Suita Conjecture and the Ohsawa-Takegoshi Extension Theorem

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DMV-PTM Joint Meeting
Poznań, September 17–20, 2014

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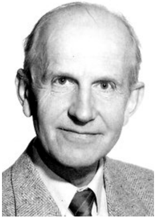
Topics

- Suita conjecture (1972) from one-dimensional complex analysis
- Optimal constant in the Ohsawa-Takegoshi extension theorem (1987) from several complex variables
- Mahler conjecture (1938) and Bourgain-Milman inequality (1987) from convex analysis

Link: Hörmander's L^2-estimate for $\bar{\partial}$-equation



Lars Hörmander (24 I 1931 - 25 XI 2012)

• *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. 113 (1965), 89–152

• *An Introduction to Complex Analysis in Several Variables*, Van Nostrand, 1966 (1st ed.)

Suita Conjecture

Green function for bounded domain D in $\mathbb{C}$:

$$\begin{cases} \Delta G_D(\cdot, z) = 2\pi\delta_z \\ G_D(\cdot, z) = 0 \text{ on } \partial D \text{ (if } D \text{ is regular)} \end{cases}$$

$$c_D(z) := \exp \lim_{\zeta \to z} (G_D(\zeta, z) - \log|\zeta - z|)$$

(logarithmic capacity of $\mathbb{C} \setminus D$ w.r.t. z)

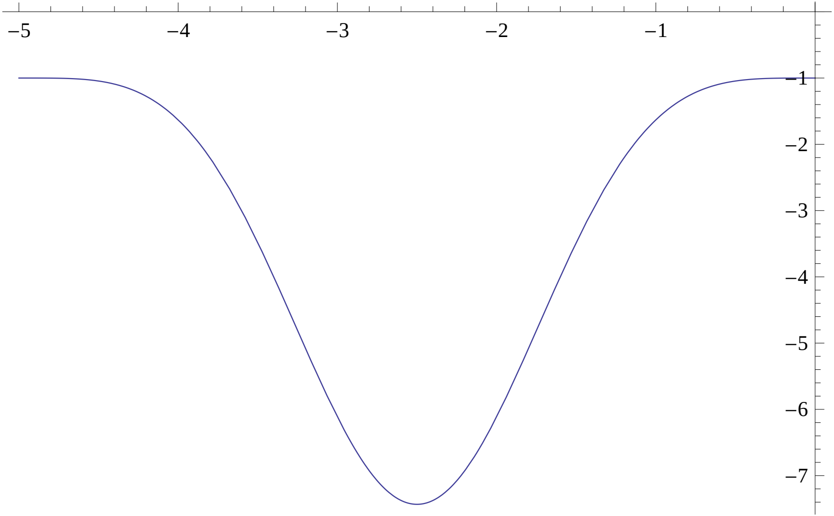
$c_D|dz|$ is an invariant metric (Suita metric)

$$Curv_{c_D|dz|} = -\frac{(\log c_D)_{z\bar{z}}}{c_D^2}$$

Suita Conjecture (1972) $\quad Curv_{c_D|dz|} \leq -1$

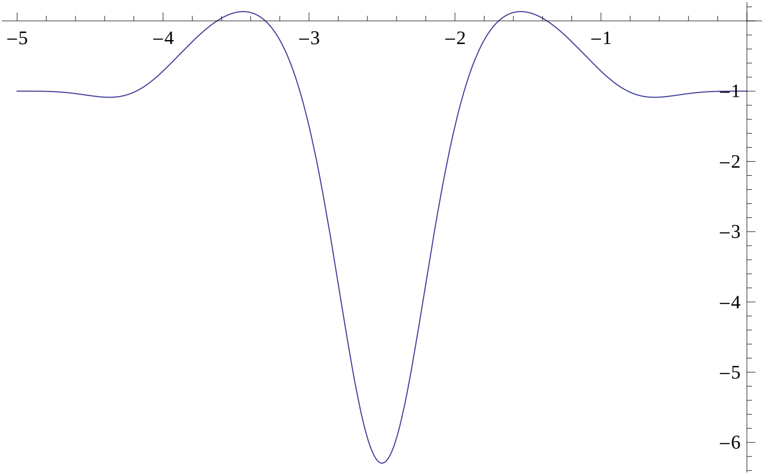
- "$=$" if D is simply connected
- "$<$" if D is an annulus (Suita)
- Enough to prove for D with smooth boundary
- "$=$" on ∂D if D has smooth boundary

We essentially ask whether $Curv_{c_D|dz|}$ satisfies the maximum principle.
In applied math. and physics it is in general a hard problem to compute the Green function for multiply connected domains, even numerically.



