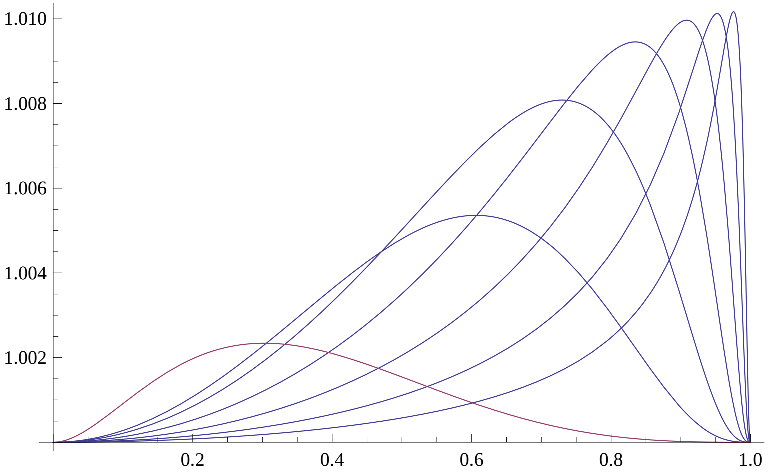
so that

$$K_\Omega((b,0)) = \frac{m+1+(1-m)b^2}{\pi^2 m(1-b^2)^3}.$$

Since for $t \in \mathbb{R}$ and $a \in \Delta$ the mapping

$$\Omega \ni z \longmapsto \left(e^{it}\frac{(1-|a|^2)^{1/2m}}{(1-\bar{a}z_2)^{1/m}}z_1, \frac{z_2-a}{1-\bar{a}z_2}\right)$$

is a holomorphic automorphism of Ω, $F_\Omega((b,0))$ for $b \in (0,1)$ attains all values of F_Ω in Ω.



$F_\Omega((b,0))$ in $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ for $m = 1/2, 4, 8, 16, 32, 64, 128$

$$\sup_{0<b<1} F_\Omega((b,0)) \to 1.010182\ldots \text{ as } m \to \infty$$

(highest value of F_Ω obtained so far in arbitrary dimension)

Theorem 3 For $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1)$ one has

$$\lambda(I_\Omega((b,0))) = \frac{\pi^2}{6}(1-b)^4\big((1-b)^4 + 8b\big)$$

and

$$\lambda(I_\Omega((b,0)))K_\Omega((b,0)) = 1 + b^2\frac{(1-b)^4}{(1+b)^4}.$$

The Bergman kernel for this ellipsoid was found by Hahn-Pflug (1988):

$$K_\Omega(w) = \frac{2}{\pi^2} \cdot \frac{3(1-|w|^2)^2(1+|w|^2) + 4|w_1|^2|w_2|^2(5-3|w|^2)}{\big((1-|w|^2)^2 - 4|w_1|^2|w_2|^2\big)^3},$$

so that

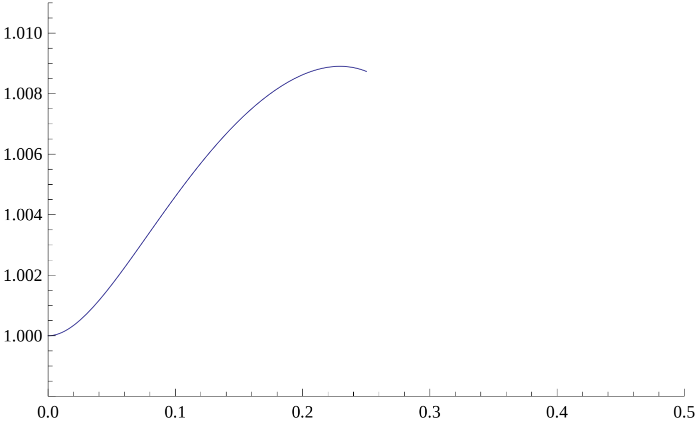
$$K_\Omega((b,0)) = \frac{6(1+b^2)}{\pi^2(1-b^2)^4}.$$

In all examples so far the function $w \mapsto \lambda(I_\Omega(w))$ is analytic. Is it true in general?

Theorem 4 For $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1/4]$ one has

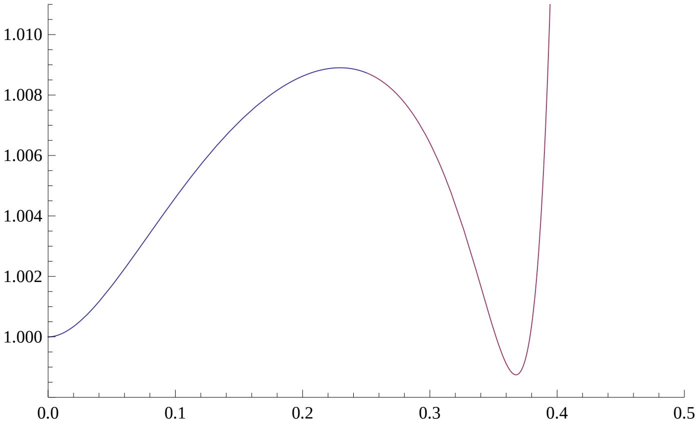
$$\lambda(I_\Omega((b,b))) = \frac{\pi^2}{6}\big(30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1\big).$$