$Curv_{c_D|dz|}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log|z|$

In general, curvatures of invariant metrics do not satisfy the maximum principle: for example the curvature of the Bergman metric for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log |z|$ looks as follows



Reformulation of the Suita conjecture:

$$\frac{\partial^2}{\partial z \partial \bar{z}} (\log c_D) = \pi K_D, \qquad \text{(Suita)}$$

where K_D is the Bergman kernel on the diagonal:

$$K_D(z) := \sup\{|f(z)|^2 : f \in \mathcal{O}(D), \int_D |f|^2 d\lambda \leq 1\}.$$

(Bergman kernel really is the reproducing kernel for the L^2 holomorphic functions:

$$f(w) = \int_D f \overline{K_D(\cdot, w)} d\lambda, \quad f \in \mathcal{O} \cap L^2(D), \; w \in D.)$$

Therefore the Suita conjecture is equivalent to

$$c_D^2 \leq \pi K_D.$$

Ohsawa (1995) observed that it is really an extension problem: for $z \in D$ find holomorphic f in D such that $f(z) = 1$ and

$$\int_D |f|^2 d\lambda \leq \frac{\pi}{(c_D(z))^2}.$$

Using the methods of the original proof of the Ohsawa-Takegoshi extension theorem he showed the estimate

$$c_D^2 \leq C\pi K_D$$

with $C = 750$.

$C = 2$ (B., 2007)
$C = 1.95388\ldots$ (Guan-Zhou-Zhu, 2011)

Ohsawa-Takegoshi Extension Theorem

A function $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$, $\Omega \subset \mathbb{C}^n$, is called *plurisubharmonic* (psh) if it is u.s.c and subharmonic on every complex line.
Equivalently, $(\partial^2\varphi/\partial z_j \partial \bar{z}_k) \geq 0$.

A domain $\Omega \subset \mathbb{C}^n$ is called *pseudoconvex* (pscvx) if there exists a plurisubharmonic exhaustion function in Ω, i.e. $\varphi \in PSH(\Omega)$ such that $\{\varphi \leq t\} \subset\subset \Omega$ for every $t \in \mathbb{R}$.

(Analogy to convex functions and domains.)

Ohsawa-Takegoshi Extension Theorem (1987)
Ω bounded pscvx domain in $\mathbb{C}^n$, φ psh in Ω
H complex affine subspace of $\mathbb{C}^n$
f holomorphic in $\Omega' := \Omega \cap H$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq C\pi \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda',$$

where C depends only on n and the diameter of Ω.

Ohsawa-Takegoshi Extension Theorem (1987)
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where C depends only on n and the diameter of Ω.

Siu / Berndtsson (1996)
If $\Omega \subset \mathbb{C}^{n-1} \times \{|z_n| < 1\}$ and $H = \{z_n = 0\}$ then $C = 4$.

Problem Can we improve it to $C = 1$?

This can be treated as a multidimensional version of the Suita conjecture.

B.-Y. Chen (2011) Ohsawa-Takegoshi extension theorem can be proved using directly Hörmander's estimate for $\bar\partial$-equation!

$\bar{\partial}$ - Equation

For a complex-valued function u of n complex variables we define

$$\bar{\partial} u = \frac{\partial u}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial u}{\partial \bar{z}_n} d\bar{z}_n.$$

u is holomorphic if and only if $\bar{\partial} u = 0$. For a (0,1)-form

$$\alpha = \alpha_1 d\bar{z}_1 + \cdots + \alpha_n d\bar{z}_n$$

we set

$$\bar{\partial}\alpha = \bar{\partial}\alpha_1 \wedge d\bar{z}_1 + \cdots + \bar{\partial}\alpha_n \wedge d\bar{z}_n.$$

We will consider the equation

$$\bar{\partial} u = \alpha.$$

Since $\bar{\partial}^2 = 0$, the necessary condition is $\bar{\partial}\alpha = 0$, that is

$$\frac{\partial \alpha_j}{\partial \bar{z}_k} = \frac{\partial \alpha_k}{\partial \bar{z}_j}.$$

Hörmander's Estimate

Theorem (Hörmander, 1965)
Ω pscvx in $\mathbb{C}^n$, φ smooth, strongly psh in Ω
$\alpha = \sum_j \alpha_j d\bar{z}_j \in L^2_{loc,(0,1)}(\Omega)$, $\bar{\partial}\alpha = 0$
Then one can find $u \in L^2_{loc}(\Omega)$ with $\bar{\partial}u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq \int_\Omega |\alpha|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} d\lambda.$$

Here $|\alpha|^2_{i\partial\bar{\partial}\varphi} = \sum_{j,k} \varphi^{j\bar{k}} \bar{\alpha}_j \alpha_k$, where $(\varphi^{j\bar{k}}) = (\partial^2\varphi/\partial z_j \partial\bar{z}_k)^{-1}$ is the length of α w.r.t. the Kähler metric $i\partial\bar{\partial}\varphi$.

Hörmander's estimate for $(0,1)$-forms is a great tool for constructing holomorphic functions (even in one variable!).
For $\alpha = \bar{\partial}\chi$ and any solution u to

$$\bar{\partial}u = \alpha$$

the function $f = \chi - u$ is holomorphic.

Building up on Donnelly-Fefferman, Berndtsson and B.-Y. Chen one can show:

Theorem (B., 2013) Ω pscvx in $\mathbb{C}^n$, φ smooth, strongly psh in Ω
$\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
$\psi \in W^{1,2}_{loc}(\Omega)$ locally bounded from above, s.th.

$$|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \begin{cases} \leq 1 & \text{in } \Omega \\ \leq \delta < 1 & \text{on } \operatorname{supp}\alpha \end{cases}.$$

Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}) e^{2\psi - \varphi} d\lambda \leq \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}} \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda.$$

Remarks 1. Setting $\psi \equiv 0$ we recover the Hörmander estimate.

2. This theorem implies previous estimates for $\bar\partial$ due to Donnelly-Fefferman and Berndtsson with optimal constants.

3. Most importantly: it gives the Ohsawa-Takegoshi extension theorem with optimal constant.

Theorem (B., 2013) Ω pscvx in $\mathbb{C}^n$, φ smooth, strongly psh in Ω,
$\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
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Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2(1 - |\bar\partial\psi|^2_{i\partial\bar\partial\varphi})e^{2\psi-\varphi}d\lambda \leq \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}} \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi}d\lambda.$$

Proof By approximation we may assume that φ, ψ are bounded in Ω
u minimal solution to $\bar\partial u = \alpha$ in $L^2(\Omega, e^{\psi-\varphi})$
$\Rightarrow u \perp \ker\bar\partial$ in $L^2(\Omega, e^{\psi-\varphi})$
$\Rightarrow v := ue^{\psi} \perp \ker\bar\partial$ in $L^2(\Omega, e^{-\varphi})$ (twisting)
$\Rightarrow v$ minimal solution to $\bar\partial v = \beta := e^{\psi}(\alpha + u\bar\partial\psi)$ in $L^2(\Omega, e^{-\varphi})$

$$\text{Hörmander} \quad \Rightarrow \quad \int_\Omega |v|^2 e^{-\varphi}d\lambda \leq \int_\Omega |\beta|^2_{i\partial\bar\partial\varphi} e^{-\varphi}d\lambda$$

Therefore

$$\int_\Omega |u|^2 e^{2\psi-\varphi} d\lambda \le \int_\Omega |\alpha + u\,\bar\partial\psi|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda$$
$$\le \int_\Omega \left(|\alpha|^2_{i\partial\bar\partial\varphi} + 2|u|\sqrt{H}|\alpha|_{i\partial\bar\partial\varphi} + |u|^2 H\right) e^{2\psi-\varphi} d\lambda,$$

where $H = |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}$. For $t > 0$ we will get

$$\int_\Omega |u|^2(1-H)e^{2\psi-\varphi} d\lambda$$
$$\le \int_\Omega \left[|\alpha|^2_{i\partial\bar\partial\varphi}\left(1 + t^{-1}\frac{H}{1-H}\right) + t|u|^2(1-H)\right] e^{2\psi-\varphi} d\lambda$$
$$\le \left(1 + t^{-1}\frac{\delta}{1-\delta}\right)\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda$$
$$+ t\int_\Omega |u|^2(1-H)e^{2\psi-\varphi} d\lambda.$$

We will obtain the required estimate if we take $t := 1/(\delta^{-1/2} + 1)$.