Since $K_\Omega((b,b)) = \dfrac{2(3-6b^2+8b^4)}{\pi^2(1-4b^2)^3}$, we get the following picture:



$F_\Omega((b,b))$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $b \in [0, 1/4]$

Since $K_\Omega((b,b)) = \dfrac{2(3-6b^2+8b^4)}{\pi^2(1-4b^2)^3}$, we get the following picture:



$F_\Omega((b,b))$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $b \in [0, 1/4]$

By Theorem 0 $b \mapsto F_\Omega((b,b))$ cannot be analytic on $(0, 1/2)$!

Theorem 4 For $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1/4]$ one has

$$\lambda(I_\Omega((b,b))) = \frac{\pi^2}{6}\left(30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1\right).$$

For $b \in [1/4, 1/2)$

$$\begin{aligned}
\lambda(I_\Omega((b,b))) = {} & \frac{2\pi^2 b(1-2b)^3\left(-2b^3 + 3b^2 - 6b + 4\right)}{3(1-b)^2} \\
& + \frac{\pi\left(30b^{10} - 124b^9 + 238b^8 - 176b^7 - 260b^6 + 424b^5 - 76b^4 - 144b^3 + 89b^2 - 18b + 1\right)}{6(1-b)^2} \\
& \quad \times \arccos\left(-1 + \frac{4b-1}{2b^2}\right) \\
& + \frac{\pi(1-2b)\left(-180b^7 + 444b^6 - 554b^5 + 754b^4 - 1214b^3 + 922b^2 - 305b + 37\right)}{72(1-b)}\sqrt{4b-1} \\
& + \frac{4\pi b(1-2b)^4\left(7b^2 + 2b - 2\right)}{3(1-b)^2}\arctan\sqrt{4b-1} \\
& + \frac{4\pi b^2(1-2b)^4(2-b)}{(1-b)^2}\arctan\frac{1-3b}{(1-b)\sqrt{4b-1}}.
\end{aligned}$$

By $\chi_-(b)$, resp. $\chi_+(b)$, denote $\lambda(I_\Omega((b,b)))$ for $b \le 1/4$, resp. $b \ge 1/4$. Then at $b = 1/4$

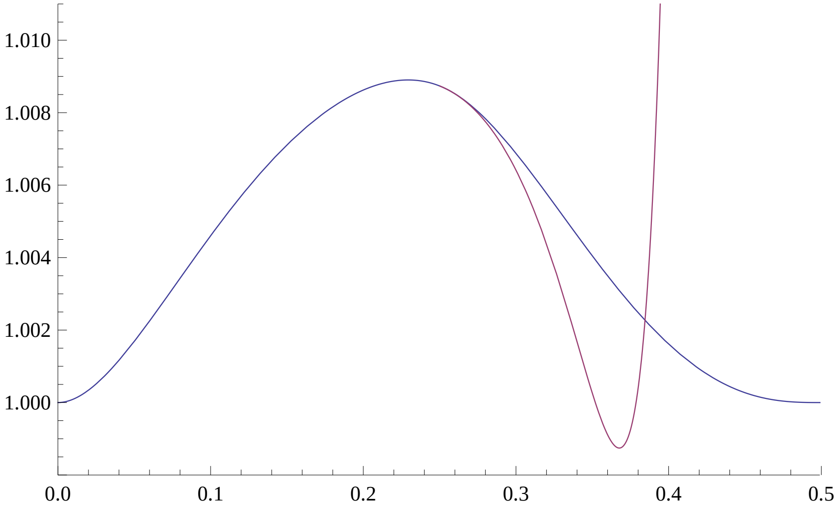
$$\chi_- = \chi_+ = \frac{15887}{196608}\pi^2, \qquad \chi'_- = \chi'_+ = -\frac{3521}{6144}\pi^2,$$

$$\chi''_- = \chi''_+ = -\frac{215}{1536}\pi^2, \quad \chi^{(3)}_- = \chi^{(3)}_+ = \frac{1785}{64}\pi^2,$$

but

$$\chi^{(4)}_- = \frac{1549}{16}\pi^2, \qquad \chi^{(4)}_+ = \infty.$$

Corollary For $\Omega = \{|z_1| + |z_2| < 1\}$ the function $w \mapsto \lambda(I_\Omega(w))$ is not $C^{3,1}$ at $w = (1/4, 1/4)$.



$F_\Omega((b,b))$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $b \in [0, 1/2)$

Thank you!