Theorem (Ohsawa-Takegoshi with optimal constant, B. 2013)
Ω pscvx in $\mathbb{C}^{n-1} \times D$, where $0 \in D \subset \mathbb{C}$,
φ psh in Ω, f holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

(For $n = 1$ and $\varphi \equiv 0$ we obtain the Suita conjecture.)

Crucial ODE Problem Find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ s.th. $h' < 0$, $h'' > 0$,

$$\lim_{t\to\infty}(g(t) + \log t) = \lim_{t\to\infty}(h(t) + \log t) = 0$$

and

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1.$$

Solution

$$h(t) := -\log(t + e^{-t} - 1)$$
$$g(t) := -\log(t + e^{-t} - 1) + \log(1 - e^{-t}).$$

Guan-Zhou recently gave another proof of the Ohsawa-Takegoshi with optimal constant (and obtained some generalizations) but used essentially the same ODE.

They also answered the following, more detailed problem posed by Suita:

Theorem (Guan-Zhou) Let M be a Riemann surface admitting a non-constant bounded subharmonic function. Then one has equality in the Suita conjecture (at any point) if and only if $M \equiv \Delta \setminus F$, where F is a closed polar subset of Δ.

A General Lower Bound for the Bergman Kernel

Theorem Assume that Ω is pscvx in $\mathbb{C}^n$. Then for $t \le 0$ and $w \in \Omega$

$$K_\Omega(w) \ge \frac{1}{e^{-2nt}\lambda(\{G_{\Omega,w} < t\})},$$

where

$$G_\Omega(\cdot, w) = G_{\Omega,w} = \sup\{u \in PSH^-(\Omega),\ \overline{\lim_{z\to w}}(u(z) - \log|z-w|) < \infty\}$$

is the pluricomplex Green function with pole at w.

For $n = 1$ letting $t \to -\infty$ this gives the Suita conjecture:

$$K_\Omega(w) \ge \frac{c_\Omega(w)^2}{\pi}.$$

Proof 1 (sketch) Using the Donnelly-Fefferman estimate for $\bar\partial$ one can show that

$$K_\Omega(w) \geq \frac{|f(w)|^2}{||f||^2} \geq \frac{c_{n,a}}{\lambda(\{G_{\Omega,w} < -a\})},$$

where

$$c_{n,a} = \frac{\mathrm{Ei}(na)^2}{(\mathrm{Ei}(na) + \sqrt{C})^2}, \qquad \mathrm{Ei}(a) = \int_a^\infty \frac{e^{-s}}{s}\, ds.$$

Tensor power trick $\widetilde\Omega := \Omega^m \subset \mathbb{C}^{nm}$, $\widetilde w := (w, \dots, w)$, $m \gg 0$

$$K_{\widetilde\Omega}(\widetilde w) = (K_\Omega(w))^m, \quad \lambda_{2nm}(\{G_{\widetilde\Omega,\widetilde w} < -a\}) = (\lambda_{2n}(\{G_{\Omega,w} < -a\})^m.$$

$$(K_\Omega(w))^m \geq \frac{c_{nm,a}}{(\lambda_{2n}(\{G_{\Omega,w} < -a\}))^m}$$

but

$$\lim_{m\to\infty} c_{nm,a}^{1/m} = e^{-2na}. \quad \square$$

Proof 2 (Lempert) By Maitani-Yamaguchi / Berndtsson's result on log-(pluri)subharmonicity of the Bergman kernel for sections of a pseudoconvex domain it follows that $\log K_{\{G_{\Omega,w}<t\}}(w)$ is convex for $t \in (-\infty, 0]$. Therefore

$$t \longmapsto 2nt + \log K_{\{G_{\Omega,w}<t\}}(w)$$

is convex and bounded, hence non-decreasing. It follows that

$$K_\Omega(w) \geq e^{2nt} K_{\{G_{\Omega,w}<t\}}(w) \geq \frac{e^{2nt}}{\lambda(\{G_{\Omega,w} < t\})}. \quad \square$$

Three proofs of the Suita conjecture:

1. One-dimensional (ODE)
2. Infinitely-dimensional (tensor power trick)
3. Two-dimensional (Lempert)

Berndtsson-Lempert Proof 2 can be improved to obtain the Ohsawa-Takegoshi extension theorem with optimal constant (one has to use Berndtsson's positivity of direct image bundles).

Theorem Assume Ω is pscvx in $\mathbb{C}^n$. Then for $t \leq 0$ and $w \in \Omega$

$$K_\Omega(w) \geq \frac{1}{e^{-2nt}\lambda(\{G_{\Omega,w} < t\})}.$$

What happens when $t \to -\infty$ for arbitrary n?
For convex domains one can use Lempert's theory to obtain:

Theorem If Ω is a convex domain in $\mathbb{C}^n$ then for $w \in \Omega$

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega(w))},$$

$I_\Omega(w) = \{\varphi'(0)\colon \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$ (Kobayashi indicatrix).

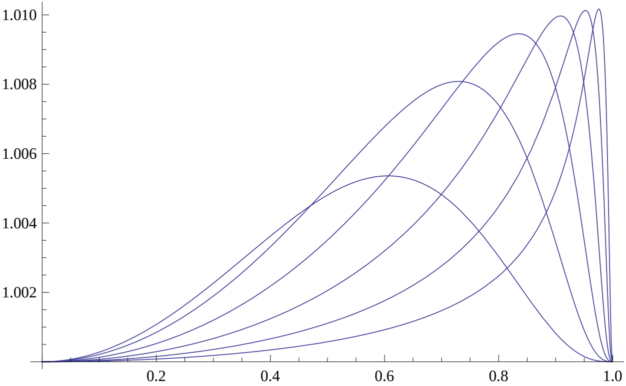
Multidimensional version of the Suita conjecture (B.-Zwonek)
If $\Omega \subset \mathbb{C}^n$ is pscvx and $w \in \Omega$ then

$$K_\Omega(w) \geq \frac{1}{\lambda(I^A_\Omega(w))},$$

$I^A_\Omega(w) = \{X \in \mathbb{C}^n : \overline{\lim}_{\zeta\to 0}\left(G_{\Omega,w}(w+\zeta X) - \log|\zeta|\right) \leq 0\}$
(Azukawa indicatrix)

For convex domains we also have the upper bound:

Theorem (B.-Zwonek) Ω convex, $w \in \Omega \Rightarrow K_\Omega(w) \le \dfrac{4^n}{\lambda(I_\Omega(w))}$.



$\big(K_\Omega(w)\lambda(I_\Omega(w))\big)^{1/2}$ for $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$, $w = (0, b)$, $0 < b < 1$
$m = 4, 8, 16, 32, 64, 128$

$\sup_\Omega \to 1.010182\ldots$ as $m \to \infty$

Theorem Assume Ω is pscvx in $\mathbb{C}^n$. Then for $t \le 0$ and $w \in \Omega$

$$K_\Omega(w) \ge \frac{1}{e^{-2nt}\lambda(\{G_{\Omega,w} < t\})}.$$

Conjecture For pseudoconvex Ω the function $t \mapsto e^{2nt}\lambda(\{G_{\Omega,w} < t\})$ is increasing.

Theorem (B.-Zwonek) Conjecture is true for $n = 1$.

Proof: isoperimetric inequality

For arbitrary n the conjecture is equivalent to the following *pluripotential isoperimetric inequality*:

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G_{\Omega,w}|} \ge 2\lambda(\Omega)$$

for smooth, strongly pseudoconvex Ω.

Possible future interest: compact Kähler manifolds.

Mahler Conjecture

K - convex symmetric body in $\mathbb{R}^n$

$$K' := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } x \in K\}$$

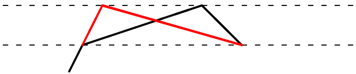
Mahler volume $:= \lambda(K)\lambda(K')$

Mahler volume is an invariant of the Banach space defined by K: it is independent of linear transformations and of the choice of inner product.

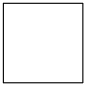
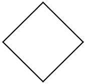
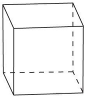
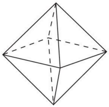
Blaschke-Santaló Inequality (1949) Mahler volume is maximized by balls

Mahler Conjecture (1938) Mahler volume is minimized by cubes

True for $n = 2$:



Hansen-Lima bodies: starting from an interval they are produced by taking products of lower dimensional HL bodies and their duals.

$n = 2$  $\simeq$  $n = 3$  $\not\simeq$ 

Equivalent SCV formulation (Nazarov, 2012)

For $u \in L^2(K')$ we have

$$|\widehat{u}(0)|^2 = \left| \int_{K'} u \, d\lambda \right|^2 \leq \lambda(K') ||u||^2_{L^2(K')} = (2\pi)^{-n} \lambda(K') ||\widehat{u}||^2_{L^2(\mathbb{R}^n)}$$

with equality for $u = \chi_{K'}$. Therefore

$$\lambda(K') = (2\pi)^n \sup_{f \in \mathcal{P}} \frac{|f(0)|^2}{||f||^2_{L^2(\mathbb{R}^n)}},$$

where $\mathcal{P} = \{\widehat{u} \colon u \in L^2(K')\} \subset \mathcal{O}(\mathbb{C}^n)$. By the Paley-Wiener thm

$$\mathcal{P} = \{f \in \mathcal{O}(\mathbb{C}^n) \colon |f(z)| \leq Ce^{C|z|}, \ \ |f(iy)| \leq Ce^{q_K(y)}\},$$

where q_K is the Minkowski function for K. Therefore the Mahler conjecture is equivalent to finding $f \in \mathcal{P}$ with $f(0) = 1$ and

$$\int_{\mathbb{R}^n} |f(x)|^2 d\lambda(x) \leq n! \left(\frac{\pi}{2}\right)^n \lambda(K).$$

Bourgain-Milman Inequality

Bourgain-Milman (1987) There exists $c > 0$ such that

$$\lambda(K)\lambda(K') \geq c^n \frac{4^n}{n!}.$$

Mahler Conjecture: $c = 1$

G. Kuperberg (2006) $c = \pi/4$

Nazarov (2012) SCV proof using Hörmander's estimate ($c = (\pi/4)^3$)

Consider the tube domain $T_K := \mathrm{int} K + i\mathbb{R}^n \subset \mathbb{C}^n$. Then

$$\left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda_n(K))^2} \leq K_{T_K}(0) \leq \frac{n!}{\pi^n} \frac{\lambda_n(K')}{\lambda_n(K)}.$$

Therefore

$$\lambda_n(K)\lambda_n(K') \geq \left(\frac{\pi}{4}\right)^{3n} \frac{4^n}{n!}.$$

The upper bound $K_{T_K}(0) \le \frac{n!}{\pi^n} \frac{\lambda_n(K')}{\lambda_n(K)}$ easily follows from Rothaus' formula (1968):

$$K_{T_K}(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{d\lambda}{J_K},$$

where

$$J_K(y) = \int_K e^{-2x\cdot y} d\lambda(x).$$

To show the lower bound $K_{T_K}(0) \ge \left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda_n(K))^2}$ we can use the estimate:

$$K_{T_K}(0) \ge \frac{1}{\lambda_{2n}(I_{T_K}(0))}$$

and

Proposition $I_{T_K}(0) \subset \frac{4}{\pi}(K + iK)$

Conjecture $K_{T_K}(0) \ge \left(\frac{\pi}{4}\right)^n \frac{1}{(\lambda_n(K))^2}$

This would be optimal, since we have equality for cubes.

However, one can check that for $K = \{|x_1| + |x_2| + |x_3| \leq 1\}$ we have

$$K_{T_K}(0) > \left(\frac{\pi}{4}\right)^3 \frac{1}{(\lambda_3(K))^2}.$$

This shows that Nazarov's proof of the Bourgain-Milman inequality cannot give the Mahler conjecture directly.

Thank you!